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Stancu type generalization of modified Srivastava-Gupta operators

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Abstract. In this paper, we introduce a Stancu type generalization of modified Srivastava-Gupta operators. We obtain the moments of the operators and then prove the basic convergence theorem. Next, the Voronovskaja type asymptotic formula and some direct results for the above operators are discussed. Also, the rate of convergence and weighted approximation by these operators in terms of modulus of continuity are studied. Then, we obtain point-wise estimates using the Lipschitz type maximal function and two parameter Lipschitz-type space. Further, we study the A-statistical convergence of these operators. Lastly, we give better estimations of the above operators using King type approach.

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Key Words and Phrases: Srivastava-Gupta operators, rate of convergence, modulus of continuity, weighted approximation, pointwise estimates, A-statistical convergence.

1. Introduction

In the year 2003, Srivastava and Gupta [33] introduced a general family of summation-integral type operators $\{G_{n,c}\}$ which includes some well-known operators as special cases. They obtained the rate of convergence for functions of bounded variation. For the details of special cases in [33], we refer the readers to [13], [20] and [31].

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For $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq M(1+t)^\gamma \text{ for some } M > 0, \gamma > 0\}$, Srivastava and Gupta proposed a certain family of positive linear operators defined by

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^\infty p_{n+c,k-1}(t,c)f(t)dt + p_{n,0}(x,c)f(0), \tag{1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x) \tag{2}$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1+cx)^{-n/c}, & c \in N. \end{cases}$$

Verma and Agrawal [35] introduced the generalized form of the operators (1) and studied some of its approximation properties. Deo [3] gave a modification of these operators and established the rate of convergence and Voronovskaja type asymptotic result. Recently, Acar et al. [1] introduced Stancu type generalization of the operators (1) and obtained an estimate of the rate of convergence for functions having derivatives of bounded variation and also studied the simultaneous approximation for these operators.
Yadav [36] proposed the modification of the operators (1) using the King approach as

$$G^*_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^\infty p_{n+c,k-1}(t,c)f\left(\frac{(n-c)t}{n}\right)dt + p_{n,0}(x,c)f(0) \tag{3}$$

and studied its moment estimates, direct estimate, asymptotic formula and statistical convergence. Recently, Maheshwari [22] obtained the rate of convergence for the functions having bounded derivatives on every finite subinterval of $[0,\infty)$ for the operators (3). Very recently, Neer et al. [29] introduced the Bezier variant of the operators (3) and studied the direct approximation result and estimate of the rate of convergence of these operators for functions of bounded variation.
In [34], Stancu introduced the positive linear operators $P_n^{(\alpha,\beta)} : C[0,1] \to C[0,1]$ by modifying the Bernstein polynomial as

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} b_{n,k}(x)f\left(\frac{k+\alpha}{n+\beta}\right),$$

where $b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k},\ x \in [0,1]$ is the Bernstein basis function and α, β are any two real numbers which satisfy the condition that $0 \leq \alpha \leq \beta$.
In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [1], [2], [32] etc.

For $f \in C_\gamma[0,\infty)$, $0 \leq \alpha \leq \beta$ we introduce the following Stancu type generalization of the operators (3):

$$\begin{aligned} G_{n,c}^{*(\alpha,\beta)}(f;x) &= n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^\infty p_{n+c,k-1}(t,c) f\left(\frac{(n-c)t+\alpha}{n+\beta}\right)dt \\ &\quad + p_{n,0}(x,c) f\left(\frac{\alpha}{n+\beta}\right) \end{aligned} \tag{4}$$

For $\alpha = \beta = 0$, we denote $G_{n,c}^{*(\alpha,\beta)}(f;x)$ by $G_{n,c}^{*}(f;x)$.

The goal of the present paper is to study the basic convergence theorem, Voronovskaja type asymptotic result, local approximation theorem, rate of convergence, weighted approximation, pointwise estimation and A-statistical convergence of the operators (4). Further, to obtain better approximation, we also propose modification of the operators (4) using King type approach.

2. Moment Estimates

Lemma 1. *[36] For $G_{n,c}^*(t^m;x)$, $m = 0,1,2,$ one has*

(i) $G_{n,c}^*(1;x) = 1;$

(ii) $G_{n,c}^*(t;x) = x;$

(iii) $G_{n,c}^*(t^2;x) = \frac{(n^2-c^2)x^2+2x(n-c)}{n(n-2c)}$, *for* $n > 2c$.

Lemma 2. *For the operators $G_{n,c}^{*(\alpha,\beta)}(f;x)$ as defined in (4), the following equalities hold:*

(i) $G_{n,c}^{*(\alpha,\beta)}(1;x) = 1;$

(ii) $G_{n,c}^{*(\alpha,\beta)}(t;x) = \frac{nx+\alpha}{n+\beta};$

(iii) $G_{n,c}^{*(\alpha,\beta)}(t^2;x) = \left\{\frac{n(n^2-c^2)}{(n-2c)(n+\beta)^2}\right\}x^2 + \left\{\frac{2n((n-c)+\alpha(n-2c))}{(n-2c)(n+\beta)^2}\right\}x + \frac{\alpha^2}{(n+\beta)^2}$, *for* $n > 2c$.

Proof. For $x \in [0,\infty)$, in view of Lemma 1, we have

$$G_{n,c}^{*(\alpha,\beta)}(1;x) = 1.$$

Next, for $f(t) = t$, again applying Lemma 1, we get

$$\begin{aligned} G_{n,c}^{*(\alpha,\beta)}(t;x) &= n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^\infty p_{n+c,k-1}(t,c)\left(\frac{(n-c)t+\alpha}{n+\beta}\right)dt + p_{n,0}(x,c)\left(\frac{\alpha}{n+\beta}\right) \\ &= \frac{n}{n+\beta}G_{n,c}^*(t,x) + \frac{\alpha}{n+\beta} = \frac{nx+\alpha}{n+\beta} \end{aligned}$$

Proceeding similarly, we have

$$\begin{aligned}
G_{n,c}^{*(\alpha,\beta)}(t^2;x) &= n\sum_{k=1}^{\infty}p_{n,k}(x,c)\int_0^{\infty}p_{n+c,k-1}(t,c)\left(\frac{(n-c)t+\alpha}{n+\beta}\right)^2dt+p_{n,0}(x,c)\left(\frac{\alpha}{n+\beta}\right)^2\\
&= \left(\frac{n}{n+\beta}\right)^2G_{n,c}^{*}(t^2,x)+\frac{2n\alpha}{(n+\beta)^2}G_{n,c}^{*}(t,x)+\left(\frac{\alpha}{n+\beta}\right)^2\\
&= \left\{\frac{n(n^2-c^2)}{(n-2c)(n+\beta)^2}\right\}x^2+\left\{\frac{2n((n-c)+\alpha(n-2c))}{(n-2c)(n+\beta)^2}\right\}x+\frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

Lemma 3. *For $f\in C_B[0,\infty)$ (space of all bounded and continuous functions on $[0,\infty)$ endowed with norm $\parallel f\parallel=\sup\{|f(x)|:x\in[0,\infty)\}$), $\parallel G_{n,c}^{*(\alpha,\beta)}(f;x)\parallel\le\parallel f\parallel$.*

Proof. In view of (4) and Lemma 2, the proof of this lemma easily follows.

Remark 1. *For every $x\ge 0, n>2c$, we have*

$$G_{n,c}^{*(\alpha,\beta)}\left((t-x);x\right) = \frac{\alpha-\beta x}{n+\beta},$$

and

$$\begin{aligned}
G_{n,c}^{*(\alpha,\beta)}\left((t-x)^2;x\right) &= \left\{\frac{nc(2n-c)+\beta^2(n-2c)}{(n-2c)(n+\beta)^2}\right\}x^2\\
&\quad+\left\{\frac{2n(n-c)-2\alpha\beta(n-2c)}{(n-2c)(n+\beta)^2}\right\}x+\frac{\alpha^2}{(n+\beta)^2},\ n>2c\\
&= \gamma_{n,c}^{(\alpha,\beta)}(x), \text{(say)}.
\end{aligned}$$

3. Main Results

Theorem 4. *(Voronovskaja type theorem) Let $f\in C_B[0,\infty)$. If f',f'' exists at a fixed point $x\in[0,\infty)$, we have*

$$\lim_{n\to\infty}n\left(G_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)\right)=(\alpha-\beta x)f'(x)+x(1+cx)f''(x).$$

Proof. Let $x\in[0,\infty)$ be fixed. From the Taylor's theorem, we may write

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}f''(x)(t-x)^2+\xi(t,x)(t-x)^2, \tag{5}$$

where $\xi(t,x)$ is the peano form of the remainder and $\lim_{t\to x}\xi(t,x)=0$.
Applying $G_{n,c}^{*(\alpha,\beta)}(f,x)$ on both sides of (5), we have

$$n\left(G_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)\right) = nf'(x)G_{n,c}^{*(\alpha,\beta)}\left((t-x);x\right)+\frac{1}{2}nf''(x)G_{n,c}^{*(\alpha,\beta)}\left((t-x)^2;x\right)$$

$$+nG_{n,c}^{*(\alpha,\beta)}\left((t-x)^2\xi(t,x);x\right).$$

In view of Remark 1, we have

$$\lim_{n\to\infty} nG_{n,c}^{*(\alpha,\beta)}\left((t-x);x\right)=\alpha-\beta x \tag{6}$$

and

$$\lim_{n\to\infty} nG_{n,c}^{*(\alpha,\beta)}\left((t-x)^2;x\right)=2x(1+cx). \tag{7}$$

Now, we shall show that

$$\lim_{n\to\infty} nG_{n,c}^{*(\alpha,\beta)}\Bigg(\xi(t,x)(t-x)^2;x\Bigg)=0$$

By using Cauchy-Schwarz inequality, we have

$$G_{n,c}^{*(\alpha,\beta)}\Bigg(\xi(t,x)(t-x)^2;x\Bigg)\leq\sqrt{G_{n,c}^{*(\alpha,\beta)}(\xi^2(t,x);x)}\sqrt{G_{n,c}^{*(\alpha,\beta)}((t-x)^4;x)}. \tag{8}$$

We observe that $\xi^2(x,x)=0$ and $\xi^2(.,x)\in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} G_{n,c}^{*(\alpha,\beta)}(\xi^2(t,x);x)=\xi^2(x,x)=0, \tag{9}$$

in view of fact that $G_{n,c}^{*(\alpha,\beta)}((t-x)^4;x)=O\left(\dfrac{1}{n^2}\right)$. Now, from (8) and (9) we obtain

$$\lim_{n\to\infty} nG_{n,c}^{*(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right)=0. \tag{10}$$

From (6), (7) and (10), we get the required result.

3.1. Local approximation

For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f,\delta) \;=\; \inf_{g\in W^2}\{\parallel f-g\parallel+\delta\parallel g''\parallel\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g', g''\in C_B[0,\infty)\}$. By, p. 177, Theorem 2.4 in [4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}), \tag{11}$$

where

$$\omega_2(f,\sqrt{\delta}) \;=\; \sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x)\mid$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)} |\, f(x+h)-f(x)\,|,$$

we denote the usual modulus of continuity of $f \in C_B[0,\infty)$.

Theorem 5. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, we have*

$$|\, G_{n,c}^{*(\alpha,\beta)}(f;x) - f(x)\,| \leq C\omega_2\left(f, \delta_{n,c}^{(\alpha,\beta)}(x)\right) + \omega\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right),$$

where C is an absolute constant and

$$\delta_{n,c}^{(\alpha,\beta)}(x) = \left(G_{n,c}^{*(\alpha,\beta)}((t-x)^2;x) + \left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right)^{1/2}.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\overline{G}_{n,c}^{*(\alpha,\beta)}$ defined by

$$\overline{G}_{n,c}^{*(\alpha,\beta)}(f;x) = G_{n,c}^{*(\alpha,\beta)}(f;x) - f\left(\frac{nx+\alpha}{n+\beta}\right) + f(x). \tag{12}$$

From Lemma 2, we observe that the operators $\overline{G}_{n,c}^{*(\alpha,\beta)}$ are linear and reproduce the linear functions.
Hence

$$\overline{G}_{n,c}^{*(\alpha,\beta)}((t-x);x) = 0. \tag{13}$$

Let $g \in W^2$. By Taylor's theorem, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv, \ \ t \in [0,\infty).$$

Applying $\overline{G}_{n,c}^{*(\alpha,\beta)}$ on both sides of the above equation and using (13), we have

$$\overline{G}_{n,c}^{*(\alpha,\beta)}(g;x) = g(x) + \overline{G}_{n,c}^{*(\alpha,\beta)}\left(\int_x^t (t-v)g''(v)dv;x\right).$$

Thus, by (12) we get
$|\overline{G}_{n,c}^{*(\alpha,\beta)}(g;x) - g(x)|$

$$\begin{aligned}
&\leq G_{n,c}^{*(\alpha,\beta)}\left(\left|\int_x^t (t-v)g''(v)dv\right|;x\right) + \left|\int_x^{\frac{nx+\alpha}{n+\beta}} \left(\frac{nx+\alpha}{n+\beta}-v\right)g''(v)dv\right| \\
&\leq G_{n,c}^{*(\alpha,\beta)}\left(\int_x^t |t-v||g''(v)|dv;x\right) + \int_x^{\frac{nx+\alpha}{n+\beta}} \left|\frac{nx+\alpha}{n+\beta}-v\right||g''(v)|dv
\end{aligned}$$

$$\begin{aligned}
&\leq \left[G^{*(\alpha,\beta)}_{n,c}((t-x)^2;x)+\left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right] \parallel g'' \parallel \\
&\leq \left(\delta^{(\alpha,\beta)}_{n,c}(x)\right)^2 \parallel g'' \parallel . \qquad (14)
\end{aligned}$$

On other hand, by (12) and Lemma 3, we have

$$|\overline{G}^{*(\alpha,\beta)}_{n,c}(f;x)| \leq 3 \parallel f \parallel . \qquad (15)$$

Using (14) and (15) in (12), we obtain

$$\begin{aligned}
|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)| &\leq |\overline{G}^{*(\alpha,\beta)}_{n,c}(f-g;x)|+|(f-g)(x)|+|\overline{G}^{*(\alpha,\beta)}_{n,c}(g;x)-g(x)| \\
&\quad +\left|f\left(\frac{nx+\alpha}{n+\beta}\right)-f(x)\right| \\
&\leq 4 \parallel f-g \parallel + \left(\delta^{(\alpha,\beta)}_{n,c}(x)\right)^2 \parallel g'' \parallel + \left|f\left(\frac{nx+\alpha}{n+\beta}\right)-f(x)\right|.
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2$, we get

$$| G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x) | \leq K_2\left(f,(\delta^{(\alpha,\beta)}_{n,c}(x))^2\right)+\omega\left(f,\frac{|\alpha-\beta x|}{n+\beta}\right).$$

In view of (11), we get

$$| G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x) | \leq C\omega_2\left(f,\delta^{(\alpha,\beta)}_{n,c}(x)\right)+\omega\left(f,\frac{|\alpha-\beta x|}{n+\beta}\right).$$

Hence, the proof is completed.

3.2. Rate of convergence

Let $\omega_b(f,\delta)$ denote the modulus of continuity of f on the closed interval $[0,b], b>0$, and defined as

$$\omega_b(f,\delta)=\sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,b]}|f(t)-f(x)|.$$

We observe that for a function $f\in C_B[0,\infty)$, the modulus of continuity $\omega_b(f,\delta)$ tends to zero. Now, we give a rate of convergence theorem for the operators $G^{*(\alpha,\beta)}_{n,c}$.

Theorem 6. *Let $f\in C_B[0,\infty)$ and $\omega_{b+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,b+1]\subset[0,\infty)$, where $b>0$. Then, for every $n>2c$,*

$$|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)|\leq 4M_f(1+b^2)\gamma^{(\alpha,\beta)}_{n,c}(x)+2\omega_{b+1}\left(f,\sqrt{\gamma^{(\alpha,\beta)}_{n,c}(x)}\right),$$

where $\gamma^{(\alpha,\beta)}_{n,c}(x)$ is defined in Remark 1 and M_f is a constant depending only on f.

Proof. For $x \in [0, b]$ and $t > b + 1$. Since $t - x > 1$, we have

$$|f(t) - f(x)| \leq M_f(2 + t^2 + x^2) \leq M_f(t - x)^2(2 + 2x + 2x^2) \leq 4M_f(1 + b^2)(t - x)^2.$$

For $x \in [0, b]$ and $t \leq b + 1$, we have

$$|f(t) - f(x)| \leq \omega_{b+1}(f, |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right)\omega_{b+1}(f, \delta), \delta > 0.$$

From the above, we have

$$|f(t) - f(x)| \leq 4M_f(1 + b^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{b+1}(f, \delta), \delta > 0.$$

Thus, by applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|G^{*(\alpha,\beta)}_{n,c}(f; x) - f(x)| &\leq 4M_f(1 + b^2)(G^{*(\alpha,\beta)}_{n,c}(t - x)^2; x) \\
&\quad +\omega_{b+1}(f, \delta)\left(1 + \frac{1}{\delta}(G^{*(\alpha,\beta)}_{n,c}(t - x)^2; x)^{\frac{1}{2}}\right) \\
&\leq 4M_f(1 + b^2)\gamma^{(\alpha,\beta)}_{n,c}(x) + 2\omega_{b+1}\left(f, \sqrt{\gamma^{(\alpha,\beta)}_{n,c}(x)}\right),
\end{aligned}$$

on choosing $\delta = \sqrt{\gamma^{(\alpha,\beta)}_{n,c}(x)}$. This completes the proof of the theorem.

3.3. Weighted approximation.

Let C_ν be the space of all continuous functions on $[0, \infty)$ with the norm $\| f \|_\nu = \sup\limits_{x\in[0,\infty)} \dfrac{|f(x)|}{\nu(x)}$ and $C^0_\nu = \{f \in C_\nu : \lim\limits_{x\to\infty} \dfrac{|f(x)|}{\nu(x)} < \infty\}$, where $\nu(x)$ is a weight function. In what follows we consider $\nu(x) = 1 + x^2$.

Theorem 7. *For each $f \in C^0_\nu$, we have*

$$\lim_{n\to\infty} \| G^{*(\alpha,\beta)}_{n,c}(f) - f \|_\nu = 0.$$

Proof. From [8], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| G^{*(\alpha,\beta)}_{n,c}(t^k; x) - x^k \|_\nu = 0, \ k = 0, 1, 2. \tag{16}$$

Since $G^{*(\alpha,\beta)}_{n,c}(1; x) = 1$, the condition in (16) holds for $k = 0$.
By Lemma 2, we have

$$\| G^{*(\alpha,\beta)}_{n,c}(t; x) - x) \|_\nu = \sup_{x\in[0,\infty)} \frac{|G^{*(\alpha,\beta)}_{n,c}(t; x) - x|}{1 + x^2}$$

$$\leq \frac{\beta}{n+\beta}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha}{n+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}$$
$$\leq \frac{\alpha+\beta}{n+\beta},$$

which implies that the condition in (16) holds for $k=1$.
Similarly, we can write for $n>2c$

$$\begin{aligned}\parallel G^{*(\alpha,\beta)}_{n,c}(t^2;x)-x^2\parallel_\nu &= \sup_{x\in[0,\infty)}\frac{|G^{*(\alpha,\beta)}_{n,c}(t^2;x)-x^2|}{1+x^2}\\ &\leq \left|\frac{n(n^2-c^2)}{(n-2c)(n+\beta)^2}-1\right|+\left|\frac{2n((n-c)+\alpha(n-2c))}{(n-2c)(n+\beta)^2}\right|\\ &\quad+\frac{\alpha^2}{(n+\beta)^2},\end{aligned}$$

which implies that $\lim\limits_{n\to\infty}\parallel G^{*(\alpha,\beta)}_{n,c}(t^2;x)-x^2\parallel_\nu=0$, the equation (16) holds for $k=2$.
This completes the proof of theorem.

Now we give the following theorem to approximate all functions in C^0_ν. Such type of results are given in [9] for locally integrable functions.

Theorem 8. *For each $f\in C^0_\nu$ and $\sigma>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)|}{(1+x^2)^{\sigma+1}}=0.$$

Proof. For any fixed $x_0>0$,

$$\sup_{x\in[0,\infty)}\frac{|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)|}{(1+x^2)^{\sigma+1}} = \sup_{x\leq x_0}\frac{|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)|}{(1+x^2)^{\sigma+1}}+\sup_{x>x_0}\frac{|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)|}{(1+x^2)^{\sigma+1}}$$

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|G^{*(\alpha,\beta)}_{n,c}(f;x)-f(x)|}{(1+x^2)^{\sigma+1}} &\leq \parallel G^{*(\alpha,\beta)}_{n,c}(f)-f\parallel_{C[0,x_0]}\\ &\quad+\parallel f\parallel_\nu\sup_{x>x_0}\frac{|G^{*(\alpha,\beta)}_{n,c}(1+t^2;x)|}{(1+x^2)^{\sigma+1}}+\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{\sigma+1}}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 6. By Lemma 2, for any fixed $x_0>0$, it is easily prove that

$$\sup_{x>x_0}\frac{|G^{*(\alpha,\beta)}_{n,c}(1+t^2;x)|}{(1+x^2)^{\sigma+1}}\to 0$$

as $n\to\infty$. We can choose $x_0>0$ so large that the last part of the above inequality can be small.
Hence the proof is completed.

3.4. Pointwise Estimates

In this section, we establish some pointwise estimates of the rate of convergence of the operators $G_{n,c}^{*(\alpha,\beta)}$. First, we give the relationship between the local smoothness of f and local approximation.
We know that a function $f \in C[0,\infty)$ is in Lip_M (α) on E, $\alpha \in (0,1]$, E$\subset$ $[0,\infty)$ if it satisfies the condition

$$|f(t)-f(x)| \leq M|t-x|^{\alpha},\ t \in [0,\infty) \ \ and \ \ x \in E,$$

where M is a constant depending only on α and f.

Theorem 9. *Let $f \in C[0,\infty) \cap Lip_M$ (α), $E \subset [0,\infty)$ and $\alpha \in (0,1]$. Then, we have*

$$|G_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)| \ \ \leq \ \ M\biggl(\Bigl(\gamma_{n,c}^{(\alpha,\beta)}(x)\Bigr)^{\alpha/2} + 2d^{\alpha}(x,E)\biggr), \ \ x \in [0,\infty),$$

where M is a constant depending on α and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E) = \inf\{|t-x| : t \in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $x_0 \in \overline{E}$ such that

$$d(x,E) = |x-x_0|.$$

By our hypothesis and the monotonicity of $G_{n,c}^{*(\alpha,\beta)}$, we get

$$\begin{aligned} |G_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)| &\leq G_{n,c}^{*(\alpha,\beta)}(|f(t)-f(x_0)|;x) + G_{n,c}^{*(\alpha,\beta)}(|f(x)-f(x_0)|;x) \\ &\leq M\left(G_{n,c}^{*(\alpha,\beta)}(|t-x_0|^{\alpha};x) + |x-x_0|^{\alpha}\right) \\ &\leq M\left(G_{n,c}^{*(\alpha,\beta)}(|t-x|^{\alpha};x) + 2|x-x_0|^{\alpha}\right). \end{aligned}$$

Now, applying Hölder's inequality with $p = \dfrac{2}{\alpha}$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$, we obtain

$$|G_{n,c}^{*(\alpha,\beta)}((f;x)-f(x)| \leq M\left(\{G_{n,c}^{*(\alpha,\beta)}(|t-x|^2;x)\}^{\alpha/2} + 2d^{\alpha}(x,E)\right),$$

from which the desired result immediate.

Next, we obtain the local direct estimate of the operators defined in (4), using the Lipschitz-type maximal function of order α introduced by B. Lenze [19] as

$$\widetilde{\omega}_{\alpha}(f,x) = \sup_{t \neq x,\ t \in [0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^{\alpha}}, \ \ x \in [0,\infty) \ \text{ and } \ \alpha \in (0,1]. \tag{17}$$

Theorem 10. *Let $f \in C_B[0,\infty)$ and $0 < \alpha \leq 1$. Then, for all $x \in [0,\infty)$ we have*

$$|G^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)| \leq \widetilde{\omega}_\alpha(f,x)\left(\gamma^{(\alpha,\beta)}_{n,c}(x)\right)^{\alpha/2}.$$

Proof. From the equation (17), we have

$$|G^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)| \leq \widetilde{\omega}_\alpha(f,x) G^{*(\alpha,\beta)}_{n,c}(|t-x|^\alpha;x).$$

Applying the Hölder's inequality with $p = \dfrac{2}{\alpha}$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$, we get

$$|G^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)| \leq \widetilde{\omega}_\alpha(f,x) G^{*(\alpha,\beta)}_{n,c}((t-x)^2;x)^{\frac{\alpha}{2}} \leq \widetilde{\omega}_\alpha(f,x)\left(\gamma^{(\alpha,\beta)}_{n,c}(x)\right)^{\alpha/2}.$$

Thus, the proof is completed.

For $a, b > 0$, Özarslan and Aktuğlu [30] consider the Lipschitz-type space with two parameters:

$$Lip^{(a,b)}_M(\alpha) = \left(f \in C[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\alpha}{(t + ax^2 + bx)^{\alpha/2}};\ x, t \in [0,\infty) \right),$$

where M is any positive constant and $0 < \alpha \leq 1$.

Theorem 11. *For $f \in Lip^{(a,b)}_M(\alpha)$. Then, for all $x > 0$, we have*

$$|G^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)| \leq M\left(\frac{\gamma^{(\alpha,\beta)}_{n,c}(x)}{ax^2 + bx}\right)^{\alpha/2}.$$

Proof. First we prove the theorem for $\alpha = 1$. Then, for $f \in Lip^{(a,b)}_M(1)$, and $x \in [0,\infty)$, we have

$$\begin{aligned} |G^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)| &\leq G^{*(\alpha,\beta)}_{n,c}(|f(t) - f(x)|;x) \\ &\leq M G^{*(\alpha,\beta)}_{n,c}\left(\frac{|t-x|}{(t + ax^2 + bx)^{1/2}};x\right) \\ &\leq \frac{M}{(ax^2 + bx)^{1/2}} G^{*(\alpha,\beta)}_{n,c}(|t-x|;x). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} |G^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)| &\leq \frac{M}{(ax^2 + bx)^{1/2}}\left(G^{*(\alpha,\beta)}_{n,c}((t-x)^2;x)\right)^{1/2} \\ &\leq M\left(\frac{\gamma^{(\alpha,\beta)}_{n,c}(x)}{ax^2 + bx}\right)^{1/2}. \end{aligned}$$

Thus the result holds for $\alpha = 1$.
Now, we prove that the result is true for $0 < \alpha < 1$. Then, for $f \in Lip_M^{(a,b)}(\alpha)$, and $x \in [0, \infty)$, we get

$$|G_{n,c}^{*(\alpha,\beta)}(f;x) - f(x)| \leq \frac{M}{(ax^2+bx)^{\alpha/2}} G_{n,c}^{*(\alpha,\beta)}(|t-x|^{\alpha};x).$$

Taking $p = \frac{1}{\alpha}$ and $q = \frac{p}{p-1}$, applying the Hölders inequality, we have

$$|G_{n,c}^{*(\alpha,\beta)}(f;x) - f(x)| \leq \frac{M}{(ax^2+bx)^{\alpha/2}} \left(G_{n,c}^{*(\alpha,\beta)}(|t-x|;x) \right)^{\alpha}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|G_{n,c}^{*(\alpha,\beta)}(f;x) - f(x)| \leq M \left(\frac{\gamma_{n,c}^{(\alpha,\beta)}(x)}{ax^2+bx} \right)^{\alpha/2}.$$

Thus, the proof is completed.

3.5. Statistical convergence

Let $A = (a_{nk}), (n, k \in N)$, be a non-negative infinite summability matrix. For a given sequence $x := (x)_n$, the A-transform of x denoted by $Ax : ((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$$

provided the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_n x_n = L$. The sequence $x = (x)_n$ is said to be a A- statistically convergent to L i.e. $st_A - \lim_n (x)_n = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \epsilon} a_{nk} = 0$. If we replace A by C_1 then A is a Cesáro matrix of order one and A- statistical convergence is reduced to the statistical convergence. Similarly, if $A = I$, the identity matrix, then A- statistical convergence coincides with the ordinary convergence. It is to be noted that the concept of A-statistical convergence may also be given in normed spaces. Many researchers have investigated the statistical convergence properties for several sequences and classes of linear positive operators (see [5], [6], [7], [10], [23], [28]). In the following result we prove a weighted Korovkin theorem via A-statistical convergence.
Throughout this section, let us assume that $e_i(t) = t^i, i = 0, 1, 2$.

Theorem 12. *Let (a_{nk}) be a non-negative regular infinite summability matrix and $x \in [0, \infty)$. Let $\nu_\varsigma \geq 1$ be a continuous function such that*

$$\lim_{x \to \infty} \frac{\nu(x)}{\nu_\varsigma(x)} = 0.$$

Then, for all $f \in C^0_\nu$, we have

$$st_A - \lim_n \| G^{*(\alpha,\beta)}_{n,c}(f) - f \|_{\nu_\varsigma} = 0.$$

Proof. From ([7] p. 195, Th. 6), it is enough to show that

$$st_A - \lim_n \| G^{*(\alpha,\beta)}_{n,c}(e_i) - e_i \|_\nu = 0.$$

From Lemma 2, we get

$$st_A - \lim_n \| G^{*(\alpha,\beta)}_{n,c}(e_0) - e_0 \|_\nu = 0.$$

Again by using Lemma 2, we have

$$\begin{aligned} \| G^{*(\alpha,\beta)}_{n,c}(e_1) - e_1 \|_\nu &\leq \frac{\beta}{(n+\beta)} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha}{n+\beta} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq \frac{\alpha+\beta}{n+\beta}. \end{aligned}$$

For any given $\epsilon > 0$, let us define the following sets:

$$S := \left\{ n : \| G^{*(\alpha,\beta)}_{n,c}(e_1) - e_1 \|_\nu \geq \epsilon \right\},$$

$$S_1 := \left\{ n : \frac{\alpha}{n+\beta} \geq \frac{\epsilon}{2} \right\}$$

and

$$S_2 := \left\{ n : \frac{\beta}{n+\beta} \geq \frac{\epsilon}{2} \right\}.$$

Then, we get $S \subseteq S_1 \cup S_2$ which implies that

$$\sum_{k\in S} a_{nk} \leq \sum_{k\in S_1} a_{nk} + \sum_{k\in S_2} a_{nk}$$

and hence

$$st_A - \lim_n \| G^{*(\alpha,\beta)}_{n,c}(e_1) - e_1 \|_\nu = 0.$$

Similarly, we have

$$\| G^{*(\alpha,\beta)}_{n,c}(e_2) - e_2 \|_\nu \leq \left(\frac{n(n^2-c^2)}{(n-2c)(n+\beta)^2} - 1 \right) + \frac{2n((n-c)+\alpha(n-2c))}{(n-2c)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Now, we define the following sets:

$$U := \left\{ n : \| G^{*(\alpha,\beta)}_{n,c}(e_2) - e_2 \|_\nu \geq \epsilon \right\},$$

$$U_1 := \left\{ n : \left(\frac{n(n^2-c^2)}{(n-2c)(n+\beta)^2} - 1 \right) \geq \frac{\epsilon}{3} \right\},$$

$$U_2 := \left\{ n : \frac{2n((n-c)+\alpha(n-2c))}{(n-2c)(n+\beta)^2} \geq \frac{\epsilon}{3} \right\}$$

and

$$U_3 := \left\{ n : \frac{\alpha^2}{(n+\beta)^2} \geq \frac{\epsilon}{3} \right\}.$$

Then, we get $U \subseteq U_1 \cup U_2 \cup U_3$ which implies that

$$\sum_{k\in U} a_{nk} \leq \sum_{k\in U_1} a_{nk} + \sum_{k\in U_2} a_{nk} + \sum_{k\in U_3} a_{nk}$$

and hence

$$st_A - \lim_n \parallel G_{n,c}^{*(\alpha,\beta)}(e_2) - e_2 \parallel_\nu = 0.$$

This completes the proof of the theorem.

4. Better Estimates

It is well known that the classical Bernstein polynomial preserve constant as well as linear functions. To make the convergence faster, King [18] proposed an approach to modify the Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t) = t^i, i = 0, 1, 2$. As the operator $G_{n,c}^{*(\alpha,\beta)}(f;x)$ defined in (4) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions.
For this purpose the modification of (4) is defined as

$$\begin{aligned} \overline{G}_{n,c}^{*(\alpha,\beta)}(f;x) &= n\sum_{k=1}^{\infty} p_{n,k}(r_n(x),c) \int_0^\infty p_{n+c,k-1}(t,c) f\left(\frac{(n-c)t+\alpha}{n+\beta}\right) dt \\ &\quad + p_{n,0}(r_n(x),c) f\left(\frac{\alpha}{n+\beta}\right), \end{aligned} \tag{18}$$

where $r_n(x) = \frac{(n+\beta)x-\alpha}{n}$ for $x \in I_n = [\frac{\alpha}{n+\beta}, \infty)$ and $n > 2c$.

Lemma 13. *For each* $x \in I_n$, *by simple computations, we have*

(i) $\overline{G}_{n,c}^{*(\alpha,\beta)}(1;x) = 1$;

(ii) $\overline{G}_{n,c}^{*(\alpha,\beta)}(t;x) = x$;

(iii) $\overline{G}_{n,c}^{*(\alpha,\beta)}(t^2;x)=\dfrac{(n^2-c^2)}{n(n-2c)}x^2+\dfrac{2n(n-c)-2\alpha c(2n-c)}{n(n-2c)(n+\beta)}x+\dfrac{\alpha^2c(2n-c)-2\alpha n(n-c)}{n(n-2c)(n+\beta)^2}.$

Consequently, for each $x\in I_n$, we have the following equalities

$$\overline{G}_{n,c}^{*(\alpha,\beta)}(t-x;x)=0$$

$$\begin{aligned}\overline{G}_{n,c}^{*(\alpha,\beta)}((t-x)^2;x) &= \frac{c(2n-c)}{n(n-2c)}x^2+\frac{2n(n-c)-2c\alpha(2n-c)}{n(n-2c)(n+\beta)}x\\ &\quad+\frac{\alpha^2c(2n-c)-2\alpha n(n-c)}{n(n-2c)(n+\beta)^2}\\ &= \zeta_{n,c}^{(\alpha,\beta)}(x), \textit{(say)}. \qquad (19)\end{aligned}$$

Theorem 14. *Let $f\in C_B(I_n)$ and $x\in I_n$. Then for $n>2c$, there exists a positive constant C' such that*

$$|\overline{G}_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)|\leq C'\omega_2\left(f,\sqrt{\zeta_{n,c}^{(\alpha,\beta)}(x)}\right),$$

where $\zeta_{n,c}^{(\alpha,\beta)}(x)$ is given by (19).

Proof. Let $g\in W^2$ and $x,t\in I_n$. Using the Taylor's expansion we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-v)g''(v)dv.$$

Applying $\overline{G}_{n,c}^{*(\alpha,\beta)}$ on both sides and using Lemma 13, we get

$$\overline{G}_{n,c}^{*(\alpha,\beta)}(g;x)-g(x) = \overline{G}_{n,c}^{*(\alpha,\beta)}\left(\int_x^t(t-v)g''(v)dv;x\right).$$

Obviously, we have $\left|\int_x^t(t-v)g''(v)dv\right|\leq(t-x)^2\|g''\|$. Therefore

$$|\,\overline{G}_{n,c}^{*(\alpha,\beta)}(g;x)-g(x)\,|\leq\overline{G}_{n,c}^{*(\alpha,\beta)}((t-x)^2;x)\parallel g''\parallel=\zeta_{n,c}^{(\alpha,\beta)}(x)\parallel g''\parallel.$$

Since $|\,\overline{G}_{n,c}^{*(\alpha,\beta)}(f;x)\,|\leq\|f\|$, we get

$$\begin{aligned}|\,\overline{G}_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)\,| &\leq |\,\overline{G}_{n,c}^{*(\alpha,\beta)}(f-g;x)\,|+|\,(f-g)(x)\,|+|\,\overline{G}_{n,c}^{*(\alpha,\beta)}(g;x)-g(x)\,|\\ &\leq 2\|f-g\|+\zeta_{n,c}^{(\alpha,\beta)}(x)\|g''\|.\end{aligned}$$

Finally, taking the infimum over all $g\in W^2$ and using (11) we obtain

$$|\,\overline{G}_{n,c}^{*(\alpha,\beta)}(f;x)-f(x)\,|\leq C'\omega_2\left(f,\sqrt{\zeta_{n,c}^{(\alpha,\beta)}(x)}\right),$$

which proves the theorem.

Theorem 15. *Let $f \in C_B(I_n)$. If f', f'' exists at a fixed point $x \in I_n$, then we have*

$$\lim_{n\to\infty} n\left(\overline{G}^{*(\alpha,\beta)}_{n,c}(f;x) - f(x)\right) = x(1+cx)f''(x).$$

The proof follows along the lines of Theorem 4.

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