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A New Type Coefficient Conjugate on the Gradient Methods for Impulse Noise Removal in Images

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Abstract. The conjugate gradient methods are outstanding by choosing a suitable for coefficient conjugate. In this paper, a modified version of the conjugate gradient algorithm suggested by Hideaki and Yasushi [4] is proposed in order to show that the new method is globally convergent, under standard assumptions. To exemplify the efficiency of the new method, its performance is examined for impulse noise removal in images.

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Key Words and Phrases: Derivative coefficient conjugate gradient, Convergence property, Image restoration problems

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1. Introduction

Image declaring [16] from noisy data is a fundamental problem in image processing. Two-phase approach, consists of two phases. Firstly, accurate detection of the location of impulse noise using a median-type filter (AMF) [17]. Let x be the true image with M-by-N pixels, and $x_{ij}{}_{i,j=1}^{M,N}$ denote the gray level of x, y signifying the observed noisy image of x corrupted by the salt-and-pepper noise, $\bar{y}$ is defined by the image obtained byapplying the adaptive median filter method to the noisy image y in the first phase. Secondly, recovering the noise pixels by minimizing the following functional:

$$f_\alpha(u) = \sum_{(i,j)\in N} [|u_{i,j} - y_{i,j}| + \frac{\beta}{2}(S^1_{i,j} + S^2_{i,j})] \tag{1}$$

Where $u_{i,j} = \lfloor u_{i,j} \rfloor_{(i,j)\in N}$ is a column vector of length $|N|$, β is the regularization parameter and:

$$S^1_{i,j} = 2 \sum_{(m,n)\in P_{i,j}\cap N^c} \varphi_\alpha(u_{i,j} - y_{m,n}),\ S^2_{i,j} = \sum_{(m,n)\in P_{i,j}\cap N} \varphi_\alpha(u_{i,j} - y_{m,n}) \tag{2}$$

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The noise candidate indices set $N^c = \{(i,j) \in A/\bar{y}_{i,j} \neq y_{i,j} \; and \; y_{i,j} = s_{max} \; or \; s_{min}\}$, s_{max} is the maximum of the noisy pixel and s_{min} denotes the minimum of the noisy pixel. $A = 1,2,3,...,M \times 1,2,3,...,N$, $V_{i,j} = (V_{i,j} \cap N^c) \cup (V_{i,j} \cap N)$ is the neighborhood of (i,j), and φ_α is an edge preserving potential function having the parameter α, example of such $\varphi_\alpha = \sqrt{\alpha + x^2}, \alpha > 0$. Similar optimization problems arise with non-smooth regularizations, where $F_\alpha(u)$ is of the form (1) with $S^1_{i,j} + S^2_{i,j}$ smooth and $|u_{i,j} - y_{i,j}|$ non-smooth at zero. In formula, the function which is minimized is a half-quadratic smooth approximation of $F_\alpha(u)$ as:

$$f_\alpha(u) = \sum_{(i,j)\in N} [(2 \times S^1_{i,j} + S^2_{i,j})] \tag{3}$$

More details can be found in [16],[17].

The minimization of formula (3) has been accomplished by the method of conjugated gradients such as:

$$f(x^*) = \min_{x \in R^N} f(u). \tag{4}$$

In formula [15], the k-th iteration, a step-length α_k is obtained by a line search technique and the next iterate is set to:

$$u_{k+1} = u_k + \alpha_k d_k. \tag{5}$$

If f is a convex quadratic, its one-dimensional minimizer along the ray $u_k + \alpha_k d_k$ can be computed analytically, and is given by:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k}. \tag{6}$$

For general non-linear functions, it is necessary to use an iterative procedure. More details can be found in formula [14].

In the convergence analysis and implementations of conjugate gradient methods, the Wolfe condition is often used to find the step length α_k satisfying:

$$f(u_k + a_k d_k) \leq f(u_k) + \delta \alpha_k g_k^T d_k \tag{7}$$

$$d_k^T g(u_k + a_k d_k) \geq \sigma \; d_k^T g_k \tag{8}$$

where $0 < \delta < \sigma < 1$. More details can be found in [13].

The conjugate gradient direction for the next iteration has the following form:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{9}$$

where β_k is a scalar. Two famous ways of choosing β_k are:

$$\beta_k^{FR} = \frac{\|g_{k+1}|^2}{\|g_k|^2}, \; \beta_{k+1}^{DY} = \frac{\|g_{k+1}|^2}{d_k^T y_k} \tag{10}$$

These were given by Fletcher-Reeves (FR) method [4], and the Dai - Yuan (DY) method [2] independently. Its algorithms have important properties including the globally convergent property.]

Recently, Hideaki and Yasushi [11] and Basim [8] proposed conjugate gradient methods which significantly differ in using both their gradient and function values with higher accuracy in the approximation of the curvature, in which the parameters are specified as follows:

$$\beta_k^{HY} = \frac{\|g_{k+1}|^2}{2(f_k - f_{k+1})/a_k}, \ \beta_k^{B} = \frac{\|g_{k+1}|^2}{(f_k - f_{k+1})/a_k - g_k^T d_k/2} \tag{11}$$

These algorithms are significantly efficient in their computational efficiency also. The reported improved performance for this modification for unconstrained problems is a quadratic model in order to optimize the benefits of the original conjugate gradient methods.

The main goal of this study is derivation of the new coefficient conjugate and to think differently of the denominator $d_k^T G v_k$ based on the quadratic model. Our method is found to be numerically coherent and also efficient in image restoration.

2. Deriving new coefficient conjugate for conjugate gradient methods

We will assume that f is quadratic and exact line search are being used. This is a studies on the behaviors of one of the family of conjugate gradient optimization methods, which was introduced by Hideaki and Yasushi in 2011. Hideaki and Yasushi choice of β_k is:

$$\beta_k = \frac{g_{k+1}^T Q s_k}{d_k^T Q s_k} \tag{12}$$

where Q is Hessian matrix and where β_k is satisfies the conjugacy condition:

$$d_{k+1}^T Q d_k = 0 \tag{13}$$

To modify this method we will introduce a good approximation to the $d_k^T Q s_k$. So by Taylor formula we have:

$$f(u) = f(u_{k+1}) + g_{k+1}^T (u - u_{k+1}) + \frac{1}{2}(u - u_{k+1})^T Q(u_k)(u - u_{k+1}) \tag{14}$$

Define the gradient step:

$$g_{k+1} = g_k + Q(u_k) s_k \tag{15}$$

Note that equation (14) and equation (15) result that:

$$1/2 s_k^T Q(u_k) s_k = (f_{k+1} - f_k - g_k^T s_k) \tag{16}$$

Now, by equation (16) and equation (6) we can see that:

$$d_k^T Q(u_k) s_k = (f_{k+1} - f_k)/a_k - 3/2 d_k^T g_k \tag{17}$$

Using (12) and (17), we obtain the following formula for computing β_k:

$$\beta_k = \frac{g_{k+1}^T y_k}{(f_{k+1} - f_k)/\alpha_k - 3/2 d_k^T g_k} \tag{18}$$

On the other hand, using (6) in (17) and putting in (12), then got another formula:

$$\beta_k = \frac{g_{k+1}^T y_k}{(f_{k+1} - f_k)/\alpha_k + 3/2 d_k^T y_k} \tag{19}$$

Since f is quadratic function and exact line search is employed, then:

$$\beta_k = \frac{\|g_{k+1}|^2}{(f_{k+1} - f_k)/\alpha_k - 3/2 d_k^T g_k} \tag{20}$$

and

$$\beta_k = \frac{\|g_{k+1}|^2}{(f_{k+1} - f_k)/\alpha_k + 3/2 d_k^T y_k} \tag{21}$$

Given our formula for modified **HY** formula, so-called **BNC** and **BTC**.

At this point, we describe a new algorithm, called **BNC** and **BTC**.

Algorithms BNC and BTC.

Stage 1. An initial point u_1. Set $d_1 = -g_1$. If $\|g_1| = 10^{-6}$, then stop.
Stage 2. Determine the $\alpha_k > 0$ satisfying the Wolfe conditions (7) and (8).
Stage 3. Let $x_{k+1} = x_k + \alpha_k d_k$ and $g_{k+1} = g(x_{k+1})$. If $\|g_{k+1}| = 10^{-6}$, then stop.
Stage 4. Compute β_k by the formulae (20-21), then generate d_{k+1} by (9).
Stage 5. Set $k = k + 1$ and continue with step 2.

The nice property for any a good algorithm should accomplish the sufficient descent condition. Specially, in this study will provide additional property are very vital in the convergence proving. Summarizing the above discussion, the following theorem is obtained.

Theorem 1. *Let x_k and d_k be generated by BNC method, then we obtain:*

$$d_{k+1}^T g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k \tag{22}$$

Proof. It is visible that $d_k = -g_k$ satisfies $d_1^T g_1 < 0$. Suppose that $d_k^T g_k < 0$ for any k. Directly from (11) and (21) we conclude that:

$$\begin{aligned} d_{k+1}^T g_{k+1} &= -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1} \\ &= -\beta_k((f_{k+1} - f_k)/a_k - 3/2 d_k^T g_k) + \beta_k d_k^T g_{k+1} \end{aligned} \tag{23}$$

Since the gradient of the function (14) is (15), we can conclude that:

$$d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k \tag{24}$$

Because $d_k^T g_k < 0$, then we obtain:

$$d_{k+1}^T g_{k+1} < 0 \tag{25}$$

so the proof is completed. **A similar** result holds for the **BTC** formula.

3. Convergence Analysis

In order to analyze the global convergence of the BDC and BTC methods, we begin by impose the following assumptions :

[1] The level set "$D = u/f(u) \leq f(u_0)$" is bounded, where u_0 is the starting point.

[2] The gradientis is Lipschitz continuous in some neighborhood D which contains L_0; i.e. there exist L, such that:

$$\|g(\upsilon) - g(\omega)| \leq L\|\upsilon - \omega|, \ \forall \upsilon, \omega \in D. \tag{26}$$

For more details see [2],[5].

In [18], Zoutendijk introduced the general result are important feature in proving the convergence results.

Lemma 1. *Let the Assumptions* (1) *and* (2) *holds true. If the direction* d_k *is descent and the sequence generated by (5) where* α_k *satisfies (7) and (8), then:*

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k|^2} < \infty. \tag{27}$$

Theorem 2. *Suppose that assumptions (1) and (2) hold. If formula* β_k *satisfies (21), then Then we have:*

$$\lim_{k\to\infty} inf|g_k| = 0. \tag{28}$$

Proof. The prove is by induction $\|g_k| \neq 0$ for all $k \in n$. Rewriting (9) as $d_{k+1} + g_{k+1} = \beta_k d_k$. Accordingly, squaring both sides, we have:

$$|d_{k+1}|^2 + |g_{k+1}|^2 + 2d_{k+1}^T g_{k+1} = (\beta_k)^2 |d_k|^2 \tag{29}$$

Using (23) to (27) implies:

$$|d_{k+1}|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} |d_k|^2 - 2d_{k+1}^T g_{k+1} - |g_{k+1}|^2 \tag{30}$$

Dividing both sides of (28) by $(d_{k+1}^T g_{k+1})^2$, we get:

$$\begin{aligned} \frac{|d_{k+1}|^2}{(d_{k+1}^T g_{k+1})^2} &= \frac{|d_k|^2}{(d_k^T g_k)^2} - \frac{|g_{k+1}|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}} \\ &\leq \frac{|d_k|^2}{(d_k^T g_k)^2} - (\frac{|g_{k+1}|}{(d_{k+1}^T g_{k+1})} + \frac{1}{|g_{k+1}|^2}) + \frac{1}{|g_{k+1}|^2} \\ &\leq \frac{|d_k|^2}{(d_k^T g_k)^2} + \frac{1}{|g_{k+1}|^2} \end{aligned} \tag{31}$$

Hence, we obtained:

$$\frac{|d_{k+1}|^2}{(d_{k+1}^T g_{k+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{|g_i|^2} \tag{32}$$

Assume that there exists $c_1 > 0$ such that $|g_k| \geq c_1$ for all $k \in n$. Then:

$$\frac{|d_{k+1}|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2} \tag{33}$$

Equation (31) shows that:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{|d_k|^2} = \infty \tag{34}$$

Based on Lemma 1, we get $\lim_{k\to\infty} inf|g_k| = 0$ holds. **A similar** result holds for the **BTC** formula.

4. Numerical results

The main goal we carry out some numerical experiments for minimization of 4 test images taken from [3]. To test and compare the computation effect of the proposed BNC, BTC in this paper with FR method whose results be given by [4]. There are different methods than this methods that we can see in [6, 7, 9, 10, 12]. In the numerical experiments, BNC, BTC and FR methods uses the following parameters: $\delta = 0.0001$ and $\sigma = 0.5$.

Numerical results of the BNC and FR are listed in Table 1. Here denotes the NI/NF/PSNR denote the number of iteration, function evaluations and PSNR (peak signal to noise ratio), which is defined as:

$$PSNR = 10log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \tag{35}$$

where $u_{i,j}^r$ and $u_{i,j}^*$ denote the pixel values of the restored image and the original image, respectively. We stop the iterations, if:

$$\frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \; and \; |f(u_k)| \leq 10^{-4}(1 + |f(u_k)|) \tag{36}$$

For more details see [1],[3].

Table 1: Numerical results for the NI , NF and PSNR.

Image	Noise level r (%)	FR-Method			BNC-Method			BTC-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Le	50	82	153	30.5529	45	93	30.5049	15	33	35.7945
	70	81	155	27.4824	46	91	27.4686	15	35	34.3739
	90	108	211	22.8583	53	100	22.8008	15	33	31.6911
ho	50	52	53	30.6845	28	56	34.7098	15	32	39.1522
	70	63	116	31.2564	37	73	31.344	15	31	36.3978
	90	111	214	25.287	49	96	24.927	14	30	34.52
El	50	35	36	33.9129	26	47	33.8859	16	32	35.0033
	70	38	39	31.864	28	49	31.8548	16	31	33.456
	90	65	114	28.2019	39	70	28.2239	14	28	32.6548
c512	50	59	87	35.5359	30	65	35.4408	12	26	42.0829
	70	78	142	30.6259	35	75	30.7049	12	27	40.3123
	90	121	236	24.3962	45	96	24.935	13	31	36.9659

As we can see from Table (1), our proposed algorithm is competitive and promising has a very great advantage over the FR method in term of the the number ofiterations and the number of function evaluations.

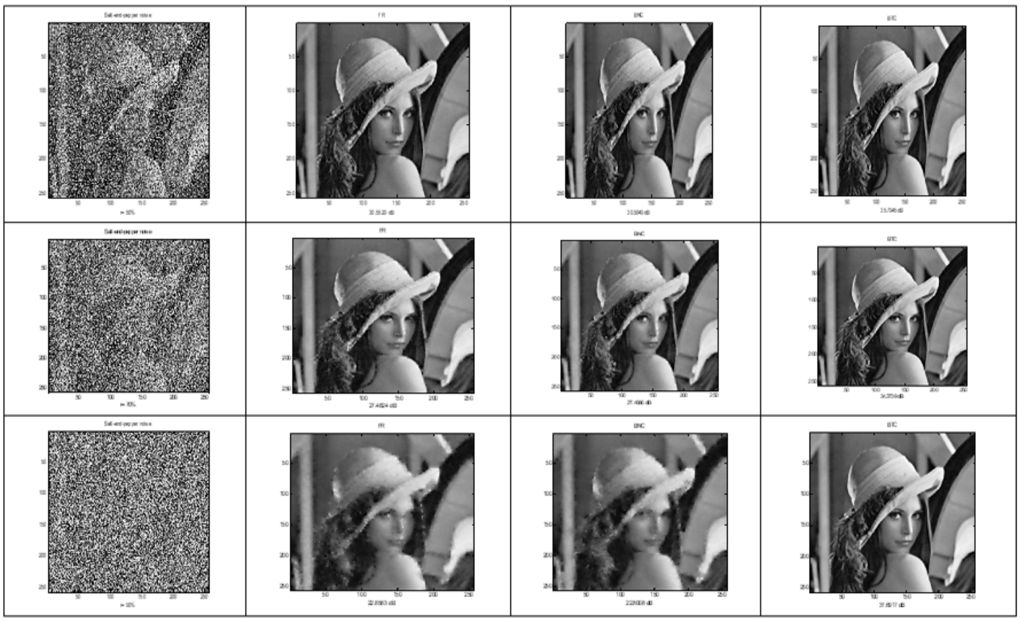

Figure 1: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 256×256 Lena image.

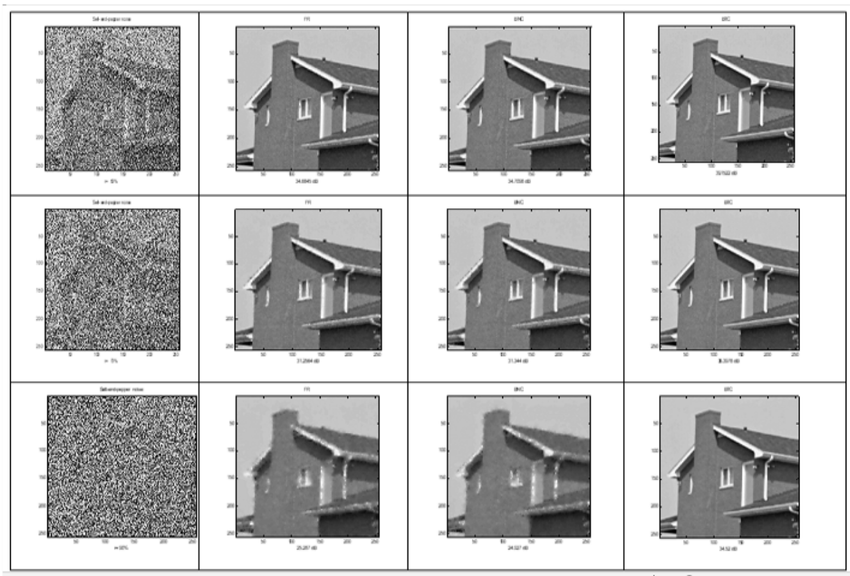

Figure 2: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 256×256 House image.

Figure 3: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 512×512 Elaine image.

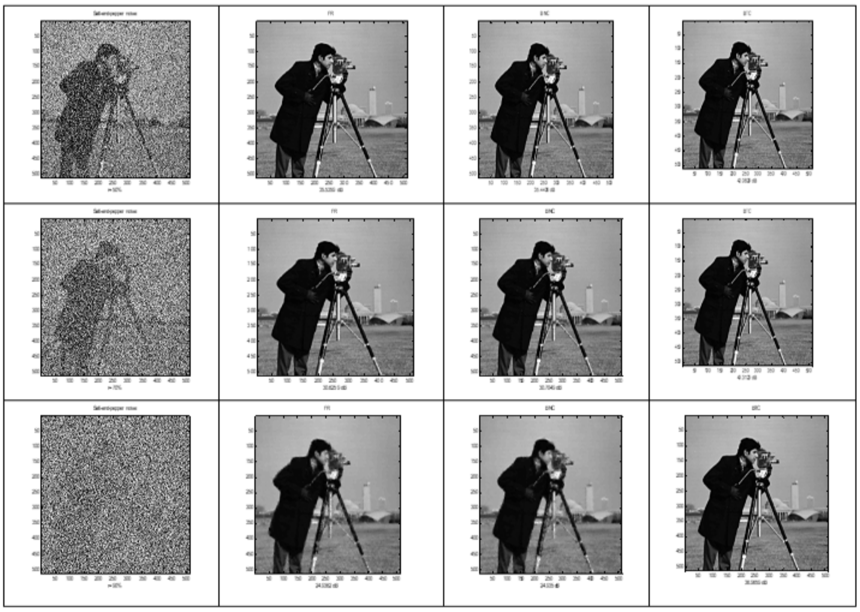

Figure 2: From left to right: 50, 70, 90% noise, FR method, BNC and BTC method 512×512 Cameraman image.

5. Conclusions

We presented a powerful conjugate gradient technique. In addition to meeting the adequate descent criterion, the proposed approach is globally convergent. According to numerical findings, the approach operates well in practice and is superior than the widely used FR method. We also looked at our approach's aptitude for resolving several practical problems. In this manner, a typical issue from applications for image processing was taken into account. We demonstrated the acceptability of the image that our approach restored.

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