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On ωb-open sets and b-Lindelöf spaces

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Abstract. In this paper, we introduce and investigate a new class of sets called ωb-open sets which is weaker than both ω-open sets and b-open sets. Moreover, we obtain a characterization and preserving theorems of b-Lindelöf spaces.

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Key words: b-open set, ω-open set, , b-Lindelöf space

1. Introduction

Throughout this paper, (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be a denoted by $Cl(A)$ and $Int(A)$, respectively. Let (X, τ) be a space and let A be a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [8] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X, τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in the same way as $Cl(A)$ and $Int(A)$, respectively, will be denoted by $Cl_\omega(A)$ and $Int_\omega(A)$, respectively. Several characterizations of ω-closed subsets were provided in [1,8,9].

Andrijević [4] introduced a new class of generalized open sets in a topological space, the so-called b-open sets. This type of sets was discussed by [7] under the name of γ-open sets. The class of b-open sets is contained in the class of semi-preopen sets and contains all semi-open sets and preopen sets. The class of b-open sets generates the same topology as the class of preopen sets. Since the advent of these notions, several research paper with interesting results in different respects came to existence see [2, 3, 5, 10, 11].

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Definition 1.1. A subset A of a space X is said to be b-open [4] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.

The complement of a b-open set is said to be b-closed [4]. The intersection of all b-closed sets of X containing A is called the b-closure of A and is denoted by $bCl(A)$. The union of all b-open sets of X contained in A is called the b-interior of A and is denoted by $bInt(A)$. The family of all b-open (resp. b-closed) subsets of a space X is denoted by $BO(X)$ (resp. $BC(X)$) and the collection of all b-open subsets of X containing a fixed point x is denoted by $BO(X,x)$. In this paper, we introduce a new generalization of ω-open set and b-open set and investigate some properties of this set. Moreover, we obtain a characterization and preserving theorems of b-Lindelöf spaces.

2. ωb-open sets

In this section we introduce the following notion:

Definition 2.1. A subset A of a space X is said to be ωb-open if for every $x \in A$, there exists a b-open subset $U_x \subseteq X$ containing x such that $U_x - A$ is countable. The complement of an ωb-open subset is said to be ωb-closed.

Lemma 2.2. *For a subset of a topological space, both ω-openness and b-openness imply ωb-openness.*

Proof. (1) Assume A is ω-open then, for each $x \in A$, there is an open set containing x such that $U_x - A$ is countable set. Since every open set is b-open, A is ωb-open.
(2) Let A be b-open. For each $x \in A$, there exists a b-open set $U_x = A$ such that $x \in U_x$ and $U_x - A = \phi$. Therefore, A is ωb-open.

The following diagram shows the implications for properties of subsets

$$\begin{array}{ccc} \text{open set} & \longrightarrow & b\text{-open set} \\ \downarrow & & \downarrow \\ \omega\text{-open set} & \longrightarrow & \omega b\text{-open set} \end{array}$$

The converses need not be true as shown by the following examples.

Example 2.3. *Let $X = \{a,b,c\}$ and $\tau = \{X, \phi, \{a\}\}, \{b\}, \{a,b\}\}$, then $BO(X) = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,c\}\}$. Then $\{c\}$ is ω-open (since X is a countable set) and it is not b-open.*

Example 2.4. *Let $X = \mathbb{R}$ with the usual topology τ. Let $A = \mathbb{Q}$ be the set of all rational numbers. Then A is b-open but it is not ω-open.*

Lemma 2.5. *A subset A of a space X is ωb-open if and only if for every $x \in A$, there exists a b-open subset U containing x and a countable subset C such that $U - C \subseteq A$.*

Proof. Let A be ωb-open and $x \in A$, then there exists a b-open subset U_x containing x such that $|U_x - A|$ is countable. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exists a b-open subset U_x containing x and a countable subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is countable set.

Theorem 2.6. *Let X be a space and $C \subseteq X$. If C is ωb-closed, then $C \subseteq K \cup B$ for some b-closed subset K and a countable subset B.*

Proof. If C is ωb-closed, then $X - C$ is ωb-open and hence for every $x \in X - C$, there exists a b-open set U containing x and a countable set B such that $U - B \subseteq X - C$. Thus $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$. Let $K = X - U$. Then K is b-closed such that $C \subseteq K \cup B$.

Lemma 2.7. *[4] Let (X, τ) be a topological space.*

1. *The intersection of an open set and a b-open set is a b-open set.*
2. *The union of any family of b-open sets is a b-open set.*

Proposition 2.8. *The intersection of an ωb-open set and an ω-open set is ωb-open.*

Proof. Let A be an ωb-open set and B an ω-open set in a space X. Let x be any point of $A \cap B$. Since A is ωb-open, there exists a b-open set U_A containing x such that $|U_A - A|$ is countable. Since B is ω-open, there exists an open set U_B containing x such that $|U_B - B|$ is countable. By Lemma 2.7, $U_A \cap U_B$ is a b-open set containing x and

$$\begin{aligned}(U_A \cap U_B) - (A \cap B) &= (U_A \cap U_B) \cap [(X - A) \cup (X - B)] \\ &= [U_A \cap U_B \cap (X - A)] \cup [U_A \cap U_B \cap (X - B)] \\ &\subseteq (U_A \cap (X - A)) \cup (U_B \cap (X - B)).\end{aligned}$$

Since $(U_A \cap (X - A)) \cup (U_B \cap (X - B))$ is a countable set, $|(U_A \cap U_B) - (A \cap B)|$ is countable. This shows that $A \cap B$ is ωb-open.

Corollary 2.9. *The intersection of an ωb-open set with an open set is ωb-open.*

The intersection of two ωb-open sets is not always ωb-open.

Example 2.10. *Let $X = \mathbb{R}$ with the usual topology τ. Let $A = \mathbb{Q}$ be the set of all rational numbers and $B = [0, 1)$. Then A and B are ωb-open, but $A \cap B$ is not ωb-open, since each b-open containing 0 is uncountable set.*

Proposition 2.11. *The union of any family of ωb-open sets is ωb-open.*

Proof. If $\{A_\alpha : \alpha \in \Lambda\}$ is a collection of ωb-open subsets of X, then for every $x \in \cup_{\alpha\in\Lambda} A_\alpha$, $x \in A_\beta$ for some $\beta \in \Lambda$. Hence there exists a b-open subset U of X containing x such that $U \backslash A_\beta$ is countable. Now as $U \backslash (\cup_{\alpha\in\Lambda} A_\alpha) \subseteq U \backslash A_\beta$ and thus $U \backslash (\cup_{\alpha\in\Lambda} A_\alpha)$ is countable. Therefore, $\cup_{\alpha\in\Lambda} A_\alpha$ is ωb-open.

The intersection of all ωb-closed sets of X containing A is called the ωb-closure of A and is denoted by $\omega bCl(A)$. And the union of all ωb-open sets of X contained in A is called the ωb-interior and is denoted by $\omega bInt(A)$.

Theorem 2.12. *If each non-empty b-open set of a space X is uncountable, then $bCl(A) = \omega bCl(A)$ for each open set A of X.*

Proof. Clearly $\omega bCl(A) \subseteq bCl(A)$. On the other hand, let $x \in bCl(A)$ and B be an ωb-open subset containing x. Then by Lemma 2.5, there exists a b-open set V containing x and a countable set C such that $V - C \subseteq B$. Thus $(V - C) \cap A \subseteq B \cap A$ and so $(V \cap A) - C \subseteq B \cap A$. Since $x \in V$ and $x \in bCl(A)$, $V \cap A \neq \phi$ and $V \cap A$ is b-open since V is b-open and A is open. By the hypothesis each non-empty b-open set of a space X is uncountable and so is $(V \cap A) - C$. Thus $B \cap A$ is uncountable. Therefore, $B \cap A \neq \phi$ which means that $x \in \omega bCl(A)$.

Corollary 2.13. *If each non-empty b-open set of a space X is uncountable, then $bInt(A) = \omega bInt(A)$ for each closed set A of X.*

Definition 2.14. A function $f : X \to Y$ is said to be quasi b-open if the image of each b-open set in X is open in Y.

Proposition 2.15. *If $f : X \to Y$ is quasi b-open, then the image of an ωb-open set of X is ω-open in Y.*

Proof. Let $f : X \to Y$ be quasi b-open and W an ωb-open subset of X. Let $y \in f(W)$, there exists $x \in W$ such that $f(x) = y$. Since W is ωb-open, there exists a b-open set U such that $x \in U$ and $U - W = C$ is countable. Since f is quasi b-open, $f(U)$ is open in Y such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$ is countable. Therefore, $f(W)$ is ω-open in Y.

3. b-lindelöf spaces

Definition 3.1. (1) [6] A space X is said to be b-Lindelöf if every b-open cover of X has a countable subcover.
(2) A subset A of a space X is said to be b-Lindelöf relative to X if every cover of A by b-open sets of X has a countable subcover.

Theorem 3.2. *If X is a space such that every b-open subset of X is b-Lindelöf relative to X, then every subset is b-Lindelöf relative to X.*

Proof. Let B be an arbitrary subset of X and let $\{U_i : i \in I\}$ be a cover of B by b-open sets of X. Then the family $\{U_i : i \in I\}$ is a b-open cover of the b-open set $\cup\{U_i : i \in I\}$ by Lemma 2.7. Hence by hypothesis there is a countable subfamily $\{U_{i_j} : j \in \mathbb{N}\}$ which covers $\cup\{U_i : i \in I\}$. This subfamily is also a cover of the set B.

Theorem 3.3. *For any space X, the following properties are equivalent:*

1. *X is b-Lindelöf;*
2. *Every ωb-open cover of X has a countable subcover.*

Proof. (1)⇒ (2): Let $\{U_\alpha : \alpha \in \Lambda\}$ be any ωb-open cover of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is ωb-open,there exists a b-open set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)}$ and $V_{\alpha(x)} \backslash U_{\alpha(x)}$ is countable. The family $\{V_{\alpha(x)} | x \in X\}$ is a b-open cover of X and X is b-Lindelöf. There exists a countable subset, says $\alpha(x_1), \alpha(x_2), \cdots \alpha(x_n), \cdots$ such that $X = \cup\{V_{\alpha(x_i)} | i \in \mathbf{N}\}$. Now, we have

$$\begin{aligned} X &= \cup_{i\in\mathbf{N}} \{(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}) \cup U_{\alpha(x_i)}\} \\ &= [\cup_{i\in\mathbf{N}}(V_{\alpha(x_i)} \backslash U_{\alpha(x_i)})] \cup [\cup_{i\in\mathbf{N}} U_{\alpha(x_i)}]. \end{aligned}$$

For each $\alpha(x_i)$, $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)}$ is a countable set and there exists a countable subset $\Lambda_{\alpha(x_i)}$ of Λ such that $V_{\alpha(x_i)} \backslash U_{\alpha(x_i)} \subseteq \cup\{U_\alpha | \alpha \in \Lambda_{\alpha(x_i)}\}$. Therefore, we have $X \subseteq [\cup_{i\in\mathbf{N}}(\cup\{U_\alpha | \alpha \in \Lambda_{\alpha(x_i)}\})] \cup [\cup_{i\in\mathbf{N}} U_{\alpha(x_i)}]$.
(2)⇒ (1): Since every b-open is ωb-open, the proof is obvious.

Definition 3.4. A function $f : X \to Y$ is said to be ωb-continuous if $f^{-1}(V)$ is ωb-open in X for each open set V in Y.

Theorem 3.5. *Let f be an ωb-continuous function from a space X onto a space Y. If X is b-Lindelöf, then Y is Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an ωb-open cover of X. Since X is b-Lindelöf, by Theorem 3.3, X has a countable subcover, say $\{f^{-1}(V_{\alpha i})\}_{i=1}^{\infty}$ and $V_{\alpha i} \in \{V_\alpha : \alpha \in \Lambda\}$. Hence $\{V_{\alpha i}\}_{i=1}^{\infty}$ is a countable subcover of Y. Hence Y is Lindelöf.

Definition 3.6. A function $f : X \to Y$ is said to be γ-continuous [7] (resp. ω-continuous [9]) if $f^{-1}(V)$ is b-open (resp. ω-open) for each open set V in Y.

Since the notion of b-open sets and the notion of γ-open sets are same, we will use the term b-continuous functions instead of γ-continuous functions.

Corollary 3.7. *Let f be a b-continuous (or ω-continuous) function from a space X onto a space Y. If X is b-Lindelöf, then Y is Lindelöf.*

Definition 3.8. A function $f : X \to Y$ is said to be ωb^*-continuous if $f^{-1}(V)$ is ωb-open in X for each b-open set V in Y.

Now we state the following theorem whose proof is similar to Theorem 3.5.

Theorem 3.9. *Let f be an ωb^*-continuous function from a space X onto a space Y. If X is b-Lindelöf, then Y is b-Lindelöf.*

Proposition 3.10. *An ωb-closed subset of a b-Lindelöf space X is b-Lindelöf relative to X.*

Proof. Let A be an ωb-closed subset of X. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by b-open sets of X. Now for each $x \in X - A$, there is a b-open set V_x such that $V_x \cap A$ is countable. Since $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X - A\}$ is a b-open cover of X and X is b-Lindelöf, there exists a countable subcover $\{U_{\alpha_i} : i \in \mathbb{N}\} \cup \{V_{x_i} : i \in \mathbb{N}\}$. Since $\cup_{i\in N}(V_{x_i} \cap A)$ is countable, so for each $x_j \in \cup(V_{x_i} \cap A)$, there is $U_{\alpha(x_j)} \in \{U_\alpha : \alpha \in \Lambda\}$ such that $x_j \in U_{\alpha(x_j)}$ and $j \in \mathbb{N}$. Hence $\{U_{\alpha_i} : i \in \mathbb{N}\} \cup \{U_{\alpha(x_j)} : j \in \mathbb{N}\}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$ and it covers A. Therefore, A is b-Lindelöf relative to X.

Corollary 3.11. *If a space X is b-Lindelöf and A is ω-closed (or b-closed), then A is b-Lindelöf relative to X.*

Definition 3.12. A function $f : X \to Y$ is said to be ωb-closed if $f(A)$ ωb-closed in Y for each b-closed set A of X.

Theorem 3.13. *If $f : X \to Y$ is an ωb-closed surjection such that $f^{-1}(y)$ is b-Lindelöf relative to X and Y is b-Lindelöf, then X is b-Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be any b-open cover of X. For each $y \in Y$, $f^{-1}(y)$ is b-Lindelöf relative to X and there exists a countable subset $\Lambda_1(y)$ of Λ such that $f^{-1}(y) \subset \cup\{U_\alpha : \alpha \in \Lambda_1(y)\}$. Now we put $U(y) = \cup\{U_\alpha : \alpha \in \Lambda_1(y)\}$ and $V(y) = Y - f(X - U(y))$. Then, since f is ωb-closed, $V(y)$ is an ωb-open set in Y containing y such that $f^{-1}(V(y)) \subset U(y)$. Since $V(y)$ is ωb-open, there exists a b-open set $W(y)$ containing y such that $W(y) - V(y)$ is a countable set. For each $y \in Y$, we have $W(y) \subset (W(y) - V(y)) \cup V(y)$ and hence

$$\begin{aligned} f^{-1}(W(y)) &\subset f^{-1}(W(y) - V(y)) \cup f^{-1}(V(y)) \\ &\subset f^{-1}(W(y) - V(y)) \cup U(y). \end{aligned}$$

Since $W(y) - V(y)$ is a countable set and $f^{-1}(y)$ is b-Lindelöf relative to X, there exists a countable set $\Lambda_2(y)$ of Λ such that

$$f^{-1}(W(y) - V(y)) \subset \cup\{U_\alpha : \alpha \in \Lambda_2(y)\}$$

and hence

$$f^{-1}(W(y)) \subset [\cup\{U_\alpha : \alpha \in \Lambda_2(y)\}] \cup [U(y)].$$

Since $\{W(y) : y \in Y\}$ is a b-open cover of the b-Lindelöf space Y, there exist countable points of Y, say, $y_1, y_2, ..., y_n, ...$ such that
$Y = \cup\{W(y_i) : i \in N\}$. Therefore, we obtain

$$\begin{aligned} X &= \cup_{i\in N} f^{-1}(W(y_i)) = \cup_{i\in N}[\cup_{\alpha\in\Lambda_2(y_i)} U_\alpha) \cup (\cup_{\alpha\in\Lambda_1(y_i)} U_\alpha)] \\ &= \cup\{U_\alpha : \alpha \in \Lambda_1(y_i) \cup \Lambda_2(y_i), i \in N\}. \end{aligned}$$

This shows that X is b-Lindelöf.

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