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Global existence of solutions for a system modelling electromigration of ions through biological cell membranes with L^1 data

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Abstract. The aim of this work is to show the existence of weak solutions and supersolutions for a nonlinear system modelling Ions migration through biological cells membranes with L^1- Data. In the first step, we describe the mathematical model after that we define an approximating scheme. Under simplifying assumptions on the model equation, we prove some L^1 a priori estimates, then we prove that the solution of the truncated system converges to the solution of our main problem.

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1. Introduction

Mathematical models is an abstract model that uses mathematical language to describe the behaviour of a system. Mathematical models are used particularly in the natural sciences and engineering disciplines such as physics, biology, and electrical engineering in order to solve a complicated or the difficult nonlinear systems [1, 8, 13, 14, 7, 2], so one of the models that we are interested in is the ions electro-migration through biological cell membranes. Recently some several authors have introduced this model [15, 12, 10, 5, 19, 11]. Concerning those who have obtained the numerical results, here are some references [3, 4, 6].

These kinds of models have been studied by many researchers in the biophysical litterature, [12, 10]. For more understanding this model, we will begin by a simple description of this phenomena that arise across membranes.

A membrane, in simple terms, may be defined as a phase that acts as a barrier to prevent mass movement but allows restricted and/or regulated passage of one or several

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species through it. It can be a solid or liquid containing ionized or ionizable groups, or it can be completely un-ionized. Functionally, all membranes are active when used as barriers to separate two other phases unless they are too porous or too fragile.

Passing through the barrier of a cell is to move materials into and out of a cell and this is important in cell communication and normal cell function. For example the cell of the nervous system function properly. The ions water, proteins, macromolecules and Nutrients need to be able to pass in and out of the cells.

The first proposal that cellular membranes might contain a lipid bilayer was made in 1925 by two Dutch scientists, E. Gorter and F. Grendel [9], these two reaserchers extracted the membrane lipids from a known number of red blood cells, corresponding to a known surface area of plasma membrane. They then determined the surface area occupied by a monolayer of the extracted lipid spread out at an air-water interface. The surface area of the lipid monolayer turned out to be twice that occupied by the erythrocyte plasma membranes, leading to the conclusion that the membranes consisted of lipid bilayers rather than monolayers.

The cells of the nervous system form networks. They are like all the cells that are able to function because they can control the substances inside of the cell and out of the cell, all this materials move in and out of the cell by passing to the plasma membrane. The plasma membrane surround the cell and separate the interior (intracellular)from the exterior (extracellular) of the cell environment.

The impermeability of the cell membranes is composed of a lipid bilayer, which is a universal component of all cell membrane, its role is critical because its structural components provide the barrier that marks the boundaries of a cell. The structure is called a lipid bilayer, because it is composed of two layers of fat cells organized in two sheets. The lipid bilayer is typically nanometers thick and surrounds all the cells providing the cell membrane structure. The phospholipids organize themselves in a bilayer to hide their hydrophobic tail regions and expose the hydrophilic regions to water. This organization is spontaneous, meaning it is a natural process and does not require energy. This structure forms the layer that is the wall between the inside and outside of the cell.

One of the mechanisms for getting in and out of the cell, we have the diffusion across the lipid bilayer. Since membranes are held together weak forces, certain molecules can slip between the lipids in the bilayer and across from one side to the other. This spontaneous process is termed diffusion. This process allows molecules, that are small and lipophilic (lipid soluble),including most drugs, to easily enter and exit cells. More on this later.

The electrochemical equilibrium of the electro-diffusion system is the result of delicate balance between concentration gradients and electrostatic forces and requires a true compromise; microscopic electro-neutrality does not hold in a boundary layer around the location of membrane impermeability. This implies the presence of excess positive or negative changes on either side of the membrane and causes a nonzero electrostatic potential difference across the membrane. In turn, a portion of the permeable salt is excluded from the compartment confining the large, charge-carrying protein, which causes a nonzero concentration gradient across the membrane which is the key component of the biological world.

In this work, we consider a class of models of ions migration through biological cell membranes. where the concentrations satisfy the Nernst Planck flux equation, including a kinetic reaction terms and the potential is given by the Poisson equation, for all $1 \leq i \leq NS$

$$\begin{cases} \dfrac{\partial \omega_i}{\partial t} - d_i \Delta \omega_i - m_i div(\omega_i \nabla \phi) = S_i(\omega, \phi) & \text{on } Q_T \\ -\varepsilon \Delta \phi = F(\omega_1, .., \omega_{NS}) & \text{on } Q_T \\ -d_i \dfrac{\partial \omega_i}{\partial \upsilon} - m_i \omega_i \dfrac{\partial \phi}{\partial \upsilon} = 0 & \text{in } \Sigma_T \\ \phi(t,x) = 0 & \text{in } \Sigma_T \\ \phi(0,x) = \phi_0(x) & \text{on } \Omega \\ \omega_i(0,x) = \omega_{i,0}(x) & \text{on } \Omega \end{cases} \tag{1}$$

where Ω denotes an open and bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For each i, ω_i is the concentration of the i species which has diffusion coefficients d_i which are non-negative inside the channel and a valency z_i. ϕ is the electrical potential which describes the Coulomb interaction in a mean-field approximation, m_i is the electric mobility that depends on the the universal gas constant, the charge carried by a mole of each species, the diffusion coefficient and also on the local temperature.

The normal exterior derivative on $\partial\Omega$ is denoted by ∂_υ and Δ denotes the Laplacian operator on Ω. Also, we have $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$ with T is a nonnegative constant.

We set

$$F(\omega) = \frac{\sum\limits_{i=1}^{NS} z_i \omega_i}{1 + \varepsilon \sum\limits_{i=1}^{NS} \omega_i} - f$$

where $\omega = (\omega_1, .., \omega_{NS})$, f is the fixed charges concentration and the dimensionless parameter ε is given by $\sqrt{\varepsilon} = \dfrac{\lambda_D}{l}$, the l denotes the reference length scale and λ_D is the Debye screening length of the reference solution defined by the following [18]

$$\lambda_D = \left(\frac{\epsilon_s k T}{2e^2 \overline{\omega}}\right)^{\frac{1}{2}}$$

where ϵ_s is the dielectric permittivity of the solution (roughly equal to that of the solvent) and assume to be constant, k denotes the Boltzmann constant, T the absolute temperature, e the elementary charge and $\overline{\omega}$ is a reference concentration of ions. In order to describe our result and to more illustre it, we have the following example. We are interested in the suicide substrate system, represented by Walsh and al. [19]

$$E + S \rightleftharpoons^{k_1}_{k_{-1}} X \rightarrow^{k_2} Y \rightarrow^{k_3} E + P, \quad Y \rightarrow^{k_4} E_i$$

where E, S and P stand for enzyme, substrate, and product, respectively; X and Y, enzyme- substrate intermediates; E_i, inactivated enzyme; and the ks are positive rate constants.

We denote the concentrations of the reactants by

$$\omega_1 = [E], \omega_2 = [S], \omega_3 = [X], \omega_4 = [Y], \omega_5 = [E_i], \omega_6 = [P].$$

Then, the basic suicide substrate reaction model becomes

$$\begin{cases} \dfrac{\partial \omega_1}{\partial t} - d_1 \Delta \omega_1 - m_1 div(\omega_1 \nabla \phi) = -k_1 \omega_1 \omega_2 + k_{-1}\omega_3 + k_3 \omega_4 & \text{on } Q_T \\ \dfrac{\partial \omega_2}{\partial t} - d_2 \Delta \omega_2 - m_2 div(\omega_2 \nabla \phi) = -k_1 \omega_1 \omega_2 + k_{-1}\omega_3 & \text{on } Q_T \\ \dfrac{\partial \omega_3}{\partial t} - d_3 \Delta \omega_3 - m_3 div(\omega_3 \nabla \phi) = k_1 \omega_1 \omega_2 - (k_{-1} + k_2)\omega_3 & \text{on } Q_T \\ \dfrac{\partial \omega_4}{\partial t} - d_4 \Delta \omega_4 - m_4 div(\omega_4 \nabla \phi) = k_2 \omega_3 - (k_3 + k_4)\omega_4 & \text{on } Q_T \\ \dfrac{\partial \omega_5}{\partial t} - d_5 \Delta \omega_5 - m_5 div(\omega_5 \nabla \phi) = k_4 \omega_4 & \text{on } Q_T \\ \dfrac{\partial \omega_6}{\partial t} - d_6 \Delta \omega_6 - m_6 div(\omega_6 \nabla \phi) = k_3 \omega_4 & \text{on } Q_T \\ -d_i \dfrac{\partial \omega_i}{\partial \upsilon} - m_i \omega_i \dfrac{\partial \phi}{\partial \upsilon} = 0 & \text{in } \Sigma_T \text{ for all } 1 \le i \le 6 \\ -\varepsilon \Delta \phi = \dfrac{\sum\limits_{i=1}^{NS} z_i \omega_i}{1 + \varepsilon \sum\limits_{i=1}^{NS} \omega_i} - f & \text{on } Q_T \\ \phi(t,x) = 0 & \text{in } \Sigma_T \\ \phi(0,x) = \phi_0(x) & \text{on } \Omega \\ \omega_i(0,x) = \omega_{i,0}(x) & \text{on } \Omega \text{ for all } 1 \le i \le 6 \end{cases}$$

2. The main result

2.1. Assumptions

At first, we introduce the notion of weak solution of the problem (1), so let us begin by giving some hypothesis on the nonlinearities and also the initial data. These two main properties are ensured by the following assumptions.

For all $i \in \{1, ..., NS\}$ and $\forall r \in [0, +\infty)^{NS}$, the nonnegativity of solutions is preserved if and only if the quasi-positive condition is verified

$$(H1) \quad S_i(r_1, r_2, ..., r_{i-1}, 0, r_{i+1}, ..., r_{NS}) \ge 0, \quad \text{for all } r = (r_1, r_2, ..., r_{NS}) \in [0, +\infty)^{NS}$$

Furthermore, we restrict ourselves to the case of nonnegative solutions satisfying the triangular structure, which means that

$$(H2): \begin{cases} \sum\limits_{1 \le i \le NS} S_i(r) \le C(1 + \sum\limits_{1 \le i \le NS} r_i) \quad \forall r \in [0, +\infty)^{NS} \\ \qquad \text{where } C \ge 0, \text{ and } \forall i = 1, ..., NS \end{cases}$$

Since we allow the nonlinearities to depend on (t,x), let us assume that for all $i = 1, ..., NS$

$$(H3): \begin{cases} S_i : Q_T \times [0,+\infty)^{NS} \to \mathbb{R} \text{ is measurable; } S_i(.,0) \in L^1(Q_T) \\ \exists K : Q_T \times [0,+\infty) \to [0,+\infty) \text{ with } \forall M > 0,\ K(.,M) \in L^1(Q_T) \\ \text{and a.e } (t,x) \in Q_T,\ \forall r, \hat{r} \in [0,+\infty)^{NS} \text{ with } |r|,\ |\hat{r}| \leq M, \\ |S_i(t,x,r) - S_i(t,x,\hat{r})| \leq K(t,x,M)|r - \hat{r}|. \end{cases}$$

Then, we make the following assumptions

$$\omega_{i,0} \in L^1(\Omega), \text{ such that } \omega_{i,0} \geq 0. \tag{2}$$

and

$$\phi_0 \in L^\infty(\Omega). \tag{3}$$

and there exists a function $\Theta \in L^\infty(Q_T)$, such that

$$\begin{cases} |F(t,x,r)| \leq \Theta(t,x) \ a.e. \ (t,x) \in Q_T, \\ \forall r \in [0,\infty)^{NS}, \end{cases} \tag{4}$$

Now, we clarify in which sense we want to solve our problem. In the following, we define the notion of weak solution.

Definition 1. $(\omega,\phi) = (\omega_1, ..., \omega_{NS}, \phi)$ *is said to be a weak solution of* (1) *if, for all* $1 \leq i \leq NS$

$$\begin{cases} \omega \in C([0,T]; L^1(\Omega)^{NS}) \cap L^1(0,T,W^{1,1}(\Omega)^{NS}),\ \phi \in L^\infty(0,T,W_0^{1,\infty}(\Omega)),\ S_i(\omega,\phi) \in L^1(Q_T) \\ \text{for all } v \in C^1(Q_T) \text{ such that } v(T,.) = 0 \\ -\int_{Q_T} \omega_i \dfrac{\partial v}{\partial t} + d_i \int_{Q_T} \nabla\omega_i \nabla v + m_i \int_{Q_T} \omega_i \nabla\phi\nabla v - \int_\Omega \omega_i(0,x)v(0,x)dx = \int_{Q_T} S_i(\omega,\phi)v \\ \text{for all } \theta \in D(\Omega) \text{ and } t \in]0,T[\\ \int_\Omega \varepsilon\nabla\phi\nabla\theta = \int_\Omega F(\omega)\theta \\ \phi(0,x) = \phi_0(x) \qquad\qquad \text{in } \Omega \\ \omega_i(0,x) = \omega_{i,0}(x) \qquad\qquad \text{in } \Omega \end{cases} \tag{5}$$

The principal result of this paper is the following theorem

Theorem 1. *We assume that* $(H1)-(H3)$, (2), (3) *and* (4) *hold. Then there exists a weak solution* (ω,ϕ) *of* (1) *satisfying* $\omega_i \geq 0$ *in* Q_T *for all* $1 \leq i \leq NS$.

3. Proof of the main result

In this paper, we organized the steps of our work as follows. At first, we will prove the nonnegativity and the L^1 bound of solutions uniformly in time, after that, we will show that the nonlinear terms are also bounded in $L^1(Q_T)$, where here we add some assumptions on the nonlinearities, in order to obtain the desire estimation. The second main purpose is to give an approximate problem using the truncated functions not only on the nonlinearities but also on the initial data, where we will be inspired from the method of Michel Pierre [16].

3.1. Existence of global weak supersolutions for bounded L^1-nonlinearities

Now, we need to approximate the system (1). For this, we truncate the nonlinear terms S_i as follows $S_i^n = T_n o S_i$ where the truncated function $T_n : \mathbb{R} \to \mathbb{R}$ is given by

$$T_n(\sigma) = \sigma \text{ if } \sigma \in (-\sigma_n, n),\ T_n(\sigma) = -\sigma_n \text{ if } \sigma < -\sigma_n \text{ and } T_n(\sigma) = n \text{ if } \sigma > n$$

where $\sigma_n = (NS)n$.
Also we need to truncate the initial data, so, we set $\omega_{i,0}^n = \inf\{\omega_{i,0}, n\}$ for all $i = 1, ..., NS$.
The next step is to consider an approximated system of (1), namely classical solutions $(\omega_n, \phi_n) = (\omega_{1,n}, ..., \omega_{NS,n}, \phi_n)$ of

$$\begin{cases} \textit{for all } 1 \le i \le NS & \\ \dfrac{\partial \omega_{i,n}}{\partial t} - d_i \Delta \omega_{i,n} - m_i div(\omega_{i,n} \nabla \phi_n) = S_i^n(\omega_n, \phi_n) & \text{on } Q_T \\ -d_i \dfrac{\partial \omega_{i,n}}{\partial \upsilon} - m_i \omega_{i,n} \dfrac{\partial \phi_n}{\partial \upsilon} = 0 & \text{in } \Sigma_T \\ -\varepsilon \Delta \phi_n = F(\omega_n) & \text{on } Q_T \\ \phi_n(t,x) = 0 & \text{in } \Sigma_T \\ \phi_n(0,x) = \phi_0(x) & \text{on } \Omega \\ \omega_{i,n}(0,x) = \omega_{i,0}^n(x) & \text{on } \Omega \end{cases} \tag{6}$$

Where S_i^n are essentially truncations of the nonlinearities S_i and $\omega_{i,0}^n$ tends to $\omega_{i,0}$ *in* $L^1(\Omega)$. Moreover, we assume that S_i^n have the same properties $(H1) - (H3)$ as S_i, and we choose the nonlinearities in such a way that they will be uniformly bounded for each n.

First of all, we will need to prove the nonnegativity of ω_n, for that we introduce the function $Z_n = (Z_{1,n}, Z_{2,n}, ..., Z_{NS,n})$ which is defined by

$$Z_{i,n} = \omega_{i,n} e^{\frac{m_i}{d_i}\phi_n} \qquad 1 \le i \le NS$$

and we have,

$$p_{i,n} = e^{\frac{m_i}{d_i}\phi_n} \quad \textit{and} \quad q_{i,n} = \frac{1}{p_{i,n}},$$

where the terms $(q_{i,n})_{1 \le i \le NS}$, $(p_{i,n})_{1 \le i \le NS}$ are uniformly bounded by a constant that not depends on n.
Then, the concentrations $(Z_{i,n})_{1 \le i \le NS}$ and the potential ϕ_n will satisfy the following system

$$\begin{cases} \dfrac{\partial (q_{i,n} Z_{i,n})}{\partial t} - d_i div(q_{i,n} \nabla Z_{i,n}) = S_i^n(Z_n, \phi_n) \ \textit{in } Q_T \\ -\varepsilon \Delta \phi_n = F(q_n Z_n) \ \textit{in } Q_T \\ \dfrac{\partial Z_{i,n}}{\partial \upsilon} = 0 \ \textit{on } \Sigma_T \\ \phi_n(t,x) = 0 \ \textit{on } \Sigma_T \\ \phi_n(0,x) = \phi_0(x) \ \textit{on } \Omega \\ Z_{i,n}(0,x) = Z_{i,0}^n(x) \ \textit{on } \Omega \end{cases} \tag{7}$$

where $S_i^n = T_n o \hat{S}_i$ and the nonlinearities $\hat{S}_i^n$ are defined in $\mathbb{R}^{NS}$ by

$$\hat{S}_i(r) = \hat{S}_i(r_1, r_2, .., r_{NS}) = \begin{cases} S_i(r_1, r_2, ..., r_{NS}) \text{ if } (r_1, r_2, ..., r_{NS}) \in [0, +\infty)^{NS} \\ S_i(r_1, ..., r_{j-1}, 0, r_{j+1}, ..., r_m) \text{ if } r_j \leq 0. \end{cases} \tag{8}$$

Now, we introduce the function $sign^-$ defined on $\mathbb{R}$ by

$$sign^- r = \begin{cases} -1 \text{ if } r < 0 \\ 0 \text{ if r} \geq 0 \end{cases}$$

as $sign^-$ is an increasing function, we consider the convex function $j_\varepsilon \in C^2(\mathbb{R})$ such that

$$j_\varepsilon^{'}(r) \to sign^- r \text{ when } \varepsilon \to 0$$

Also we put $v = j_\varepsilon^{'}(Z_i)$ as a test function in (7), then we have

$$\int_0^T \int_\Omega \frac{\partial(q_{i,n} Z_{i,n})}{\partial t} j_\varepsilon^{'}(Z_{i,n}) = -d_i \int_0^T \int_\Omega q_{i,n} \nabla Z_{i,n} \nabla (j_\varepsilon^{'}(Z_{i,n})) + \int_0^T \int_\Omega S_i^n(Z_n, \phi_n) j_\varepsilon^{'}(Z_{i,n})$$

we denote by I_1 and I_2 the two members in the right side of previous equality, and by using the convexity of the function j_ε, we deduce that

$$\begin{aligned} I_1 &= -d_i \int_0^T \int_\Omega q_{i,n} \nabla Z_{i,n} \nabla (j_\varepsilon^{'}(Z_{i,n})) \\ &= -d_i \int_0^T \int_\Omega q_{i,n} |\nabla Z_{i,n}|^2 j_\varepsilon^{"}(Z_{i,n}) \leq 0. \end{aligned}$$

Concerning the second member I_2, we define the second term I_2, then, we deduce

$$\begin{aligned} \lim_{\varepsilon \to 0} I_2 &= \lim_{\varepsilon \to 0} \int_0^T \int_\Omega S_i^n(Z_n, \phi_n) j_\varepsilon^{'}(Z_{i,n}) \\ &= \lim_{\varepsilon \to 0} \int_{[Z_{i,n} \geq 0]} S_i^n(Z_n, \phi_n) j_\varepsilon^{'}(Z_{i,n}) + \lim_{\varepsilon \to 0} \int_{[Z_{i,n} < 0]} S_i^n(Z_n, \phi_n) j_\varepsilon^{'}(Z_{i,n}) \\ &= \lim_{\varepsilon \to 0} \int_{[Z_{i,n} < 0]} S_i^n(Z_n, \phi_n) j_\varepsilon^{'}(Z_{i,n}). \end{aligned}$$

By using $(H1)$, we have

$$\begin{aligned} \lim_{\varepsilon \to 0} I_2 &= -\int_{[Z_{i,n} < 0]} S_i^n(Z_n, \phi_n) \\ &= -\int_{[Z_{i,n} < 0]} T_n(S_i(Z_{1,n}, ..., Z_{i-1,n}, 0, Z_{i+1,n}, ..., Z_{NS,n}, \phi)) \leq 0 \end{aligned}$$

Then, we obtain

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \frac{\partial(q_{i,n} Z_{i,n})}{\partial t} j_\varepsilon^{'}(Z_{i,n}) \leq 0$$

which means that by passing to the limit, we obtain

$$\int_0^T \int_\Omega \frac{\partial(q_{i,n}Z_{i,n})}{\partial t} sign^-(Z_{i,n}) \leq 0$$

therefore

$$\int_0^T \int_\Omega \frac{\partial(q_{i,n}Z_{i,n})^-}{\partial t} \leq 0$$

this implies the following inequality

$$\int_\Omega (q_{i,n}Z_{i,n})^-(t,x) \leq \int_\Omega (q_{i,n}Z_{i,n})^-(0,x)$$

as $(q_{i,n}Z_{i,n})(0,x) \geq 0$ for almost everywhere then we deduce

$$\int_\Omega (q_{i,n}Z_{i,n})^-(t,x) \leq 0.$$

Finally $(q_{i,n}Z_{i,n})^-(t,x) = 0$ and then $Z_{i,n} \geq 0$, for all $i = 1, ..., NS$.

3.1.1. A priori estimate

First, we start by proving the following lemmas, where we are going to use the fact that $\omega_{i,n} = q_{i,n}Z_{i,n}$.

Lemma 1. *Assume that* $(H2)$ *and* (7) *are satisfied. Then we have the following result*

$$\int_\Omega \sum_{1\leq i\leq NS} (q_{i,n}Z_{i,n})(t) \leq e^{tC} \int_\Omega \sum_{1\leq i\leq NS} (q_{i,0}Z_{i,0}) + k(e^{tC} - 1)$$

Proof. We sum the NS equations. Then, we have

$$\sum_{1\leq i\leq NS} \frac{\partial(q_{i,n}Z_{i,n})}{\partial t} - div(\sum_{1\leq i\leq NS} d_i q_{i,n} \nabla Z_{i,n}) = \sum_{1\leq i\leq NS} S_i^n(q_n Z_n)$$

and by using $(H2)$, we have the existence of a positive constant denoted by C such that

$$\sum_{1\leq i\leq NS} \frac{\partial(q_{i,n}Z_{i,n})}{\partial t} - div(\sum_{1\leq i\leq NS} d_i q_{i,n} \nabla Z_{i,n}) \leq C(1 + \sum_{1\leq i\leq NS} q_{i,n}Z_{i,n})$$

Now, we set $V_n(t) = \sum_{1\leq i\leq NS} (q_{i,n}Z_{i,n})(t)$, so by integrating on Ω, we obtain

$$\int_\Omega \frac{\partial V_n(t)}{\partial t} - \int_{\partial\Omega} \sum_{1\leq i\leq NS} d_i q_{i,n} \frac{\partial Z_{i,n}}{\partial \upsilon} \leq C \int_\Omega (1 + V_n(t))$$

Since $\dfrac{\partial Z_{i,n}}{\partial \upsilon} = 0$ for all $i = 1, ..., NS$, we have

$$\int_\Omega \frac{\partial V_n(t)}{\partial t} \leq \int_\Omega C[1 + V_n(t)]$$

Here, we integrate over $(0, t)$, for each t in the existence interval, we get

$$\begin{aligned} \int_{Q_T} \frac{\partial}{\partial s}(V_n(s)e^{-sC}) &\leq \int_{Q_T} Ce^{-sC} \\ \int_\Omega V_n(t)e^{-tC} &\leq \int_\Omega V_n(0) + \frac{1}{C}\int_\Omega C(1 - e^{-tC}) \end{aligned}$$

and we put $k = meas(\Omega)$, which give us the following estimate

$$\int_\Omega V_n(t) \leq e^{tC} \int_\Omega V_n(0) + k(e^{tC} - 1).$$

According to the definition of the initial data, It follows that the total mass $\int_\Omega V_n(t)$ is bounded on any interval.

Remark 1. *Let ϕ_n be the unique solution of the elliptic problem*

$$\begin{cases} -\varepsilon\Delta\phi_n = F(q_n Z_n) & on \quad Q_T \\ \phi_n(t, x) = 0 & on \quad \Sigma_T \\ \phi_n(0, x) = \phi_0(x) & on \quad \Omega, \end{cases} \tag{9}$$

where ϕ_n is the solution of the Poisson equation.

Lemma 2. *There exists a constant C depends only on T and on the $L^\infty-$norm of ϕ_0, such that*

$$||\phi_n||_{L^\infty(0,T;W_0^{1,\infty}(\Omega))} \leq C.$$

Proof. We have $\forall t \in]0, T[$, ϕ_n is the unique solution of the elliptic problem (9) Where ϕ_n satisfies

$$\phi_n(t, x) = \int_\Omega H(s, x)\theta^n(t, s)ds$$

and θ^n is given by

$$\theta^n(t, s) = F(t, s, q_n Z_n), \quad s \in \Omega,$$

where H denotes the Green's function associated to (9) . Then we have

$$||F(t, s, q_n Z_n)||_{L^\infty(Q_T)} \leq C$$

hence

$$||\phi_n||_{L^\infty(0,T;W_0^{1,\infty}(\Omega))} \leq C.$$

Concerning the nonlinearities $(S_i^n)_{1\leq i\leq NS}$, we may indeed show the L^1 bounded of those nonlinearities S_i^n for all T. The proof needs to give more restrictive assumptions on the nonlinear terms $(S_i^n)_{1\leq i\leq NS}$. We assume that
There exists a lower triangular invertible matrix $A = (a_{i,j})_{1\leq i,j\leq NS}$ with nonnegative coefficients, such that

$$\begin{cases} \exists b \in (0,+\infty)^{NS}, \, \forall (t,x,r) \in (0,T)\times\Omega\times[0,+\infty)^{NS} \\ AS(t,x,r) \leq (1+\sum\limits_{1\leq i\leq NS} r_i)b \end{cases} \tag{10}$$

Where $S(r) = (S_1(r), S_2(r), ..., S_{NS}(r))$.

Proposition 1. *Assume that* $(H1)$ *and* (10) *hold. Then, if* (Z_n, ϕ_n) *is solution of* (7) *on* $(0,T)$*, there exists a nonnegative constant denoted by* C *such that, for all* $1 \leq i \leq NS$ *and for all* $n \geq 1$

$$\int_{Q_T} |S_i^n(Z_n,\phi_n)| dtdx \leq C < +\infty. \tag{11}$$

Proof. We denote by C_0 any constant depending only on the initial data and T. Then for all $t \in [0,T]$, we have $\int_\Omega (q_{i,n}Z_{i,n})(t) \leq C_0$ for all $1 \leq i \leq NS$. Now, we take the equation verified by $q_{i,n}Z_{i,n}$ and we sum the NS equations to obtain that, for $1 \leq i \leq NS$ and for $1 \leq j \leq i$, we have

$$\sum_{j=1}^{i} a_{i,j}\frac{\partial(q_{j,n}Z_{j,n})}{\partial t} - \sum_{j=1}^{i} a_{i,j}[d_j div(q_{j,n}\nabla Z_{j,n})] = \sum_{j=1}^{i} a_{i,j}S_j^n(Z_n,\phi_n)$$

we multiply this equation by $\varphi = 1$ and integrating on Q_T. Indeed, we have

$$\int_{Q_T}\sum_{j=1}^{i} a_{i,j}\frac{\partial(q_{j,n}Z_{j,n})}{\partial t} - \int_{\Sigma_T}\sum_{j=1}^{i} a_{i,j}d_j q_{j,n}\frac{\partial Z_{j,n}}{\partial \upsilon}d\sigma = \int_{Q_T}\sum_{j=1}^{i} a_{i,j}S_j^n(Z_n,\phi_n)$$

we use the boundary conditions, then we obtain

$$\int_{Q_T}\sum_{j=1}^{i} a_{i,j}\frac{\partial(q_{j,n}Z_{j,n})}{\partial t} = \int_{Q_T}\sum_{j=1}^{i} a_{i,j}S_j^n(Z_n,\phi_n),$$

therefore

$$\sum_{j=1}^{i} a_{i,j}\int_\Omega (q_{j,n}Z_{j,n})(T) = \sum_{j=1}^{i} a_{i,j}\int_{Q_T} S_j^n(Z_n,\phi_n) + \sum_{j=1}^{i} a_{i,j}\int_\Omega (q_{j,n}Z_{j,n})(0,x)$$

the nonnegativity of solutions gives us

$$-\sum_{j=1}^{i} a_{i,j}\int_{Q_T} S_j^n(Z_n,\phi_n) \leq \sum_{j=1}^{i} a_{i,j}\int_\Omega (q_{j,n}Z_{j,n})(0,x). \tag{12}$$

Now, we use (10). This leads us to the following estimate

$$\int_{Q_T} h_i(q_n Z_n) \quad \leq \quad \sum_{j=1}^{i} a_{i,j} \int_{\Omega} (q_{j,n} Z_{j,n})(0,x) + \int_{Q_T} b_i (1 + \sum_{i=1}^{NS} q_{i,n} Z_{i,n}) \tag{13}$$

where

$$h_i(q_n Z_n) = -\sum_{j=1}^{i} a_{i,j} S_j^n(Z_n, \phi_n) + b_i(1 + \sum_{i=1}^{NS} q_{i,n} Z_{i,n})$$

By using (12) and (13), we obtain

$$|| \sum_{j=1}^{i} a_{i,j} S_j^n(Z_n, \phi_n) ||_{L^1(Q_T)} \leq C.$$

Therefore, for $1 \leq i \leq NS$

$$||S_i^n(Z_n, \phi_n)||_{L^1(Q_T)} \leq C.$$

Before continuing the proof of the main result. First of all, we define the following the following set

$$D = \{\psi \in C^\infty(\bar{Q}_T); \quad \psi \geq 0; \ \psi(0,T) = 0\} \tag{14}$$

or, we choose all the Dirichlet condition as follows

$$D = \{\psi \in C^\infty(\bar{Q}_T) \ ; \quad \psi \geq 0; \ \psi(.,T) = 0, \ \psi = 0 \ on \ \Sigma_T\}. \tag{15}$$

Definition 2. *The function $(\omega, \phi) = (\omega_1, ..., \omega_{NS}, \phi)$ is called a supersolution of problem* (1)*, if*

$$\begin{cases} \omega \in C([0,T]; L^1(\Omega)^{NS}) \cap L^1(0,T; W^{1,1}(\Omega)^{NS}), \\ \phi \in L^\infty(0,T; W_0^{1,\infty}(\Omega)), S(\omega, \phi) \in L^1(Q_T)^{NS}, \ \textit{and for all } \psi \in D, \\ -\int_\Omega \omega_{i,0}\psi(0) + \int_{Q_T}[-\psi_t \omega_i + d_i \nabla\omega_i \nabla\psi + m_i \omega_i \nabla\phi\nabla\psi] \geq \int_{Q_T} S_i(\omega,\phi)\psi \\ \textit{for all } \theta \in D(\Omega) \ \textit{and } t \in]0,T[\\ \int_\Omega \varepsilon \nabla\phi\nabla\theta = \int_\Omega F(\omega)\theta \\ \phi(0,x) = \phi_0(x) \qquad\qquad\qquad\qquad \textit{in } \Omega \end{cases} \tag{16}$$

Theorem 2. *Let $(\omega_n, \phi_n) = (\omega_{1,n}, ..., \omega_{NS,n}, \phi_n)$ be a nonnegative solution to the approximate system* (6) *satisfying* (11) *and the hypothesis of Theorem 1. Then up to a subsequence of (ω_n) also denoted by ω_n converges in $L^1(Q_T)^{NS}$ and almost everywhere in Q_T to a supersolution of system* (1) *given by* (16)*, which is equivalent to the following definition*

$$\begin{cases} Z \in C([0,T]; L^1(\Omega)^{NS}) \cap L^1(0,T; W^{1,1}(\Omega)^{NS}), \\ \phi \in L^\infty(0,T; W_0^{1,\infty}(\Omega)), S(Z, \phi) \in L^1(Q_T)^{NS}, \ \textit{and for all } \psi \in D, \\ -\int_\Omega (q_{i,0} Z_{i,0})\psi(0) + \int_{Q_T}[-\psi_t (q_i Z_i) + d_i q_i \nabla Z_i \nabla\psi] \geq \int_{Q_T} S_i(Z,\phi)\psi \\ \textit{for all } \theta \in D(\Omega) \ \textit{and } t \in]0,T[\\ \int_\Omega \varepsilon \nabla\phi\nabla\theta = \int_\Omega F(qZ)\theta \\ \phi(0,x) = \phi_0(x) \qquad\qquad\qquad\qquad \textit{in } \Omega \end{cases} \tag{17}$$

Proof. (Existence of Supersolution)

Lemma 3. *[17] Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary and $\omega_{i,n}$ solution of (6). Then for every $T > 0$, the mapping*

$$(\omega_{i,0}^n, S_i^n) \in \ L^1(\Omega) \times L^1(Q_T) \to \omega_{i,n} \in L^1(Q_T). \tag{18}$$

is compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1(Q_T)$, and even into $L^1(0,T;W^{1,1}(\Omega))$, and for the compactness of the trace, we use the continuity of the trace operator from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. Then the trace mapping

$$(\omega_{i,0}^n, S_i^n) \to (\omega_{i,n})_{|\Sigma_T} \in L^1(\Sigma_T) \text{ is also compact.}$$

According to the a priori estimate (11) and to the compactness lemma, there exists $\omega \in L^1(Q_T)^{NS}$ with $\nabla\omega \in [L^1(Q_T)^N]^{NS}$ such that, up to a subsequence, one may assume that

$$\begin{cases} \omega_n \to \omega \text{ in } L^1(Q_T)^{NS} \text{ and a.e in } Q_T,\ \nabla\omega_n \to \nabla\omega \text{ in } [L^1(Q_T)^N]^{NS} \\ \omega \in L^1(0,T;W^{1,1}(\Omega))^{NS}. \end{cases} \tag{19}$$

Now, we need to prove that the nonlinearities $(S_i^n)_{1\leq\leq NS}$ and $(S_i)_{1\leq i\leq NS}$ belong to L^1. So, first we define the function η_M^n such that

$$\eta_M^n = \sup_{0\leq|r|\leq M,\ 1\leq i\leq NS} |S_i^n(t,x,r) - S_i(t,x,r)|$$

where

$$\eta_M^n \to 0 \text{ almost everywhere in } Q_T$$

and

$$\eta_M^n \to 0 \ \textit{in}\ L^1(Q_T).$$

Indeed, since

$$\eta_M^n \leq \sup_{0\leq|r|\leq M,\ 1\leq i\leq NS} \chi_{[-\sigma_n<S_i<n]}|S_i(t,x,r)|.$$

From the Lipschitz property $(H3)$, we find the following inequality

$$|r| \leq M \Rightarrow |S_i(t,x,r)| \leq |S_i(t,x,0)| + K(t,x,M)M.$$

By using the fact that $K(.,M)$, $S_i(.,0) \in L^1(Q_T)$, we obtain the required result. It means that $\eta_M^n \to 0$ in $L^1(Q_T)$ and almost everywhere in Q_T.
Now, thanks to the continuity of $S_i(.,.,r)$ where $r \in (\mathbb{R}_+)^{NS}$, we also have

$$S_i^n(\omega_n,\phi_n) \to S_i(\omega,\phi) \text{ a.e in } Q_T \tag{20}$$

and By Fatou's lemma, we obtain

$$\int_{Q_T} |S_i(\omega,\phi)| \leq \liminf_{n\to\ +\infty} \int_{Q_T} |S_i^n(\omega_n,\phi_n)| \tag{21}$$

and in particular, we have $S_i(\omega, \phi) \in L^1(Q_T)$ for all $1 \leq i \leq NS$. To accomplish the proof and in order to pass to the limit in the equation, we will need the convergence in $L^1(Q_T)$ of $S_i^n(\omega_n, \phi_n)$, which is not true in general, however we will show that we could have at least one inequality in the limit so this is the purpose of the second theorem. Now we are left with the following step, which is the boundedness of the gradient terms. For this, we need to ennunciate the following lemma.

Lemma 4. *Let (7) and (11) be satisfied and there exists $\varepsilon > 0$. Then, for all $k > 0$ and $n \geq 1$*

$$(d_i - \varepsilon) \int_{[|q_{i,n} Z_{i,n}| \leq k]} |q_{i,n} \nabla Z_{i,n}|^2 \leq Ck^2 + k[\int_\Omega |q_{i,0} Z_{i,0}| + \int_{Q_T} |S_i^n(Z_n, \phi_n)|]. \tag{22}$$

where C is a constant depending only on ε, $|\Omega|$ and T.

Proof. We may suppose S_i^n is a regular function. To show how the estimate (22) is easy to obtain, we introduce the function $j_k(r) = \int_0^r T_k(s)ds$ where $T_k(s)$ is the projection of s onto $[-k, k]$.
Multiplying the following equation by $T_k(q_{i,n} Z_{i,n})$

$$\frac{\partial(q_{i,n} Z_{i,n})}{\partial t} - d_i div(q_{i,n} \nabla Z_{i,n}) = S_i^n(Z_n, \phi_n)$$

and integrating by parts on Q_T, we get

$$\int_{Q_T} \partial_t(j_k(q_{i,n} Z_{i,n})) - d_i \int_{Q_T} T_k(q_{i,n} Z_{i,n}) div(q_{i,n} \nabla Z_{i,n}) = \int_{Q_T} T_k(q_{i,n} Z_{i,n}) S_i^n(Z_n, \phi_n)$$

Using now the boundary conditions to obtain

$$\int_{Q_T} \partial_t(j_k(q_{i,n} Z_{i,n})) + d_i \int_{Q_T} \nabla T_k(q_{i,n} Z_{i,n})(q_{i,n} \nabla Z_{i,n}) = \int_{Q_T} T_k(q_{i,n} Z_{i,n}) S_i^n(Z_n, \phi_n)$$

which implies that

$$\begin{aligned} \int_{Q_T} \partial_t(j_k(q_{i,n} Z_{i,n})) \quad &+ \quad d_i \int_{Q_T} T_k'(q_{i,n} Z_{i,n}) |q_{i,n} \nabla Z_{i,n}|^2 \\ &= \int_{Q_T} T_k(q_{i,n} Z_{i,n}) S_i^n(Z_n, \phi_n) + m_i \int_{Q_T} T_k'(q_{i,n} Z_{i,n})(q_{i,n} \nabla Z_{i,n})(q_{i,n} Z_{i,n}) \nabla \phi_n \end{aligned}$$

the integration over $(0, T)$ yields to the equality above

$$\begin{aligned} \int_\Omega j_k(q_{i,n} Z_{i,n})(t) + d_i \int_{Q_T} T_k'(q_{i,n} Z_{i,n}) |q_{i,n} \nabla Z_{i,n}|^2 \quad &= \quad \int_\Omega j_k(q_{i,0}^n Z_{i,0}^n) + \int_{Q_T} T_k(q_{i,n} Z_{i,n}) S_i^n(Z_n, \phi_n) \\ &+ \quad m_i \int_{Q_T} T_k'(q_{i,n} Z_{i,n})(q_{i,n} \nabla Z_{i,n})(q_{i,n} Z_{i,n}) \nabla \phi_n. \end{aligned}$$

Since $T_k^{'}(q_{i,n}Z_{i,n}) = 1$ for all $|q_{i,n}Z_{i,n}| \leq k$ and by using the following estimates

$$j_k(q_{i,n}Z_{i,n})(t) > 0,\ T_k(q_{i,n}Z_{i,n}) \leq k \text{ and } j_k(q_{i,0}^n Z_{i,0}^n) \leq k|q_{i,0}^n Z_{i,0}^n|,$$

we obtain

$$\begin{aligned} d_i \int_{[|q_{i,n}Z_{i,n}|\leq k]} |q_{i,n}\nabla Z_{i,n}|^2 \quad &\leq \quad k[\int_{Q_T} |S_i^n(Z_n,\phi_n)| + \int_\Omega |q_{i,0}^n Z_{i,0}^n|] \\ &+ \quad m_i \int_{Q_T} T_k^{'}(q_{i,n}Z_{i,n})(q_{i,n}\nabla Z_{i,n})(q_{i,n}Z_{i,n})\nabla\phi_n. \end{aligned}$$

As $||\nabla\phi_n||_{L^\infty(Q_T)} \leq C$, and by using Young's inequality

$$\begin{aligned} d_i \int_{[|q_{i,n}Z_{i,n}|\leq k]} |q_{i,n}\nabla Z_{i,n}|^2 \quad &\leq \quad k[\int_{Q_T} |S_i^n(Z_n,\phi_n)| + \int_\Omega |q_{i,0}^n Z_{i,0}^n|] \\ &+ \quad C_\varepsilon \int_{[|q_{i,n}Z_{i,n}|\leq k]} |q_{i,n}Z_{i,n}|^2 + \varepsilon \int_{[|q_{i,n}Z_{i,n}|\leq k]} |q_{i,n}\nabla Z_{i,n}|^2 \end{aligned}$$

consequently

$$(d_i - \varepsilon) \int_{[|q_{i,n}Z_{i,n}|\leq k]} |q_{i,n}\nabla Z_{i,n}|^2 \leq k[\int_{Q_T} |S_i^n(Z_n,\phi_n)| + \int_\Omega |q_{i,0}Z_{i,0}|] + C_\varepsilon k^2.$$

Continuing the proof of Theorem 2

Now, we fix $\eta \in (0,1)$ and we introduce $V_{i,n} = q_{i,n}Z_{i,n} + \sum_{\substack{1\leq j\leq NS \\ j\neq i}} \eta(q_{j,n}Z_{j,n})$, also we denote $U_{i,n} = T_k(V_{i,n})$.

Since we need to differentiate twice T_k, we replace T_k by a $C^2-regularized$ function, such that

$$\begin{aligned} T_k(r) \quad &= \quad r \quad \text{if } 0 \leq r \leq k-1 \\ T_k^{'}(r) \quad &= \quad 0 \ \text{ if } r \geq k \\ 0 \quad &\leq \quad T_k^{'}(r) \leq 1 \text{ if } r \geq 0 \\ -1 \quad &\leq \quad T_k^{"}(r) \leq 0 \text{ if } r \geq 0 \end{aligned}$$

when $k \to +\infty$, we have

$$\begin{aligned} T_k(r) \quad &\to \quad r \text{ a.e} \\ T_k^{'}(r) \quad &\to \quad 1 \text{ a.e} \\ T_k^{"}(r) \quad &\to \quad 0 \text{ a.e} \end{aligned}$$

this enables us to state the main result of this section which means that in order to finish the proof of Theorem 2, we propose to pass to the limit when n tends to infinity, then

$\eta \to 0$ and after that $k \to +\infty$, for this we need to use the hypothesis on the truncated function.

Our goal now is to continue the proof of the second theorem. The first step is to fix an $\eta \in (0,1)$, Then, for all $i = 1, ..., NS$, we set

$$C_{i,n} = \sum_{j \neq i} q_{j,n} Z_{j,n},\ V_{i,n} = q_{i,n} Z_{i,n} + \eta C_{i,n},\ U_{i,n} = T_k(V_{i,n}).$$

First, we have

$$\begin{aligned} -\Delta U_{i,n} &= -div(T_k^{'}(V_{i,n}) \nabla V_{i,n}) \\ &= -T_k^{"}(V_{i,n}) |\nabla V_{i,n}|^2 - T_k^{'}(V_{i,n}) \Delta V_{i,n} \end{aligned}$$

and $\dfrac{\partial U_{i,n}}{\partial t} = T_k^{'}(V_{i,n}) \dfrac{\partial V_{i,n}}{\partial t}$.
Then, we obtain

$$\begin{aligned} \frac{\partial U_{i,n}}{\partial t} - d_i \Delta U_{i,n} &= T_k^{'}(V_{i,n}) \frac{\partial V_{i,n}}{\partial t} - d_i T_k^{'}(V_{i,n}) \Delta V_{i,n} - d_i T_k^{"}(V_{i,n}) |\nabla V_{i,n}|^2 \\ &= T_k^{'}(V_{i,n}) [\frac{\partial V_{i,n}}{\partial t} - d_i \Delta V_{i,n}] - d_i T_k^{"}(V_{i,n}) |\nabla V_{i,n}|^2 \\ &= T_k^{'}(V_n) [\frac{\partial (q_{i,n} Z_{i,n})}{\partial t} + \eta \frac{\partial C_{i,n}}{\partial t} - d_i \Delta (q_{i,n} Z_{i,n}) - \eta d_i \Delta (C_{i,n})] \\ &- d_i T_k^{"}(V_{i,n}) |\nabla V_{i,n}|^2 \end{aligned}$$

and we have

$$\begin{aligned} \frac{\partial (q_{i,n} Z_{i,n})}{\partial t} - d_i \Delta (q_{i,n} Z_{i,n}) &= \frac{\partial (q_{i,n} Z_{i,n})}{\partial t} - d_i div(q_{i,n} \nabla Z_{i,n}) - d_i div(Z_{i,n} \nabla q_{i,n}) \\ &= S_i^n(Z_n, \phi_n) - d_i div(Z_{i,n} \nabla q_{i,n}) \end{aligned}$$

also, for $j = 1, ..., NS$, and $j \neq i$ we get

$$\frac{\partial (q_{j,n} Z_{j,n})}{\partial t} - d_i \Delta (q_{j,n} Z_{j,n}) = S_j^n(Z_n, \phi_n) + (d_j - d_i) div(q_{j,n} \nabla Z_{j,n}) - d_i div(Z_{j,n} \nabla q_{j,n}).$$

This yields to the following

$$\begin{aligned} \frac{\partial U_{i,n}}{\partial t} - d_i \Delta U_{i,n} &= T_k^{'}(V_{i,n}) [S_i^n(Z_n, \phi_n) + \eta \sum_{j \neq i} S_j^n(Z_n, \phi_n) - d_i div(Z_{i,n} \nabla q_{i,n} + \eta \sum_{j \neq i} Z_{j,n} \nabla q_{j,n}) \\ &+ \eta \sum_{j \neq i} (d_j - d_i) div(q_{j,n} \nabla Z_{j,n})] - d_i T_k^{"}(V_{i,n}) |\nabla V_{i,n}|^2 \end{aligned}$$

Therefore

$$\frac{\partial U_{i,n}}{\partial t} - d_i \Delta U_{i,n} = [Y_{i,n} + \eta X_{i,n}] - d_i T_k^{'}(V_{i,n}) div(Z_{i,n} \nabla q_{i,n} + \eta \sum_{j \neq i} Z_{j,n} \nabla q_{j,n}) - d_i T_k^{"}(V_{i,n}) |\nabla V_{i,n}|^2$$

where

$$\begin{aligned} Y_{i,n} &= T_k^{'}(V_{i,n})[S_i^n(Z_n,\phi_n)+\eta\sum_{j\neq i}S_j^n(Z_n,\phi_n)] \\ X_{i,n} &= T_k^{'}(V_{i,n})\sum_{j\neq i}(d_j-d_i)div(q_{j,n}\nabla Z_{j,n}) \end{aligned}$$

we may write for $\psi \in D$:

$$\int_{Q_T}\psi[\frac{\partial U_{i,n}}{\partial t}-d_i\Delta U_{i,n}]+d_i\int_{Q_T}\psi T_k^{'}(V_{i,n})div(Z_{i,n}\nabla q_{i,n}+\eta\sum_{j\neq i}Z_{j,n}\nabla q_{j,n})$$
$$=\int_{Q_T}\psi[Y_{i,n}+\eta X_{i,n}]-d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})|\nabla V_{i,n}|^2.$$

After an integration by parts, we obtain

$$\begin{aligned} &-\textstyle\int_\Omega\psi(0)U_{i,n}(0)+\int_{Q_T}[-\psi_tU_{i,n}+d_i\nabla\psi\nabla U_{i,n}]-d_i\int_{\Sigma_T}\psi\frac{\partial U_{i,n}}{\partial\upsilon} \\ &-d_i\textstyle\int_{Q_T}\nabla(\psi T_k^{'}(V_{i,n}))(Z_{i,n}\nabla q_{i,n}+\eta\sum\limits_{j\neq i}Z_{j,n}\nabla q_{j,n}) \\ &+d_i\textstyle\int_{\Sigma_T}\psi T_k^{'}(V_{i,n})(Z_{i,n}\partial_\upsilon q_{i,n}+\eta\sum\limits_{j\neq i}Z_{j,n}\partial_\upsilon q_{j,n})=\int_{Q_T}\psi[Y_{i,n}+\eta X_{i,n}]-d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})|\nabla V_{i,n}|^2. \end{aligned}$$

Using the homogeneous Neumann boundary conditions, it is clear that

$$\begin{aligned} &-\textstyle\int_\Omega\psi(0)U_{i,n}(0)+\int_{Q_T}[-\psi_tU_{i,n}+d_i\nabla\psi\nabla U_{i,n}]-d_i\int_{Q_T}\nabla(\psi T_k^{'}(V_{i,n}))(Z_{i,n}\nabla q_{i,n}+\eta\sum\limits_{j\neq i}Z_{j,n}\nabla q_{j,n}) \\ &=\textstyle\int_{Q_T}\psi[Y_{i,n}+\eta X_{i,n}]-d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})|\nabla V_{i,n}|^2, \end{aligned} \tag{23}$$

this lead us to the following

$$\begin{aligned} &-\textstyle\int_\Omega\psi(0)U_{i,n}(0)+\int_{Q_T}[-\psi_tU_{i,n}+d_i\nabla\psi\nabla U_{i,n}]-d_i\int_{Q_T}\nabla\psi T_k^{'}(V_{i,n})(Z_{i,n}\nabla q_{i,n}+\eta\sum\limits_{j\neq i}Z_{j,n}\nabla q_{j,n}) \\ &-d_i\textstyle\int_{Q_T}\psi T_k^{"}(V_{i,n})\nabla V_{i,n}(Z_{i,n}\nabla q_{i,n}+\eta\sum\limits_{j\neq i}Z_{j,n}\nabla q_{j,n})\geq\int_{Q_T}\psi[Y_{i,n}+\eta X_{i,n}] \end{aligned} \tag{24}$$

Where $-d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})|\nabla V_{i,n}|^2\geq 0$. Then (24) could be written as follows

$$\begin{aligned} &-\textstyle\int_\Omega\psi(0)U_{i,n}(0)+\int_{Q_T}[-\psi_tU_{i,n}+d_i\nabla\psi\nabla U_{i,n}]-d_i\int_{Q_T}\nabla\psi T_k^{'}(V_{i,n})(Z_{i,n}\nabla q_{i,n}+\eta\sum\limits_{j\neq i}Z_{j,n}\nabla q_{j,n}) \\ &+m_i\textstyle\int_{Q_T}\psi T_k^{"}(V_{i,n})\nabla V_{i,n}(Z_{i,n}q_{i,n}\nabla\phi_n)\geq\int_{Q_T}\psi[Y_{i,n}+\eta X_{i,n}]-\eta d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})\nabla V_{i,n}(\sum\limits_{j\neq i}Z_{j,n}\nabla q_{j,n}) \end{aligned} \tag{25}$$

We keep k and η fixed. We know that $U_{i,n}$ converges in $L^1(Q_T)$ and a.e to $U_{i,k}$ where

$$U_{i,k}=T_k(V_i),\ V_i=q_iZ_i+\eta\sum_{j\neq i}q_jZ_j$$

and then it is clear that the reaction terms converge only a.e. The point is that, Since $T_k^{'}(r) = 0$ for $r > k$ then, $Y_{i,n}$ is equal to zero on the set $[V_{i,n} > k]$. But on the complement of this set, we have

$$q_{i,n}Z_{i,n} \leq k,\ \forall j \neq i,\ q_{j,n}Z_{j,n} \leq \frac{k}{\eta}$$

By using the dominated convergence theorem, we can find that as $n \to +\infty$

$$Y_{i,n} \to Y_{i,k} = T_k^{'}(V_i)[S_i(Z,\phi) + \eta \sum_{j\neq i} S_j(Z,\phi)] \ in\ L^1(Q_T).$$

In addition we have $\nabla U_{i,n}$ converges in $L^1(Q_T)$. So from here on, everything looks good but this is not sufficient, in other words we are not able yet to pass to the limit in (25) and still to control the terms $-d_i \int_{Q_T} \psi T_k^{"}(V_{i,n})\nabla V_{i,n}(\sum\limits_{j\neq i} Z_{j,n}\nabla q_{j,n})$ and $\int_{Q_T} \psi X_{i,n}$, so this is the main point of the following lemma

Lemma 5. *There exists C depending only on k, ψ and the initial data such that*

$$|\int_{Q_T} \psi X_{i,n}| \leq C\eta^{\frac{-1}{2}}$$

where $\eta < 1$.

Proof. We have $X_{i,n} = T_k^{'}(V_{i,n}) \sum\limits_{j\neq i}(d_j - d_i)div(q_{j,n}\nabla Z_{j,n})$ and for $\psi \in D$, we integrate by parts on Q_T, then we use the boundary conditions to obtain

$$\begin{aligned} \int_{Q_T} \psi X_{i,n} &= \int_{Q_T} \psi T_k^{'}(V_{i,n})[\sum_{j\neq i}(d_j - d_i)div(q_{j,n}\nabla Z_{j,n})] \\ &= -\int_{Q_T} \nabla(\psi T_k^{'}(V_{i,n}))[\sum_{j\neq i}(d_j - d_i)q_{j,n}\nabla Z_{j,n}] \end{aligned}$$

therefore

$$-\int_{Q_T} \psi X_{i,n} = \int_{Q_T} [\nabla\psi T_k^{'}(V_{i,n}) + \psi T_k^{"}(V_{i,n})\nabla V_{i,n}][\sum_{j\neq i}(d_j - d_i)q_{j,n}\nabla Z_{j,n}]$$

In the following, we denote by $C > 0$ any constant depending only on the initial data and k, ψ but not n, η, then, we use Holder's inequality, this yields

$$|\int_{Q_T} \nabla\psi T_k^{'}(V_{i,n})(q_{j,n}\nabla Z_{j,n})| \leq C\{\int_{[V_{i,n}\leq k]} |q_{j,n}\nabla Z_{j,n}|^2\}^{\frac{1}{2}} ||\nabla\psi||_{L^2(Q_T)}$$

and we have

$$|\int_{Q_T} \psi T_k^{"}(V_{i,n})\nabla V_{i,n}(q_{j,n}\nabla Z_{j,n})| \leq C||\psi||_{L^\infty(Q_T)}\{\int_{[V_{i,n}\leq k]} |q_{j,n}\nabla Z_{j,n}|^2\}^{\frac{1}{2}}\{\int_{[V_{i,n}\leq k]} |\nabla V_{i,n}|^2\}^{\frac{1}{2}}$$

and here we bound the last term of those inequalities as follows, first, we have

$$\nabla V_{i,n} \quad = \quad q_{i,n}\nabla Z_{i,n} + \eta[\sum_{j\neq i} q_{j,n}\nabla Z_{j,n}] - [\frac{m_i}{d_i} q_{i,n} Z_{i,n} + \eta[\sum_{j\neq i}\frac{m_j}{d_j} q_{j,n} Z_{j,n}]]\nabla\phi_n$$

Note that $[V_{i,n} \leq k]$ is included in $[q_{i,n}Z_{i,n} \leq k]$, $[q_{j,n}Z_{j,n} \leq \frac{k}{\eta}]$ for all $j \neq i$. From lemma 2 and by using the result of lemma 4, we have

$$\int_{[V_{i,n}\leq k]} |q_{i,n}\nabla Z_{i,n}|^2 \leq C,\ \forall j\neq i, \int_{[V_{i,n}\leq k]} |q_{j,n}\nabla Z_{j,n}|^2 \leq \frac{C}{\eta}$$

this implies

$$|\int_{Q_T} \psi T_k^{"}(V_{i,n})\nabla V_{i,n}(q_{j,n}\nabla Z_{j,n})| \leq C\eta^{-\frac{1}{2}}.$$

Hence

$$|\int_{Q_T} \nabla\psi T_k^{'}(V_{i,n})(q_{j,n}\nabla Z_{j,n})| \leq C\eta^{-\frac{1}{2}}$$

finally we obtain the desired result, which means that

$$|\int_{Q_T} \psi X_{i,n}| \leq C\eta^{\frac{-1}{2}}$$

still now to bound the first term, so first of all, we have

$$-d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})\nabla V_{i,n}(\sum_{j\neq i} Z_{j,n}\nabla q_{j,n}) = d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})\nabla V_{i,n}(\sum_{j\neq i}\frac{m_j}{d_j} Z_{j,n} q_{j,n}\nabla\phi_n)$$

By applying the same steps as before, we obtain

$$|-d_i\int_{Q_T}\psi T_k^{"}(V_{i,n})\nabla V_{i,n}(\sum_{j\neq i} Z_{j,n}\nabla q_{j,n})| \leq C\eta^{\frac{-1}{2}}.$$

Now, thanks to the boundedness of ϕ_n in $L^\infty(0,T,W_0^{1,\infty}(\Omega))$, we conclude the existence of ϕ belongs to $L^\infty(0,T,W_0^{1,\infty}(\Omega))$, such that

$$\nabla\phi_n \to \nabla\phi \text{ for the topology } \sigma(L^\infty(Q_T), L^1(Q_T)). \tag{26}$$

Since $T_k^{'}$ has a compact support and $||T_k^{'}||_{L^\infty(Q_T)} \leq 1$ and $T_k^{'}(V_{i,n})$ tends to $T_k^{'}(V_i)$ a.e in Q_T, we get

$$T_k^{'}(V_{i,n})\nabla\phi_n \to T_k^{'}(V_i)\nabla\phi \text{ for the topology } \sigma(L^\infty(Q_T), L^1(Q_T)).$$

Next, Let us show that

$$T_k^{'}(V_{i,n})(q_{i,n}Z_{i,n})\nabla\phi_n \to T_k^{'}(V_i)(q_iZ_i)\nabla\phi \text{ in } D'(Q_T)$$

For this reason, we will prove that

$$T_k^{'}(V_{i,n})(q_{i,n}Z_{i,n})\nabla\phi_n \to T_k^{'}(V_i)(q_iZ_i)\nabla\phi \text{ for the topology } \sigma(L^1(Q_T), L^\infty(Q_T)). \quad (27)$$

So let $v \in L^\infty(Q_T)$, we have

$$\int_0^T\int_\Omega((q_{i,n}Z_{i,n})T_k^{'}(V_{i,n})\nabla\phi_n - (q_iZ_i)T_k^{'}(V_i)\nabla\phi)vdxdt = \int_0^T\int_\Omega((q_{i,n}Z_{i,n}) - (q_iZ_i))T_k^{'}(V_{i,n})\nabla\phi_n vdxdt$$
$$+\int_0^T\int_\Omega(q_iZ_i)(T_k^{'}(V_{i,n})\nabla\phi_n - T_k^{'}(V_i)\nabla\phi)vdxdt$$

Concerning the first term, we see that

$$|\int_0^T\int_\Omega((q_{i,n}Z_{i,n})-(q_iZ_i))T_k^{'}(V_{i,n})\nabla\phi_n vdxdt| \le ||v||_{L^\infty(Q_T)}||\nabla\phi_n||_{L^\infty(Q_T)}||(q_{i,n}Z_{i,n})-(q_iZ_i)||_{L^1(Q_T)}$$

then by using the L^1 convergence of $q_{i,n}Z_{i,n}$, we obtain

$$\int_0^T\int_\Omega((q_{i,n}Z_{i,n}) - (q_iZ_i))T_k^{'}(V_{i,n})\nabla\phi_n vdxdt \to 0$$

Since $T_k^{'}(V_{i,n})\nabla\phi_n$ converges to $T_k^{'}(V_i)\nabla\phi$ for the topology $\sigma(L^\infty(Q_T), L^1(Q_T))$, we get the following result
$T_k^{'}(V_{i,n})(Z_{i,n}\nabla q_{i,n} + \eta\sum_{j\neq i}Z_{j,n}\nabla q_{j,n})$ converges to $T_k^{'}(V_i)(Z_i\nabla q_i + \eta\sum_{j\neq i}Z_j\nabla q_j)$ for the topology $\sigma(L^1(Q_T), L^\infty(Q_T))$.
Otherwise, we know that from (19) and (26), we obtain

$$-\varepsilon\Delta\phi_n \to -\varepsilon\Delta\phi \ in\ D^{'}(Q_T).$$

Furthermore,

$$F(t,x,q_nZ_n) \to F(t,x,qZ) \ a.e\ in\ Q_T$$

According to (4) and by applynig the Lebesgue convergence theorem, we obtain

$$-\varepsilon\Delta\phi_n(t,.) \to -\varepsilon\Delta\phi(t,.) = F(t,.,qZ) \text{ strongly in } L^1(\Omega).$$

Now, let us look at the convergence of the term $m_i\int_{Q_T}T_k^{"}(V_{i,n})\nabla V_{i,n}(Z_{i,n}q_{i,n}\nabla\phi_n)\psi$. First, we can notice that on the set $[V_{i,n} \le k] \subset [q_{i,n}Z_{i,n} \le k]$, the terms $Z_{i,n}\nabla q_{i,n}$ are bounded in $L^\infty(Q_T)$. Indeed, on one hand $|q_{i,n}Z_{i,n}| \le k$ and on the other hand $||\nabla\phi_n||_{L^\infty(Q_T)} \le C$ (see lemma 2). Then, we deduce that for all $1 \le i \le NS$, $||q_{i,n}Z_{i,n}\nabla\phi_n||_{L^\infty(Q_T)} \le C(k)$. which imply that for a subsequence still denoted by $Z_{i,n}\nabla q_{i,n}$

$$Z_{i,n}\nabla q_{i,n} \rightharpoonup \beta \ converges\ weak\text{-}*\ in\ L^\infty(Q_T).$$

Such that $\beta \in L^\infty(Q_T)$.
Then, since $T_k^{"}$ has a compact support and by using the pointwise convergence of $T_k"(V_{i,n})$

to $T_k^{"}(V_i)$ as n tends to zero, the bounded character of $T_k^{"}$ and the weak-* convergence of $Z_{i,n}\nabla q_{i,n}$, we conclude that

$$T_k^{"}(V_{i,n})Z_{i,n}\nabla q_{i,n} \rightharpoonup T_k^{"}(V_i)\beta \textit{ converges weak-* in } L^\infty(Q_T).$$

Moreover, we recall that

$$\nabla V_{i,n} \textit{ converges to } \nabla V_i \textit{ strongly in } L^1(Q_T) \textit{ and a.e in } Q_T.$$

Where $\nabla V_i = \nabla(q_i Z_i + \eta \sum_{1\le i\le NS, j\ne i} q_j Z_j)$. Finally, we obtain the desired result

$$T_k^{"}(V_{i,n})\nabla V_{i,n} Z_{i,n}\nabla q_{i,n} \to T_k^{"}(V_i)\nabla V_i\beta \textit{ for the topology } \sigma(L^1(Q_T), L^\infty(Q_T)).$$

Now, we can let n tends to $+\infty$ in (24). By using the strong convergence in $L^1(Q_T)$ of $T_k(V_{i,n})$ to $T_k(V_i)$ and the L^1 convergence of the initial data, we obtain

$$\begin{aligned}&-\int_\Omega \psi(0)U_{i,k}(0) + \int_{Q_T}[-\psi_t U_{i,k} + d_i\nabla\psi\nabla U_{i,k}] - d_i\int_{Q_T}\nabla\psi T_k^{'}(V_i)(Z_i\nabla q_i + \eta\sum_{j\ne i} Z_j\nabla q_j)\\ &-d_i\int_{Q_T}\psi T_k^{"}(V_i)\nabla V_i\beta \ge \int_{Q_T}\psi T_k^{'}(V_i)[S_i(Z,\phi) + \eta\textstyle\sum_{j\ne i} S_j(Z,\phi)] + \epsilon(i,\eta,k,\psi)\end{aligned} \tag{28}$$

where $\epsilon(i,\eta,k,\psi) \ge -C(k,\psi)\eta^{\frac{-1}{2}}$ so that

$$\liminf_{\eta\to\ 0} \epsilon(i,\eta,k,\psi) \ge 0.$$

Let η tends to 0 in the above inequality. Since $U_{i,k} = T_k(V_i) = T_k(q_i Z_i + \eta\sum_{j\ne i}(q_j Z_j))$ converges to $T_k(q_i Z_i)$ strongly in $L^1(Q_T)$ and $T_k^{'}(q_i Z_i + \eta\sum_{j\ne i}(q_j Z_j))$ remains uniformly bounded by 1 and $T_k^{'}(q_i Z_i + \eta\sum_{j\ne i}(q_j Z_j))$ tends a.e to $T_k^{'}(q_i Z_i)$. Then, by passing to the limit in the sense of distributions, we found

$$\begin{aligned}&-\int_\Omega \psi(0)T_k(q_{i,0}Z_{i,0}) + \int_{Q_T}[-\psi_t T_k(q_i Z_i) + d_i\nabla\psi\nabla T_k(q_i Z_i)] - d_i\int_{Q_T}\nabla\psi T_k^{'}(q_i Z_i)(Z_i\nabla q_i)\\ &-d_i\int_{Q_T}\psi T_k^{"}(V_i)\nabla V_i\beta \ge \int_{Q_T}\psi T_k^{'}(V_i)S_i(Z,\phi).\end{aligned} \tag{29}$$

Finally, let $k\to+\infty$. Since $T_k^{"}(q_i Z_i)\to 0$ a.e in Q_T, $T_k(q_i Z_i)$ tends to $(q_i Z_i)$ in $L^1(Q_T)$, $T_k^{'}(q_i Z_i)$ tends a.e to 1 and $S_i \in L^1(Q_T)$, we can pass to the limit and obtain

$$-\int_\Omega (q_{i,0}Z_{i,0})\psi(0) + \int_{Q_T}[-\psi_t(q_i Z_i) + d_i\nabla(q_i Z_i)\nabla\psi] + m_i\int_{Q_T}\nabla\psi(q_i Z_i\nabla\phi) \ge \int_{Q_T}\psi S_i(Z,\phi) \tag{30}$$

finally, we obtain

$$-\int_\Omega (q_{i,0}Z_{i,0})\psi(0) + \int_{Q_T}[-\psi_t(q_i Z_i) + d_i q_i\nabla Z_i\nabla\psi] \ge \int_{Q_T} S_i(Z,\phi)\psi \tag{31}$$

3.2. Global existence of weak solutions

Theorem 3. *Let us consider system* (1) *together with* (14) *or* (15)*, with* $(H1)$ *and* $(H3)$*, with* $Z_{i,0} \in L^1(\Omega)$ *such that* $Z_{i,0} \geq 0$*, for all* $1 \leq i \leq NS$*. We assume the structure* $(H1) + (H2)$ *hold together with the a priori estimate* (11)*. Then, system* (1) *has a weak solution on* $(0, T)$ *(i.e equality holds in* (16) *or* (17)*).*

Proof. By Theorem 2, up to a subsequence, the approximate solution $(q_n Z_n)$ converge to a weak supersolution. Now, we try to prove that this supersolution is also a subsolution and according to this two results we will deduce that the supersolution (resp. subsolution) is a weak solution of the main system that we have considered in the beginning. From the compactness lemma 3, we have

$$(\omega_n, \nabla\omega_n) \to (\omega, \nabla\omega) \text{ in } [L^1(Q_T)]^{NS} \times [[L^1(Q_T)]^N]^{NS}.$$

Where $\omega_n = q_n Z_n$ and $\omega = qZ$.
Then, for all $\psi \in D$ nonnegative test function and all $i = 1, ..., NS$, we have

$$-\int_\Omega (q_{i,0} Z_{i,0})\psi(0) + \int_{Q_T} [-\psi_t(q_i Z_i) + d_i \nabla(q_i Z_i)\nabla\psi] + m_i \int_{Q_T} (q_i Z_i)\nabla\phi\nabla\psi \geq \int_{Q_T} S_i(Z, \phi)\psi$$

where $S_i(Z, \phi) \in L^1(Q_T)$. In the following step, we introduce the notations

$$W_n = \sum_{i=1}^{NS} q_{i,n} Z_{i,n}, \quad T_n = \sum_{i=1}^{NS} d_i q_{i,n} Z_{i,n}, \quad Y_n = \sum_{i=1}^{NS} m_i q_{i,n} Z_{i,n},$$

and

$$W = \sum_{i=1}^{NS} q_i Z_i, \quad T = \sum_{i=1}^{NS} d_i q_i Z_i, \quad Y = \sum_{i=1}^{NS} m_i q_i Z_i,$$

We sum the equations of the approximate problem then we get

$$-\int_\Omega \psi(0) W_n(0) + \int_{Q_T} [-\psi_t W_n + \nabla T_n \nabla\psi] + \int_{Q_T} Y_n \nabla\phi_n \nabla\psi = \int_{Q_T} \sum_{i=1}^{NS} S_i^n(Z_n, \phi_n)\psi$$

and from structure $(H2)$, we have

$$\sum_{i=1}^{NS} S_i^n(Z_n, \phi_n) \leq C(1 + W_n)$$

which means

$$C(1 + W_n) - \sum_{i=1}^{NS} S_i^n(Z_n, \phi_n) \geq 0$$

and we already know that

$$\begin{aligned} W_n &\to W \text{ in } L^1(Q_T) \\ S_i^n(Z_n, \phi_n) &\to S_i(Z, \phi) \text{ a.e in } Q_T \end{aligned}$$

using Fatou's lemma, this leads to

$$\int_{Q_T} -\psi \sum_{i=1}^{NS} S_i(Z,\phi) \leq \liminf_{n\to +\infty} \int_{Q_T} -\psi \sum_{i=1}^{NS} S_i^n(Z_n,\phi_n) \tag{32}$$

thus

$$-\int_{\Omega} \psi(0)W(0) + \int_{Q_T} [-\psi_t W + \nabla T \nabla \psi] + \int_{Q_T} Y \nabla \phi \nabla \psi \leq \int_{Q_T} \psi \sum_{i=1}^{NS} S_i(Z,\phi)$$

finally, we obtain the suitable result.

4. References

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