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On a modification of Dunkl generalization of Szász Operators via q-calculus

Vishnu Narayan Mishra[1,*], Shikha Pandey[2], Idrees A. Khan[3]

[1] *Department of Mathematics, Indira Gandhi National Tribal University, Amarkantak, Madhya Pradesh, India.*
[2] *Department of Applied Mathematics & Humanities, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat -395 007 (Gujarat), India.*
[3] *Department of Mathematics, Integral University, Lucknow-226026, Uttar Pradesh, India.*

Abstract. Theory of approximation is a very extensive field and study of approximation via q-calculus and (p, q)- calculus is of great mathematical interest with great practical importance. Positive approximation processes play an important role in approximation theory and appear in a very natural way dealing with approximation of continuous functions, especially one, which requires further qualitative properties such as monotonicity, convexity and shape preservation and so on. This paper deals with the q- form of Dunkl generalization of Szász - Beta type operators. Estimation of their moments and establishing basic approximation results which comprise weighted approximation and direct estimates in view of modulus of continuity is the aim of this paper.

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1. Introduction and Preliminaries

Approximation theory is the branch of mathematics where the focus of study is to work out a complicated function by easier to compute functions. In 1885, Weierstrass firstly obtained a significant result, which established the fact that the set of algebraic polynomials in the class of continuous real valued functions on a closed interval is dense. Weierstrass's theorem has encouraged mathematicians over the years to give too much of their attention to pathological functions with a little bit of smoothness. This theorem was proved by various mathematicians such as Picard, Fejer, Landau and de la Vallee Poussin using singular integrals.
Bernstein [5] gave the most effective proof using probabilistic method. In 1950, Szász

*Corresponding author.

Email addresses: vishnunarayanmishra@gmail.com, vishnu_narayanmishra@yahoo.co.in (V.N. Mishra), sp1486@gmail.com (S. Pandey), idrees_maths@yahoo.com (I.A. Khan)

[21] proved that for a continuous function f defined in positive semi-axis, the following polynomial sequence converges to $f(x)$,

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right). \tag{1}$$

Thereafter mathematicians have introduced various operators which gives better approximation to continuous functions [See [2], [3],[17],[24], [25], [26] etc.]
In 20^{th} century, the study of quantum calculus began when Jackson [11] defined the q-integral in a systematic way. Later on De Sole and Kac [19] presented the integral representations of q-gamma and q-beta functions. q-calculus has important applications in number theory, combinatorics, orthogonal polynomials, hypergeometric functions, mechanics, the theory of relativity and quantum theory. In approximation theory, application of q-calculus finds its way when Phillips [15] proposed q-Bernstein polynomials. Thereafter various mathematicians studied q-analogue of various operators. Different q-generalizations of Szász-Mirakjan operators were introduced and studied by Aral [4], Radu [16] and Mahmudov [13] for $0 < q \leq 1$, [14] for $q > 1$.
As a summation integral type of modification of q-Szász-Mirakyan operators, Gupta and Mahmudov [7] presented q-Szász-beta operators for $0 < q \leq 1$, $f \in C[0,\infty)$ as

$$\mathcal{B}_{n,q}(f;x) = e^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{[k]_q!}\int_0^{\infty/A}\frac{q^{k^2}t^k}{B_q(k+1,n)(1+t)_q^{n+k+1}}f(t)d_qt, \qquad A>0,\ x\in[0,\infty). \tag{2}$$

Adell et al. [10] had shown that linear positive operators having beta type probability distributions preserves shape properties (monotonocity and convexity), likewise other positive linear operator, such as Bernstein, Szász and Baskakov operators.
For polynomial approximation, Hermite polynomials forms a family of orthogonal polynomial sequence which is complete in the space of all polynomials. The generalized Hermite polynomials were defined by G. Szëgo in [[22], p380, Problem 25] as being orthogonal polynomials with respect to weight function $|x|^{2\mu}e^{-x^2}$, $\mu > -1/2$ in $(-\infty,\infty)$. In [18], M. Rosenblum has given the definition of generalized Hermite polynomial as,
let H_n^μ be the generalized Hermite polynomial of degree n, then for even values of n,

$$H_{2m}^{\mu} = (-1)^m(2m)!\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(m+\mu+\frac{1}{2})}L_m^{\mu-\frac{1}{2}}(x^2) \tag{3}$$

and for odd values of n,

$$H_{2m+1}^{\mu} = (-1)^m(2m+1)!\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(m+\mu+\frac{3}{2})}xL_m^{\mu+\frac{1}{2}}(x^2), \tag{4}$$

where L_m^γ is the $\gamma-$Laguerre polynomial of degree m. The generalized Hermite polynomials $\{H_n^\mu\}$ have a generating function (2.5.8) of [18] which involves the generalized exponential function e_μ defined by

$$e_\mu(z) = \sum_{m=0}^{\infty} \frac{z^m}{\gamma_\mu(m)}, \tag{5}$$

where $\gamma_\mu(m)$ is a generalized factorial defined as

$$\gamma_\mu(2m) = \frac{2^{2m} m! \Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} = (2m)! \frac{\Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})},$$
$$\gamma_\mu(2m+1) = \frac{2^{2m+1} m! \Gamma(m+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})} = (2m+1)! \frac{\Gamma(m+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{3}{2})}.$$

A recurrence relation holds for γ_μ,

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \qquad k \in \mathbb{N}_0,$$

where $\theta_k = \begin{cases} 0, & \text{if } k \in 2\mathbb{N} \\ 1, & \text{if } k \in 2\mathbb{N}+1 \end{cases}$.

It is apparent that $e_0(x) = e^x$ and e_μ is an entire function. The μ-binomial coefficient and μ-binomial expansion is also defined in [18] as

$$\begin{pmatrix} n \\ k \end{pmatrix}_\mu = \frac{\gamma_\mu(n)}{\gamma_\mu(k)\gamma_\mu(n-k)}, \qquad (x+y)_\mu^n = \sum_{j=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}_\mu x^j y^{n-j}. \tag{6}$$

Using the generalized exponential function, Sucu [20] defined a Dunkl analogue of Szász operators as follows

$$\mathcal{S}_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \tag{7}$$

where $\mu \geq 0,\ n \in \mathbb{N},\ x \geq 0,\ f \in C[0,\infty)$.

Since then Dunkl analogue of various operators has been studied [See [23],[8],[9] etc.]. G. İçöz and B. Çekim [8] presented the Dunkl generalization of Szász operators via q-calculus as

$$\mathcal{D}_{n,q}(f;x) := \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{[k+2\mu\theta_k]_q}{[n]_q}\right), \tag{8}$$

where $\mu > \frac{1}{2},\ n \in \mathbb{N},\ x \geq 0,\ 0 < q < 1,\ f \in C[0,\infty)$,

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)} \qquad \text{and} \qquad \gamma_{\mu,q}(n+1) = [n+1+2\mu\theta_{n+1}]_q \gamma_{\mu,q}(n).$$

Cheikh et al. [27] stated the definition of q-Dunkl analogue of exponential function as

$$E_{\mu,q}(x)=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}x^n}{\gamma_{\mu,q}(n)},$$

and explicit formula for $\gamma_{\mu,q}(n)$ is

$$\gamma_{\mu,q}(n)=\frac{(q^{2\mu+1},q^2)_{[\frac{n+1}{2}]}(q^2,q^2)_{[\frac{n}{2}]}}{(1-q)^n},$$

where $(a,q)_0=1,\ (a,q)_n:=\prod_{k=0}^{n-1}(1-aq^k)$.
Using the definitions of μ-binomial coefficient and $\mu-$binomial expansion, we get $q-$analogue of equation (6)

$$\begin{pmatrix} n \\ k \end{pmatrix}_{\mu,q}=\frac{\gamma_{\mu,q}(n)}{\gamma_{\mu,q}(k)\gamma_{\mu,q}(n-k)}, \qquad (x+y)^n_{\mu,q}=\sum_{j=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}_{\mu,q}x^jy^{n-j}.$$

Thus the first few μ-binomial polynomials are $1,\ x+y,\ x^2+\frac{[2]_q}{[2\mu+1]_q}xy+y^2,\ x^3+\frac{[3+2\mu]_q}{[1+2\mu]_q}(x^2y+xy^2)+y^3,\ x^4+4\frac{1}{[1+2\mu]_q}(x^3y+xy^3)+y^4$.
Furthermore, q-analogue of μ-beta and μ-gamma functions are defined as,

$$B_{\mu,q}(m,n)=\frac{\gamma_{\mu,q}(m-1)\gamma_{\mu,q}(n-1)}{\gamma_{\mu,q}(m+n-1)},$$

$$\Gamma_{\mu,q}(t)=\int_0^{\infty}x^{t-1}E_{\mu,q}(-qx)d_qx, \qquad t>0.$$

Now in this paper, we propose the Dunkl generalization of Szász-Beta operators via q-calculus as

$$D_{n,q}(f;x):=\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_0^{\infty/A}\frac{q^{k^2}t^k}{B_{\mu,q}(k+1,n)(1+t)^{n+k+1}_{\mu,q}}f(t)d_qt, \tag{9}$$

where $A>0,\ 0<q<1,\ f\in C[0,\infty)$.

2. Approximation Properties

In this section we analyze the convergence behaviour of the operators $D_{n,q}(f;x)$ via universal Korovkin theorem and weighted approximation theorem as in [6]. Consider the notation $F_m^{q,\mu}(n)=\prod_{i=1}^{m}[n-i+2\mu\theta_{n-i}]_q$.

Lemma 1. *The operators $D_{n,q}$ given by* (9) *satisfies the following*

$$D_{n,q}(1;x)=1, \tag{10}$$

$$D_{n,q}(t;x) = \frac{1}{F_1^{q,\mu}(n)}\left[\frac{[n]_q x}{q^2} + \frac{Cosh([n]_q x) + q^{2\mu}Sinh([n]_q x)}{qe_{\mu,q}([n]_q x)} + [2\mu]_q \frac{E_{\mu,q}(-[n]_q x)}{e_{\mu,q}([n]_q x)}\right], \tag{11}$$

$$\begin{aligned} D_{n,q}(t^2;x) = \frac{1}{F_2^{q,\mu}(n)}\Bigg[&\frac{([n]_q x)^2}{q^6} + \frac{[2]_q [n]_q x}{q^5 \cdot e_{\mu,q}([n]_q x)}\left\{(q+q^{2\mu})Sinh([n]_q x) + (1+q^{2\mu+1})Cosh([n]_q x)\right\} \\ &+\frac{[2]_q}{q^3}\frac{Cosh([n]_q x) + q^{4\mu}Sinh([n]_q x)}{e_{\mu,q}([n]_q x)} + \frac{[2]_q[2\mu]_q}{q^2}\frac{Cosh([n]_q x) - q^{2\mu}Sinh([n]_q x)}{e_{\mu,q}([n]_q x)}\Bigg], \end{aligned} \tag{12}$$

$$\begin{aligned} D_{n,q}((t-x)^2;x) = {}& x^2\left[1 - \frac{2[n]_q}{q^2 F_1^{q,\mu}(n)} + \frac{[n]_q^2}{q^6 F_2^{q,\mu}(n)}\right] \\ &+x\Bigg[\frac{[2]_q[n]_q}{q^5 F_2^{q,\mu}(n)}\frac{(q+q^{2\mu})Sinh([n]_q x) + (1+q^{2\mu+1})Cosh([n]_q x)}{e_{\mu,q}([n]_q x)} \\ &-2\frac{Cosh([n]_q x) + q^{2\mu}Sinh([n]_q x)}{qe_{\mu,q}([n]_q x)F_1^{q,\mu}(n)} - \frac{2[2\mu]_q}{F_1^{q,\mu}(n)}\frac{E_{\mu,q}(-[n]_q x)}{e_{\mu,q}([n]_q x)}\Bigg] + [2]_q\frac{Cosh([n]_q x) + q^{4\mu}Sinh([n]_q x)}{q^3 F_2^{q,\mu}(n) e_{\mu,q}([n]_q x)} \\ &+\frac{[2]_q[2\mu]_q}{q^2 F_2^{q,\mu}(n)}\frac{Cosh([n]_q x) - q^{2\mu}Sinh([n]_q x)}{e_{\mu,q}([n]_q x)}. \end{aligned} \tag{13}$$

Proof. Using the definition of generalised exponential function in the q-Gamma and q-Beta functions in [19], we can obtain the following important equality:

$$q^{k^2}\int_0^{\infty/A} \frac{t^{k+m}}{B_{\mu,q}(k+1,n)(1+t)_{\mu,q}^{n+k+1}} d_q t = \frac{\gamma_{\mu,q}(m+k)\gamma_{\mu,q}(n-m-1)q^{[2k^2-(k+m)(k+m+1)]/2}}{\gamma_{\mu,q}(k)\gamma_{\mu,q}(n-1)}. \tag{14}$$

For $f(t) = 1$, using (14) with $m = 0$, we obtain

$$\begin{aligned} D_{n,q}(1;x) &= \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\int_0^{\infty/A}\frac{q^{k^2}t^k}{B_{\mu,q}(k+1,n)(1+t)_{\mu,q}^{n+k+1}}d_q t \\ &= \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}q^{k(k-1)/2} \\ &= \frac{1}{e_{\mu,q}([n]_q x)}E_{\mu,q}([n]_q x) \\ &= 1. \end{aligned}$$

Next for $f(t) = t$, using (14) with $m = 1$ and $\theta_{k+1} = \theta_k + (-1)^k$, $[n]_q = [s]_q + q^s[n-s]_q$, $0 \le s \le n$, we obtain

$$D_{n,q}(t;x) = \frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}\int_0^{\infty/A}\frac{q^{k^2}t^{k+1}}{B_{\mu,q}(k+1,n)(1+t)_{\mu,q}^{n+k+1}}d_q t$$

$$= \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \frac{\gamma_{\mu,q}(k+1)\gamma_{\mu,q}(n-2)q^{(k^2-3k-2)/2}}{\gamma_{\mu,q}(k)\gamma_{\mu,q}(n-1)}$$

$$= \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \frac{[k+2\mu\theta_k+1+2\mu(-1)^k]_q q^{(k^2-3k-2)/2}}{[n-1+2\mu\theta_{n-1}]_q}$$

$$= \frac{e^{-1}_{\mu,q}([n]_q x)}{F_1^{q,\mu}(n)} \sum_{k=1}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} \frac{([k+2\mu\theta_k]_q + q^{k+2\mu\theta_k}[1+2\mu(-1)^k]_q)}{[k+2\mu\theta_k]_q} q^{(k^2-3k-2)/2}$$

$$= \frac{e^{-1}_{\mu,q}([n]_q x)}{F_1^{q,\mu}(n)} \left[\frac{[n]_q x}{q^2} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{k(k-1)/2} + \frac{1}{q} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} q^{k(k-1)/2} q^{2\mu\theta_k} \left\{ 1 + q[2\mu]_q(-1)^k \right\} \right]$$

$$= \frac{1}{F_1^{q,\mu}(n)} \left[\frac{[n]_q x}{q^2} + \frac{Cosh([n]_q x) + q^{2\mu} Sinh([n]_q x)}{q e_{\mu,q}([n]_q x)} + [2\mu]_q \frac{E_{\mu,q}(-[n]_q x)}{e_{\mu,q}([n]_q x)} \right].$$

Next for $f(t) = t^2$, using (14) with $m = 2$ and $\theta_{k+2} = \theta_k$ we obtain

$$D_{n,q}(t^2; x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_0^{\infty/A} \frac{q^{k^2} t^{k+2}}{B_{\mu,q}(k+1,n)(1+t)^{n+k+1}_{\mu,q}} d_q t$$

$$= \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \frac{\gamma_{\mu,q}(k+2)\gamma_{\mu,q}(n-3)q^{(k^2-5k-6)/2}}{\gamma_{\mu,q}(k)\gamma_{\mu,q}(n-1)}$$

$$= \frac{e^{-1}_{\mu,q}([n]_q x)}{F_2^{q,\mu}(n)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} \frac{([k+2+2\mu\theta_{k+2}]_q [k+1+2\mu\theta_{k+1}]_q)}{[k+2\mu\theta_k]_q} q^{(k^2-5k-6)/2}$$

$$= \frac{e^{-1}_{\mu,q}([n]_q x)}{F_2^{q,\mu}(n)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k-1)} \frac{([k+2\mu\theta_k]_q + q^{k+2\mu\theta_k}[2]_q)([k+1+2\mu\theta_{k+1}]_q)}{[k+2\mu\theta_k]_q} q^{\frac{(k^2-5k-6)}{2}}$$

$$= \frac{1}{F_2^{q,\mu}(n)} \left[\frac{([n]_q x)^2}{q^6} + \frac{[2]_q [n]_q x}{q^5 \cdot e_{\mu,q}([n]_q x)} \left\{ (q+q^{2\mu}) Sinh([n]_q x) + (1+q^{2\mu+1}) Cosh([n]_q x) \right\} \right.$$
$$\left. + \frac{[2]_q}{q^3} \frac{Cosh([n]_q x) + q^{4\mu} Sinh([n]_q x)}{e_{\mu,q}([n]_q x)} + \frac{[2]_q [2\mu]_q}{q^2} \frac{Cosh([n]_q x) - q^{2\mu} Sinh([n]_q x)}{e_{\mu,q}([n]_q x)} \right].$$

Using the above results, we can get

$$D_{n,q}((t-x)^2; x) = D_{n,q}(t^2; x) - 2x D_{n,q}(t; x) + x^2 D_{n,q}(1; x)$$

$$= \frac{1}{F_2^{q,\mu}(n)} \left[\frac{([n]_q x)^2}{q^6} + \frac{[2]_q [n]_q x}{q^5 \cdot e_{\mu,q}([n]_q x)} \left\{ (q+q^{2\mu}) Sinh([n]_q x) + (1+q^{2\mu+1}) Cosh([n]_q x) \right\} \right.$$
$$\left. + \frac{[2]_q}{q^3} \frac{Cosh([n]_q x) + q^{4\mu} Sinh([n]_q x)}{e_{\mu,q}([n]_q x)} + \frac{[2]_q [2\mu]_q}{q^2} \frac{Cosh([n]_q x) - q^{2\mu} Sinh([n]_q x)}{e_{\mu,q}([n]_q x)} \right]$$
$$- \frac{1}{F_1^{q,\mu}(n)} \left[\frac{2[n]_q x^2}{q^2} + +2x \frac{Cosh([n]_q x) + q^{2\mu} Sinh([n]_q x)}{q e_{\mu,q}([n]_q x)} + [2\mu]_q 2x \frac{E_{\mu,q}(-[n]_q x)}{e_{\mu,q}([n]_q x)} \right] + x^2$$

$$= x^2\left[1-\frac{2[n]_q}{q^2F_1^{q,\mu}(n)}+\frac{[n]_q^2}{q^6F_2^{q,\mu}(n)}\right]+x\left[\frac{[2]_q[n]_q}{q^5F_2^{q,\mu}(n)}\frac{(q+q^{2\mu})Sinh([n]_qx)+(1+q^{2\mu+1})Cosh([n]_qx)}{e_{\mu,q}([n]_qx)}\right.$$
$$\left.-2\frac{Cosh([n]_qx)+q^{2\mu}Sinh([n]_qx)}{qe_{\mu,q}([n]_qx)F_1^{q,\mu}(n)}-\frac{2[2\mu]_q}{F_1^{q,\mu}(n)}\frac{E_{\mu,q}(-[n]_qx)}{e_{\mu,q}([n]_qx)}\right]+[2]_q\frac{Cosh([n]_qx)+q^{4\mu}Sinh([n]_qx)}{q^3F_2^{q,\mu}(n)e_{\mu,q}([n]_qx)}$$
$$+\frac{[2]_q[2\mu]_q}{q^2F_2^{q,\mu}(n)}\frac{Cosh([n]_qx)-q^{2\mu}Sinh([n]_qx)}{e_{\mu,q}([n]_qx)}.$$

Theorem 1. *Let $D_{n,q}$ be the operators given by* (9). *Then for any $f\in C[0,\infty)\cap E$, the following relation:*

$$\lim_{n\to\infty}D_{n,q}(f;x)=f(x)$$

holds uniformly on each compact subset of $[0,\infty)$, *where* $E:=\left\{f: x\in[0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x\to\infty\right\}$.

Proof. The proof is based on the well-known universal Korovkin-type theorem (see details in [1],[12]). By taking into account the Korovkin's theorem, it is sufficient to show that

$$\lim_{n\to\infty}\|D_{n,q}(t^i;x)-t^i\|=0,\quad i=0,1,2.$$

Let (q_n) denote a sequence such that $0<q_n\le 1$. Since for fixed q with $0<q\le 1$, $\lim_{n\to\infty}[n]_q=1$, to ensure the convergence properties we will assume $q=q_n$ as a sequence such that $\lim_{n\to\infty}q_n=1$, and $\lim_{n\to\infty}q_n^n=c$ where $c\in(0,1)$. Therefore, we guarantee that $\lim_{n\to\infty}\frac{1}{[n]_{q_n}}=0$. For example, if we choose

$$(q_n)=(1-\frac{1}{n})$$

then $\lim_{n\to\infty}q_n^n=e^{-1}$. Hence, we obtain

$$\lim_{n\to\infty}\frac{1}{[n]_{q_n}}=0.$$

Besides, the other way is to take the sequence $q_n\in(0,1)$ such that $\lim_{n\to\infty}q_n=1$. Thus, $\lim_{n\to\infty}\frac{1}{[n]_{q_n}}=0$.
Taking $q=(q_n)$ as above we prove the following results.
Using Lemma 1, result for $i=0$ is trivial. For $i=1$ result can be obtained as

$$\lim_{n\to\infty}\|D_{n,q}(t;x)-x\|=\lim_{n\to\infty}\left\|\left(\frac{[n]_q}{q^2[n-1+2\mu\theta_{n-1}]_q}-1\right)x+\frac{Cosh([n]_qx)+q^{2\mu}Sinh([n]_qx)}{qF_1^{q,\mu}(n)e_{\mu,q}([n]_qx)}\right.$$
$$\left.+\frac{[2\mu]_q}{F_1^{q,\mu}(n)}\frac{E_{\mu,q}(-[n]_qx)}{e_{\mu,q}([n]_qx)}\right\|=0$$

For $i = 2$,

$$\lim_{n\to\infty}\|D_{n,q}(t^2;x)-x^2\| = \lim_{n\to\infty}\left\|\left(\frac{([n]_q)^2}{q^6[n-1+2\mu\theta_{n-1}]_q[n-2+2\mu\theta_{n-2}]_q}-1\right)x^2\right.$$
$$+\frac{1}{F_2^{q,\mu}(n)}\left[\frac{[2]_q[n]_qx}{q^5\cdot e_{\mu,q}([n]_qx)}\left\{(q+q^{2\mu})Sinh([n]_qx)+(1+q^{2\mu+1})Cosh([n]_qx)\right\}\right.$$
$$\left.\left.+\frac{[2]_q}{q^3}\frac{Cosh([n]_qx)+q^{4\mu}Sinh([n]_qx)}{e_{\mu,q}([n]_qx)}+\frac{[2]_q[2\mu]_q}{q^2}\frac{Cosh([n]_qx)-q^{2\mu}Sinh([n]_qx)}{e_{\mu,q}([n]_qx)}\right]\right\|$$
$$=0.$$

One can easily get the limit using the fact that as $n\to\infty$ we have $\frac{1}{[n]_{q_n}}\to 0$, $q\to 1$, $(Sinh([n]_qx)+Cosh([n]_qx)) = e_{\mu,q}([n]_qx)$ and $e_{\mu,q}(-[n]_qx)\to 0$.
Thus using Korovkin's result we can conclude that

$$\lim_{n\to\infty}\|D_{n,q}(f(t);x)-f(x)\| = 0.$$

Recalling the weighted spaces of the functions which are defined on the positive semi-axis $\mathbb{R}^+ = [0,\infty)$ as follows:

$$\mathcal{B}_\omega(\mathbb{R}^+) = \{f : |f(x)| \le \mathcal{M}_f\omega(x)\},$$
$$\mathcal{C}_\omega(\mathbb{R}^+) = \left\{f : f\in\mathcal{B}_\omega(\mathbb{R}^+)\cap C[0,\infty)\right\},$$
$$\mathcal{C}^k_\omega(\mathbb{R}^+) = \left\{f : f\in\mathcal{C}_\omega(\mathbb{R}^+) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\omega(x)} = k \text{ (k is a constant)}\right\},$$

where $\omega(x) = 1+x^2$ is a weight function and $\mathcal{M}_f$ is a constant depending only on f. One can observe that $\mathcal{C}_\omega(\mathbb{R}^+)$ is a normed space with norm defined as, $\|f\|_\omega := \sup_{x\ge 0}\frac{|f(x)|}{\omega(x)}$.

Theorem 2. *Let $D_{n,q}$ be the operators given by* (9)*. Then for any $f\in\mathcal{C}^k_\omega(\mathbb{R}^+)$, we have:*

$$\lim_{n\to\infty}\|D_{n,q}(f;x)-f(x)\|_\omega = 0.$$

Proof. Using Lemma 1, one can easily prove the theorem.

3. Main results

Here, we give the rate of convergence of the operators with the help of the usual and second order modulus of continuity and Lipschitz class functions.
Lipschitz class of order $\alpha, Lip_M(\alpha)$ $(0<\alpha\le 1,\ M>0)$, is defined as follows

$$Lip_M(\alpha) := \{f : |f(x)-f(y)| \le M|x-y|^\alpha, x,y\in[0,\infty)\}.$$

Theorem 3. *Let $f \in Lip_M(\alpha)$, then*

$$|D_{n,q}(f;x) - f(x)| \leq M(\delta_n(x))^{\alpha/2},$$

where $\delta_n(x) = D_{n,q}((t-x)^2;x)$.

Proof. For $f \in Lip_M(\alpha)$ and linearity behaviour of $D_{n,q}$,

$$\begin{aligned} |D_{n,q}(f;x) - f(x)| &\leq D_{n,q}(|f(t) - f(x)|;x) \\ &\leq MD_{n,q}(|t-x|^{\alpha};x). \end{aligned}$$

Using Hölder inequality for integral and then for sum with $p = \alpha/2$ and $q = 1 - \frac{\alpha}{2}$, we have

$$|D_{n,q}(f;x) - f(x)| \leq M\left(D_{n,q}((t-x)^2;x)\right)^{\alpha/2}.$$

Choosing $\delta_n(x) = D_{n,q}((t-x)^2;x)$, then we get the desired result.

Theorem 4. *Consider $\check{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$. Let $f \in \check{C}[0,\infty) \cap E, D_{n,q}$ operators verify the following*

$$|D_{n,q}(f;x) - f(x)| \leq (1+\sqrt{\mu}) \;\; \omega\left(f; \frac{1}{F_2^{q,\mu}(n)}\right).$$

Proof.

$$\begin{aligned} |D_{n,q}(f;x) - f(x)| &\leq D_{n,q}(|f(t) - f(x)|;x) \\ &\leq \left(1 + \frac{1}{\delta}D_{n,q}(|t-x|;x)\right)\omega(f;\delta) \\ &\leq \left(1 + \frac{1}{\delta}\sqrt{D_{n,q}((t-x)^2;x)}\right)\omega(f;\delta). \end{aligned}$$

Choosing $\delta = \frac{1}{F_2^{q,\mu}(n)}$ and $\mu = \frac{1}{\delta^2}D_{n,q}((t-x)^2;x)$, we can obtain the desired result.

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