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The r-Dowling Numbers and Matrices Containing r-Whitney Numbers of the Second Kind and Lah Numbers

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Abstract. This paper derives three forms of explicit formula for r-Dowling numbers. One of these is expressed in terms of exponential polynomial. The other two formulas are derived using an inverse relation and Faa di Bruno's formula together with certain identity of Bell polynomials of the second kind. These two formulas are expressed in terms of the r-Whitney numbers of the second kind, r-Whitney-Lah numbers, and the ordinary Lah numbers. As a consequence, a relation between r-Dowling numbers and the sums of row entries of the product of matrices containing the r-Whitney numbers of the second kind, r-Whitney-Lah numbers, and the ordinary Lah numbers is established. Moreover, a q-analogue of the explicit formula is obtained.

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1. Introduction

The Bell numbers, denoted by B_n, were defined in [5] as the sum of Stirling numbers of the second kind

$$B_n := \sum_{k=0}^{n} S(n,k). \tag{1}$$

Since the numbers $S(n,k)$ are interpreted as the number of ways to partition an n-set into k nonempty subsets, B_n can then be interpreted as the total number of ways to partition an n-set. Several properties and application were obtained for these numbers including

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generating functions, recursive formulas, explicit formula, and expression in terms of a moment of the Poisson random variable [19, 20].

By adding one parameter r, A.Z. Broder [3] defined combinatorially a generalization of $S(n,k)$, the r-Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}_r$, as follows:

$\left\{ {n \atop k} \right\}_r$:= the number of partitions of an n-set into k nonempty subsets such that the numbers $1, 2, \ldots, r$ are in distinct subsets.

These numbers possessed several properties parallel to those of the classical Stirling numbers of the second kind, which can be found in [3]. In the same paper [3] , Broder was able to derive a relation expressing $\left\{ {n \atop k} \right\}_r$ in terms of the classical Stirling numbers of the second kind:

$$\left\{ {n \atop k} \right\}_r = \sum_{j=k}^{n} \binom{n}{j} S(j,k) r^{n-j}. \tag{2}$$

Letting $r = 0$, equation (2) gives $\left\{ {n \atop k} \right\}_0 = S(n,k)$ with 0^0 defined to be 1. Parallel to the definition of Bell numbers in (1), Mezo [17] defined the r-Bell numbers as

$$B_{n,r} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r. \tag{3}$$

Mezo [17] obtained several interesting properties for these numbers analogous to those of the classical Bell numbers. It is worth mentioning that r-Bell numbers were first introduced by C.B. Corcino in [6].

Furthermore, by adding one more parameter m, Mező [16] defined the r-Whitney numbers of the first and second kind, denoted by $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$, as coefficients of the following expansions

$$m^n (x)_n = \sum_{k=0}^{n} (-1)^{n-k} w_{m,r}(n,k)(mx+r)^k, \tag{4}$$

$$\text{and } (mx+r)^n = \sum_{k=0}^{n} W(n,k) m^k (x)_k, \tag{5}$$

where $(x)_k = x(x-1)\ldots(x-k+1)$ if $k \geq 1$, with $(x)_0 = 1$. Below are the few values of $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ with $m = r = 2$:

n/k	0	1	2	3	4
0	1				
1	2	1			
2	8	6	1		
3	48	44	12	1	
4	384	400	140	20	1

n/k	0	1	2	3	4
0	1				
1	2	1			
2	4	6	1		
3	8	28	12	1	
4	16	120	100	20	1

Table 1: Few values of $w_{2,2}(n,k)$ Table 2: Few values of $W_{2,2}(n,k)$

It would be interesting to note that the numbers $W_{m,r}(n,k)$ are equivalent to the (r,β)-Stirling numbers [7] and the numbers $w_{m,r}(n,k)$ are equivalent to the numbers that appeared in [10]. One can easily verify that these numbers satisfy the following inverse relation

$$f_n = \sum_{j=0}^{n}(-1)^{n-j} w_{\beta,r}(n,j) g_j \Longleftrightarrow g_n = \sum_{j=0}^{n} W_{\beta,r}(n,j) f_j. \tag{6}$$

Analogous to (2), Cheon and Jung [4] expressed the r-Whitney numbers of the second kind $W_{m,r}(n,k)$ in terms of the classical Stirling numbers of the second kind $S(n,k)$ as

$$W_{m,r}(n,k) = \sum_{i=k}^{n} \binom{n}{i} m^{i-k} r^{n-i} S(i,k). \tag{7}$$

Replacing m by β, k by j and r by $-r$ in equation (7), yield

$$W_{\beta,-r}(n,j) = \sum_{k=j}^{n} \binom{n}{k} \beta^{k-j} (-r)^{n-k} S(k,j). \tag{8}$$

Moreover, Cheon and Jung [4] defined the r-Dowling polynomials, denoted by $D_{m,r}(n,x)$, as follows

$$D_{m,r}(n,x) = \sum_{k=0}^{n} W_{m,r}(n,k) x^k. \tag{9}$$

Taking $x = 1$, equation (9) reduces to

$$D_{m,r}(n,1) = \sum_{k=0}^{n} W_{m,r}(n,k),$$

the r-Dowling numbers. These numbers are equivalent to the (r,β)-Bell numbers in [8], denoted by $G_{n,\beta,r}$, and have also been considered in the paper [12] using the same notation $G_{n,\beta,r}$. Throughout this paper, we use $G_{n,\beta,r}$ to denote the r-Dowling numbers. It is worth mentioning that $G_{n,\beta,r}$ satisfy the following generating function

$$\sum_{n\geq 0} G_{n,\beta,r} \frac{t^n}{n!} = e^{rt} e^{\frac{1}{\beta}(e^{\beta t}-1)}. \tag{10}$$

The Lah numbers, denoted by $L(n,k)$, were defined in [5], combinatorially, as the number of ways to partition an n-set into k nonempty linearly ordered subsets. These numbers have been shown to satisfy the following relations

$$L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!} \tag{11}$$

$$L(n,k)=\sum_{j=k}^{n}s(n,j)S(j,k). \tag{12}$$

On the other hand, the r-Whitney-Lah numbers, denoted by $L_{m,r}(n,k)$, were defined by Cheon and Jung [4] parallel to (12) as follows

$$L_{m,r}(n,k)=\sum_{j=k}^{n}w_{m,r}(n,j)W_{m,r}(j,k). \tag{13}$$

Several properties of $L_{m,r}(n,k)$ have been derived through factorization of the r-Whitney-Lah matrix $[L_{m,r}(n,k)]_{n,k\geq 0}$ (see [4]) including the triangular relation

$$L_{m,r}(n,k)=L_{m,r}(n-1,k-1)+(2r+(n+k-1)m)L_{m,r}(n-1,k)$$

Below is a triangular array of values for $L_{m,r}(n,k)$ with $m=r=2$:

n/k	0	1	2	3	4
0	1				
1	4	1			
2	24	12	1		
3	192	144	24	1	
4	1920	1920	480	40	1

Table 3: Few values of $L_{2,2}(n,k)$.

In this paper, two explicit formulas for $G_{n,\beta,r}$ are derived using the two methods applied by Feng Qi [21] in expressing the Bell numbers in terms of Stirling numbers of the second kind and Lah numbers. The two methods yield exactly the same explicit formula when they are applied by Feng Qi to Bell numbers. However, when these methods are applied here to $G_{n,\beta,r}$, they give two equivalent formulas of different forms. These formulas imply two matrix relations involving r-Dowling numbers, r-Whitney numbers of the second, r-Whitney-Lah numbers and Lah numbers.

2. Expression in Terms of Exponential Polynomials

The exponential polynomial [2], denoted by $\Phi_n(x)$, appeared in the resulting expression in applying Mellin derivative $\left(x\frac{d}{dx}\right)^n$ to the function e^x. The notation for Mellin derivative would mean that the differential operator $x\frac{d}{dx}$ is applied n times to e^x. The first two applications of the operator give

$$x\frac{d}{dx}e^x = xe^x$$
$$\left(x\frac{d}{dx}\right)^2 e^x = \left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}e^x\right)$$

$$= x\frac{d}{dx}(xe^x) = \left(x^2+x\right)e^x.$$

Continuing in this manner yields

$$\left(x\frac{d}{dx}\right)^n e^x = \Phi_n(x)e^x.$$

The exponential polynomial satisfies the following generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty}\Phi_n(x)\frac{t^n}{n!}, \tag{14}$$

which can be expressed in polynomial form as

$$\Phi_n(x) = \sum_{k=0}^{n} S(n,k)x^k, \tag{15}$$

whose coefficients are the Stirling numbers of the second kind. Note that, when $x = 1/\beta$ and $t = \beta t$, (16) reduces to

$$e^{\frac{1}{\beta}(e^t-1)} = \sum_{n=0}^{\infty}\Phi_n\left(1/\beta\right)\frac{(\beta t)^n}{n!},\ \beta \neq 0. \tag{16}$$

Hence, the exponential generating function in (10) can be written as

$$\begin{aligned}\sum_{n\geq 0} G_{n,\beta,r}\frac{t^n}{n!} &= \left(\sum_{n\geq 0}\frac{(rt)^n}{n!}\right)\left(\sum_{n\geq 0}\Phi_n\left(1/\beta\right)\frac{(\beta t)^n}{n!}\right)\\ &= \sum_{n\geq 0}\left\{\sum_{k=0}^{n}\Phi_k\left(1/\beta\right)\frac{(\beta t)^k}{k!}\frac{(rt)^{n-k}}{(n-k)!}\right\}\\ &= \sum_{n\geq 0}\left\{\sum_{k=0}^{n}\binom{n}{k}\Phi_k\left(1/\beta\right)\beta^k r^{n-k}\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the following explicit formula.

Theorem 2.1. *The r-Dowling numbers can be expressed as*

$$G_{n,\beta,r} = \sum_{k=0}^{n}\binom{n}{k}\Phi_k\left(1/\beta\right)\beta^k r^{n-k}, \tag{17}$$

which is a kind of binomial combination of $\Phi_k(1/\beta)$.

Using (15), the explicit formula in (17) can further be written as

$$G_{n,\beta,r} = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} S(k,j)(1/\beta)^j \right\} \binom{n}{k} \beta^k r^{n-k}. \tag{18}$$

This gives the following matrix relation.

Theorem 2.2. *For $n \in \mathbb{N}$, the r-Dowling numbers $G_{i,\beta,r}$ equal to the sum of the entries of the ith row of the product of two matrices*

$$\left[\binom{i}{j}\beta^j r^{i-j}\right]_{(n+1)\times(n+1)} \left[S(i,j)(1/\beta)^j\right]_{(n+1)\times(n+1)}. \tag{19}$$

3. r-Whitney Numbers of the Second Kind and r-Whitney-Lah Numbers

In this section, a new explicit formula for r-Dowling numbers expressed in terms of r-Whitney Lah numbers and r-Whitney numbers of the second kind is established. As a consequence, a relation in terms of matrices involving the r-Dowling numbers, the r-Whitney-Lah numbers and the r-Whitney numbers of the second kind is obtained.

Note that equation (13) can be rewritten as follows

$$(-1)^n L_{\beta,r}(n,k) = \sum_{j=0}^{n} w_{\beta,r}(n,j) W_{\beta,r}(j,k). \tag{20}$$

Using the inverse relation of r-Whitney numbers in (6) with

$$f_n = (-1)^n L_{\beta,r}(n,k) \text{ and } g_j = (-1)^j W_{\beta,r}(j,k),$$

equation (20) yields

$$(-1)^n W_{\beta,r}(n,k) = \sum_{j=0}^{n} W_{\beta,r}(n,j)(-1)^j L_{\beta,r}(j,k); \text{ that is,}$$

$$S(n,k;\beta,r) = W_{\beta,r}(n,k) = \sum_{j=0}^{n} (-1)^{n-j} W_{\beta,r}(n,j) L_{\beta,r}(j,k).$$

Summing up both sides over k from 0 to n, gives the following theorem.

Theorem 3.1. *The explicit formula for r-Dowling numbers is given by*

$$G_{n,\beta,r} = \sum_{j=0}^{n} (-1)^{n-j} \left\{ \sum_{k=0}^{j} L_{\beta,r}(j,k) \right\} W_{\beta,r}(n,j). \tag{21}$$

For instance, when $\beta = r = 2$ and $n = 4$, we get

$$\begin{aligned} G_{4,2,2} &= \sum_{j=0}^{4} (-1)^{4-j} \left\{ \sum_{k=0}^{j} L_{2,2}(j,k) \right\} W_{2,2}(4,j) \\ &= (1)(16) - (5)(120) + (37)(100) - (361)(20) + (461)(1) \\ &= 257. \end{aligned}$$

Now, we can rewrite the sum in (21) as

$$G_{n,\beta,r} = S_0 + S_1 + S_2 + \ldots + S_n$$

where $S_j = \sum_{k=0}^{n} (-1)^{n-k} W_{\beta,r}(n,k) L_{\beta,r}(k,j)$. As a consequence, we have the following theorem.

Theorem 3.2. *For $n \in \mathbb{N}$, the r-Dowling numbers $G_{i,\beta,r}$ equal to the sum of the entries of the ith row of the product of two matrices*

$$\left[(-1)^{i-j} W_{\beta,r}(i,j)\right]_{(n+1)\times(n+1)} \left[L_{\beta,r}(i,j)\right]_{(n+1)\times(n+1)}, \tag{22}$$

whose entries are respectively the r-Whitney numbers of the second kind and the r-Whitney Lah numbers.

For instance, when $\beta = r = 2$ and $n = 3$, we get

$$\begin{aligned} &\left[(-1)^{i-j} W_{2,2}(i,j)\right]_{4\times4} \left[L_{\beta,r}(i,j)\right]_{4\times4} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -6 & 1 & 0 \\ -8 & 28 & -12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 24 & 12 & 1 & 0 \\ 192 & 144 & 24 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 6 & 1 & 0 \\ 8 & 28 & 12 & 1 \end{bmatrix} \end{aligned}$$

Summing up the entries of each row of the above matrix product, we obtain the column vector whose entries are the r-Dowling numbers

$$\begin{bmatrix} 1 \\ 2+1 \\ 4+6+1 \\ 8+28+12+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \\ 49 \end{bmatrix} = \begin{bmatrix} G_{0,2,2} \\ G_{1,2,2} \\ G_{2,2,2} \\ G_{3,2,2} \end{bmatrix}.$$

Corollary 3.3. *For $0 \le i, l \le n$, the r-Whitney numbers of the second kind satisfy the following explicit formula*

$$W_{\beta,r}(i,l) = \sum_{j=0}^{i} (-1)^{i-j} W_{\beta,r}(i,j) L_{\beta,r}(j,l);$$

that is,

$$[W_{\beta,r}(i,j)]_{n+1\times n+1} = \left[(-1)^{i-j} W_{\beta,r}(i,j)\right]_{n+1\times n+1} [L_{\beta,r}(i,j)]_{n+1\times n+1}.$$

It can easily be shown that

$$\sum_{j=i}^{n} (-1)^{n-j} w_{\beta,r}(n,j) W_{\beta,r}(j,i) = \sum_{j=i}^{n} W_{\beta,r}(n,j)(-1)^{j-i} w_{\beta,r}(j,i) = \delta_{ni}, \tag{23}$$

where δ_{ni} is the Kronecker delta. This relation implies that

$$[W_{\beta,r}(i,j)]^{-1}_{n+1\times n+1} = \left[(-1)^{i-j} w_{\beta,r}(i,j)\right]_{n+1\times n+1}. \tag{24}$$

Thus, we have

$$\begin{aligned} &\left[(-1)^{i-j} w_{\beta,r}(i,j)\right]_{n+1\times n+1} \left[(-1)^{i-j} W_{\beta,r}(i,j)\right]_{n+1\times n+1} [L_{\beta,r}(i,j)]_{n+1\times n+1} \\ &\quad = I_{n+1}. \end{aligned}$$

4. r-Whitney Numbers of the Second Kind and Lah Numbers

In this section, we will find a new explicit formula for computing r-Dowling numbers $G_{n,\beta,r}$ in terms of r-Whitney numbers of the second kind and the ordinary Lah numbers using the Faa di Bruno's formula and certain identity of Bell polynomials of the second kind. The following theorem contains the desired formula.

Theorem 4.1. *For $n \in \mathbb{N}$, the r-Dowling numbers $G_{n,r,\beta}$ equal*

$$G_{n,r,\beta} = \sum_{j=0}^{n} (-1)^{n-j} W_{\beta,-r}(n,j) \sum_{i=0}^{j} \beta^{j-i} L(j,i). \tag{25}$$

Proof. Let us recall the following identity from [1, 13] on the *nth* derivative of the exponential function $e^{\pm \frac{1}{t}}$ expressed in terms of the Lah numbers

$$\left(e^{\pm\frac{1}{t}}\right)^{(n)} = (-1)^n e^{\pm\frac{1}{t}} \sum_{k=1}^{n} (\pm 1)^k L(n,k) \frac{1}{t^{n+k}}, \tag{26}$$

the identity from [5] on Bell polynomials of the second kind

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \tag{27}$$

and the famous identity from [5] on Faá di Bruno formula described in terms of the Bell polynomials of the second kind

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)). \tag{28}$$

Replacing t by $-t$ in the generating function for the r-Dowling numbers $G_{n,\beta,r}$ in equation (10), yields

$$\sum_{n\geq 0} G_{n,\beta,r} \frac{(-t)^n}{n!} = \frac{e^{-rt} \cdot e^{\frac{1}{\beta e^{\beta t}}}}{e^{\frac{1}{\beta}}};$$

equivalently,

$$e^{\frac{1}{\beta}} \sum_{n\geq 0} (-1)^n G_{n,\beta,r} \frac{t^n}{n!} = e^{\frac{1}{\beta e^{\beta t}}} \cdot e^{-rt}. \tag{29}$$

Then taking kth derivative both sides of (29) with respect to t yields

$$e^{\frac{1}{\beta}} \sum_{n=k}^{\infty} (-1)^k G_{n,\beta,r} \frac{t^{n-k}}{(n-k)!} = \frac{d^k}{dt^k} \big(e^{\frac{1}{\beta e^{\beta t}}} \cdot e^{-rt} \big). \tag{30}$$

Taking $f(u) = e^{\frac{1}{u}}$ and $h(t) = \beta e^{\beta t}$ in (28) and making use of (26) give

$$\begin{aligned}
\frac{d^k \left(e^{\frac{1}{\beta e^{\beta t}}} \right)}{dt^k} &= \frac{d^k \big(f \circ h(t) \big)}{dt^k} \\
&= \sum_{j=1}^{k} \frac{d^j (e^{1/u})}{du^j} B_{k,j}(\beta(\beta e^{\beta t}), \beta^2(\beta e^{\beta t}), \ldots, \beta^{k-j+1}(\beta e^{\beta t})) \\
&= \sum_{j=1}^{k} (-1)^j e^{1/u} \sum_{i=1}^{j} L(j,i) \cdot \frac{1}{u^{j+i}} B_{k,j}(\beta(\beta e^{\beta t}), \beta^2(\beta e^{\beta t}), \ldots, \beta^{k-j+1}(\beta e^{\beta t})) \\
&= e^{\frac{1}{\beta e^{\beta t}}} \sum_{j=1}^{k} (-1)^j \sum_{i=1}^{j} L(j,i) \cdot \frac{1}{(\beta e^{\beta t})^{j+i}} B_{k,j}(\beta(\beta e^{\beta t}), \beta^2(\beta e^{\beta t}), \ldots, \beta^{k-j+1}(\beta e^{\beta t})),
\end{aligned}$$

where $u(t) = \beta e^{\beta t}$. Further by virtue of

$$B_{k,j}(abx_1, ab^2x_2, \ldots, ab^{k-j+1}x_{k-j+1}) = a^j b^k B_{k,j}(x_1, x_2, \ldots, x_{k-j+1})$$

and

$$B_{k,j}(\overbrace{1, 1, \ldots, 1}^{k-j+1}) = S(k,j)$$

listed in [5], [p.135], where a and b are complex numbers, we obtain

$$\begin{aligned}\frac{d^k\left(e^{\frac{1}{\beta e^{\beta t}}}\right)}{dt^k} &= e^{\frac{1}{\beta e^{\beta t}}}\sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j}L(j,i)\cdot\frac{1}{(\beta e^{\beta t})^{j+i}}\cdot(\beta e^{\beta t})^j\beta^k B_{k,j}(\overbrace{1,1,\ldots,1}^{k-j+1})\\ &= e^{\frac{1}{\beta e^{\beta t}}}\sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j}L(j,i)\cdot\frac{\beta^{k-i}}{(e^{\beta t})^i}S(k,j).\end{aligned}$$

Hence, using Leibniz formula,

$$\begin{aligned}\frac{d^n}{dz^n}\left(e^{\frac{1}{\beta e^{\beta t}}}\cdot e^{-rt}\right) &= \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}e^{\frac{1}{\beta e^{\beta t}}}\frac{d^{n-k}}{dt^{n-k}}e^{-rt}\\ &= \sum_{k=0}^{n}\binom{n}{k}\left\{e^{\frac{1}{\beta e^{\beta t}}}\sum_{j=1}^{k}(-1)^j\sum_{i=1}^{j}L(j,i)\cdot\frac{\beta^{k-i}}{(e^{\beta t})^i}S(k,j)\right\}\cdot(-r)^{n-k}e^{-rt}.\end{aligned}$$

Thus, replacing k by n and evaluating at $t=0$ in equation (30) give

$$e^{\frac{1}{\beta}}(-1)^n G_{n,\beta,r} = \sum_{k=0}^{n}\binom{n}{k}\sum_{j=1}^{k}(-1)^j e^{\frac{1}{\beta}}\sum_{i=1}^{j}L(j,i)\cdot\beta^{k-i}S(k,j)\cdot(-r)^{n-k};$$

Rearranging the above sum and using the fact that $L(0,i)=0$ for all positive integers i, we get

$$G_{n,\beta,r} = \sum_{i=0}^{n}(-1)^{n-j}\sum_{j=0}^{i}\left\{\sum_{k=j}^{n}\binom{n}{k}\beta^{k-j}(-r)^{n-k}S(k,j)\right\}\beta^{j-i}L(j,i).$$

Applying the property of r-Whitney numbers of the second kind in equation (8) yields

$$G_{n,\beta,r} = \sum_{i=0}^{n}(-1)^{n-j}\sum_{j=0}^{i}W_{\beta,-r}(n,j)\beta^{j-i}L(j,i).$$

This is exactly the formula in (25).

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2. *For $n\in\mathbb{N}$, the r-Dowling numbers $G_{i,\beta,r}$ equal to the sum of the entries of the ith row of the product of two matrices*

$$\left[(-1)^{i-j}W_{\beta,-r}(i,j)\right]_{n\times n}\left[\beta^{j-i}L(i,j)\right]_{n\times n}, \tag{31}$$

whose entries are respectively r-Whitney numbers of the second kind and the Lah numbers.

Proof. We can rewrite the formula in Theorem 4.1 as

$$G_{i,\beta,r} = \sum_{l=0}^{i}T_{il},\quad i=0,1,2,\ldots,n,$$

where

$$T_{il} = \sum_{j=0}^{i} (-1)^{i-j} W_{\beta,-r}(i,j)\beta^{j-l} L(j,l), \quad l = 0, 1, 2, \ldots i.$$

Clearly, T_{il} is the (i,l)-entry of the following product of two matrices

$$\left[(-1)^{i-j} W_{\beta,-r}(i,j)\right]_{n\times n} \left[\beta^{j-i} L(i,j)\right]_{n\times n}, \tag{32}$$

containing the r-Whitney numbers of the second kind and Lah numbers, respectively.

To illustrate this corollary, let us consider the case where $\beta = 1$, $r = 2$, $n = 6$. That is,

$$\left[(-1)^{i-j} W_{1,-2}(i,j)\right]_{6\times 6} \left[\beta^{i-j} L(i,j)\right]_{6\times 6}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & 0 & 0 & 0 \\ 8 & 7 & 3 & 1 & 0 & 0 \\ 16 & 15 & 7 & 2 & 1 & 0 \\ 32 & 31 & 15 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & 1 & 0 & 0 \\ 0 & 24 & 36 & 12 & 1 & 0 \\ 0 & 120 & 240 & 120 & 20 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \\ 8 & 19 & 9 & 1 & 0 & 0 \\ 16 & 65 & 55 & 14 & 1 & 0 \\ 32 & 211 & 285 & 125 & 20 & 1 \end{bmatrix}. \tag{33}$$

Hence, summing up the entries of each row of the matrix in (33) gives the following column vector whose entries are the r-Dowling numbers with $\beta = 1$ and $r = 2$

$$\begin{bmatrix} 1 \\ 2+1 \\ 4+5+1 \\ 8+19+9+1 \\ 16+65+55+14+1 \\ 32+211+285+125+20+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 10 \\ 37 \\ 151 \\ 674 \end{bmatrix} = \begin{bmatrix} G_{0,1,2} \\ G_{1,1,2} \\ G_{2,1,2} \\ G_{3,1,2} \\ G_{4,1,2} \\ G_{5,1,2} \end{bmatrix}.$$

Clearly, the r-Whitney numbers of the second kind $W_{\beta,r}(i,l)$ can be expressed as

$$W_{\beta,r}(i,l) = \sum_{j=0}^{i} (-1)^{i-j} W_{\beta,-r}(i,j)\beta^{j-l} L(j,l).$$

That is,

$$\left[W_{\beta,r}(i,j)\right]_{n+1\times n+1} = \left[(-1)^{i-j} W_{\beta,-r}(i,j)\right]_{n+1\times n+1} \left[\beta^{i-j} L(i,j)\right]_{n+1\times n+1}.$$

Using (24), we obtain the following matrix identity.

$$\left[(-1)^{i-j} w_{\beta,r}(i,j)\right]_{n+1\times n+1} \left[(-1)^{i-j} W_{\beta,-r}(i,j)\right]_{n+1\times n+1} \left[\beta^{i-j} L(i,j)\right]_{n+1\times n+1}$$
$$= I_{n+1}.$$

5. A q-Analogue

A q-analogue is a generalization of a known expression parameterized by a quantity q that reduces to the known expression in the limit, as $q \to 1$. For example, the q-analogue of n, $n!$, $(n)_k$ and $\binom{n}{k}$ are respectively given by

$$\begin{aligned} [n]_q &= 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}; \\ [n]_q! &= [n]_q[n-1]_q \cdots [2]_q[1]_q; \\ [n]_{k,q} &= [n]_q[n-1]_q \cdots [n-k+1]_q; \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_{k,q}}{k!}. \end{aligned}$$

Recently, R. Corcino et. al [9] defined a q-analogue of r-Whitney numbers of the second kind by means of the following recurrence relation:

$$W_{m,r}[n,k]_q = q^{m(k-1)+r} W_{m,r}[n-1,k-1]_q + [mk+r]_q W_{m,r}[n-1,k]_q. \quad (34)$$

When $q \to 1$, this will reduce to

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (mk+r)W_{m,r}(n-1,k).$$

One can easily verify that

$$W_{m,r}[n,0] = [r]_q^n.$$

The q-analogue $W_{m,r}[n,k]_q$ possessed several properties including the following relation

$$\sum_{k=0}^{n} W_{m,r}[n,k]_q [t-r|m]_{k,q} = [t]_q^n. \quad (35)$$

For the r-Whitney numbers of the first kind, their q-analogue may be defined by

$$\sum_{k=0}^{n} (-1)^{n-k} w_{m,r}[n,k]_q [t]_q^k = [t-r|m]_{n,q}. \quad (36)$$

To compute the first values of $w_{m,r}[n,k]_q$, we need to derive the triangular recurrence relation for $w_{m,r}[n,k]_q$. Using (36) and the identity

$$[t-n]_q = \frac{1}{q^n}([t]_q - [n]_q),$$

we have

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} w_{m,r}[n+1,k]_q [t]_q^k = [t-r|m]_{n+1,q} = [t-(r+nm)]_q [t-r|m]_{n+1,q}$$

$$= \left(\frac{1}{q^{r+nm}}([t]_q - [r+nm]_q) \right) \sum_{k=0}^{n} (-1)^{n-k} w_{m,r}[n,k]_q [t]_q^k$$
$$= \sum_{k=0}^{n+1} \frac{1}{q^{r+nm}} (-1)^{n-k+1} w_{m,r}[n,k-1]_q [t]_q^k + \sum_{k=0}^{n+1} \frac{-[r+nm]_q}{q^{r+nm}} (-1)^{n-k} w_{m,r}[n,k]_q [t]_q^k$$
$$= \sum_{k=0}^{n+1} \frac{(-1)^{n-k+1}}{q^{r+nm}} \left\{ w_{m,r}[n,k-1]_q + [r+nm]_q w_{m,r}[n,k]_q \right\} [t]_q^k.$$

Thus, comparing the coefficients of $[t]_q^k$, we easily obtain the following triangular recurrence relation

$$q^{r+nm} w_{m,r}[n+1,k]_q = w_{m,r}[n,k-1]_q + [r+nm]_q w_{m,r}[n,k]_q. \tag{37}$$

Now, to derive the orthogonality relations for $w_{m,r}[n,k]_q$ and $W_{m,r}[n,k]_q$, we first rewrite (36) as

$$\sum_{j=0}^{k} w_{m,r}[k,j]_q [t]_q^j = [t-r|m]_{k,q}$$

and substituting to (35) yields

$$[t]_q^n = \sum_{k=0}^{n} W_{m,r}[n,k]_q [t-r|m]_{k,q}$$
$$= \sum_{k=0}^{n} W_{m,r}[n,k]_q \sum_{j=0}^{k} (-1)^{k-j} w_{m,r}[k,j]_q [t]_q^j$$
$$= \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} (-1)^{k-j} W_{m,r}[n,k]_q w_{m,r}[k,j]_q \right\} [t]_q^j.$$

Hence, we obtain the first form of the desired orthogonality relation

$$\sum_{k=j}^{n} (-1)^{k-j} W_{m,r}[n,k]_q w_{m,r}[k,j]_q = \delta_{n,j}, \tag{38}$$

where $\delta_{n,j}$ is the well-known Kronecker delta. By applying similar argument, that is, by substituting (35) to (36), we obtain the second form of the orthogonality relation

$$\sum_{k=j}^{n} (-1)^{n-k} w_{m,r}[n,k]_q W_{m,r}[k,j]_q = \delta_{n,j}, \tag{39}$$

Furthermore, the orthogonality relations in (38) and (39) immediately imply the following inverse relations:

$$f_n = \sum_{k=0}^{n} (-1)^{n-k} w_{m,r}[n,k]_q g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} W_{m,r}[n,k]_q f_k \tag{40}$$

$$f_k = \sum_{n=k}^{\infty} (-1)^{n-k} w_{m,r}[n,k]_q g_n \Longleftrightarrow g_k = \sum_{n=k}^{\infty} W_{m,r}[n,k]_q f_n. \tag{41}$$

Parallel to Cheon and Jung [4], a q-analogue of r-Whitney-Lah numbers $L_{\beta,r}[n,k]_q$ may be defined by

$$L_{\beta,r}[n,k]_q = \sum_{j=0}^{n} w_{\beta,r}[n,j]_q W_{\beta,r}[j,k]_q. \tag{42}$$

This can be written as

$$(-1)^n L_{\beta,r}[n,k]_q = \sum_{j=0}^{n} (-1)^{n-j} w_{\beta,r}[n,j]_q (-1)^j W_{\beta,r}[j,k]_q. \tag{43}$$

Using the inverse relation in (40) with $f_n = (-1)^n L_{\beta,r}[n,k]_q$ and $g_j = (-1)^j W_{\beta,r}[j,k]_q$, relation (43) implies the following relation

$$\begin{aligned} (-1)^n W_{\beta,r}[n,k]_q &= \sum_{j=0}^{n} W_{\beta,r}[n,j]_q (-1)^j L_{\beta,r}[j,k]_q \\ W_{\beta,r}[n,k]_q &= \sum_{j=0}^{n} (-1)^{n-j} W_{\beta,r}[n,j]_q L_{\beta,r}[j,k]_q \end{aligned} \tag{44}$$

Summing up both sides of (44) over k yields

$$\begin{aligned} \sum_{k=0}^{n} W_{\beta,r}[n,k]_q &= \sum_{k=0}^{n} W_{\beta,r}[n,k]_q \sum_{j=0}^{n} (-1)^{n-j} W_{\beta,r}[n,j]_q L_{\beta,r}[j,k]_q \\ D_{\beta,r}[n]_q &= \sum_{j=0}^{n} (-1)^{n-j} \left\{ \sum_{k=0}^{j} L_{\beta,r}[j,k]_q \right\} W_{\beta,r}[n,j]_q. \end{aligned} \tag{45}$$

Remark 5.1. *The explicit formula in* (45) *implies that the* (q,r)*-Dowling numbers* $D_{\beta,r}[n]_q$ *are equal to* $\mathbf{e_i DLe}$, *where* $\mathbf{D}$ *and* $\mathbf{L}$ *are matrices whose entries are* $W_{\beta,r}[n,j]_q$ *and* $L_{\beta,r}[n,k]_q$, *respectively,* $\mathbf{e_i}$ *is the* $i-th$ *unit vector, and* $\mathbf{e}$ *is the vector with all entries equal to 1.*

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References

[1] K. N. Boyadzhiev, Lah Numbers, Laguerre Polynomials of Order Negative One, and the nth Derivative of $\exp(1/x)$, *Acta Univ. Sapientiae Math.* 8 (2016)(1), 22–31.

[2] K. N. Boyadzhiev, Exponential polynomials, Stirling numbers, and evaluation of some Gamma integrals, *Abstract and Applied Analysis* Volume 2009, Article ID 168672.

[3] A.Z. Broder, The r-Stirling numbers, *Discrete Math.* 49(1984), 241–259.

[4] G.S. Cheon and J.H Jung. (2012). r-Whitney numbers of Dowling Lattices. *Discrete Mathematics*, 15, 2337-2348.

[5] L. Comtet, *Advanced Combinatorics.* D. Reidel Publishing Company Inc., 1974.

[6] C. B. Corcino, An asymptotic formula for the r-Bell numbers, *Matimyas Mat.* 24 (2001), 9–18.

[7] R.B. Corcino, The (r, β)-Stirling numbers, *Mindanao Forum*, 14(2) (1999), 91-99.

[8] Asymptotic Normality of the (r, β)-Stirling Numbers, *Ars Combin.*, 81 (2006), 81–96.

[9] R.B. Corcino, M.J.R. Latayada, and M.A.P. Vega, *Hankel Transform of (q, r)-Dowling Numbers*, Eur. Pure Appl. Math.,12(2)(2019), 279–293.

[10] R.B. Corcino and M. Herrera, The Limit of the Differences of the Generalized Factorial, Its q- and p,q-Analogue, *Utilitas Mat.*, Vol. 72 (2007) pp. 33-49.

[11] R.B. Corcino, J.T. Malusay, J. Cillar, G. Rama, O. Silang, I. Tacoloy, Analogies of the Qi formula for some Dowling type numbers, Accepted for Publication in Utilitas Mathematica (2017).

[12] R.B. Corcino and C.B. Corcino, On Generalized Bell Polynomials, *Discrete Dynamics in Nature and Society*, 2011, Article ID 623456, doi: 10.1155/2011/623456.

[13] S. Daboul, J. Mangaldan, M.Z. Spivey, and P.J. Taylor, The Lah numbers and the nth derivative of $e^{1/x}$, *Math. Mag.* 86(2013), 39–47.

[14] L.C. Hsu and P.J-S. Shiue. "A Unified Approach to Generalized Stirling Numbers." *Advances in Applied Math.* **20** (1998): 366-384.

[15] M. Koutras, Non-Central Stirling Numbers and Applications, *Discrete Math.* 42 (1982), 73–89.

[16] I. Mező, A New formula for the Bernoulli polynomials, *Results Math.* 58(2010), 329–335.

[17] I. Mező, The r-Bell numbers, *J. Integer Seq.* 14(2011), Article 11.1.1.

[18] Gábor Nyul and Gabriella Rácz, The r-Lah numbers, *Discrete Math.* 338(2015) 1660–1666.

[19] N. Privault, Generalized Bell Polynomials and the Combinatorics of Poisson Central Moments, *The Electronic Journal of Combinatorics*, **18** (2011) #P54.

[20] S.M. Tanny, On some numbers related to Bell numbers, *Canad. Math. Bull.* **17**(15), 1975, 733-738.

[21] F. Qi, An explicit formula for the Bell numbers in terms of Lah and Stirling numbers, *Mediterr. J. Math.* 13 (2016)(5), 2795–2800.

[22] J. Riordan, *Introduction to Combinatorial Analysis,* John Wiley and Sons Inc., 1958.

[23] X.-J. Zhang, F. Qi, and W.-H. Li, Properties of Three Functions Relating to the Exponential Function and the Existence of Partitions of Unity, *Int. J. Open Probl. Comput. Sci. Math.* 5 (2012), no. 3, 122–127; Available online at https://doi.org/10.12816/0006128.