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Evaluation of Some Convolution Sums by Quasimodular Forms

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Abstract. We evaluate the convolution sums

$$\sum_{l+27m=n}\sigma(l)\sigma(m),\quad \sum_{3l+9m=n}\sigma(l)\sigma(m),\quad \sum_{l+40m=n}\sigma(l)\sigma(m),$$
$$\sum_{5l+8m=n}\sigma(l)\sigma(m),\quad \sum_{4l+10m=n}\sigma(l)\sigma(m),\quad \sum_{l+55m=n}\sigma(l)\sigma(m),$$
$$\sum_{5l+11m=n}\sigma(l)\sigma(m),\quad \sum_{l+5m=n}\sigma(l)\tau_{2,11}(m),$$
$$\sum_{5l+m=n}\sigma(l)\tau_{2,11}(m),$$
$$\sum_{11l+5m=n}\sigma(l)\tau_{2,11}(m),$$
$$\sum_{55l+m=n}\sigma(l)\tau_{2,11}(m),$$
$$\sum_{l+5m=n}\tau_{2,11}(l)\tau_{2,11}(m),$$

and for all positive integers n using the theory of quasimodular forms, we determine the number of representations of a positive integer n by the forms

$$x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+9\left(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2\right),$$
$$x_1^2+2x_2^2+x_3^2+2x_4^2+5\left(x_5^2+2x_6^2+x_7^2+2x_8^2\right),$$
$$x_1^2+x_1x_2+3x_2^2+x_3^2+x_3x_4+3x_4^2+5\left(x_5^2+x_5x_6+3x_6^2+x_7^2+x_7x_8+3x_8^2\right).$$

Key Words and Phrases: Quasimodular forms, divisor functions, convolution sums, representation number

1. Introduction

Let $\sigma(m)$ be the sum of positive divisors of a positive integer m. It is well known that divisor function σ appears in a number of remarkable identities, including relationship on the Riemann zeta function and the Eisenstein series of modular forms. It was studied by Ramanujan [22], who has found a number of important congruences and identities. It was also used in the counting of the number of nonisomorphic branched coverings of surfaces of genus g with a given ramification type σ, and in the orbitwise counting of $\mathfrak{H}(2)$, see [19]. On the other hand, the work on representation number $r(Q,n)$ of quadratic forms has been started

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by Fermat in 1640 on $Q = x^2 + y^2$. Later the formula $r(Q,n) = 4\left(\sum_{d|n \text{ d is odd}} (-1)^{\frac{d-1}{2}}\right)$ has been proved by Euler. Afterwards it was advanced by Jacobi, see [11] with the proof of

$$r(Q,n) = 8\left(\sum_{d|n4d} d\right), Q = x^2 + y^2 + z^2 + t^2.$$

It would be nice to obtain such simple formulas for other positive definite quadratic forms so that we would be able to understand the number of solutions of the equation $Q = n$ for any positive integer.

Here, we want to study some convolutions of divisor functions

$$W_N(n) := \sum_{\substack{m<n/N \\ m,n\in\mathbb{N}}} \sigma(m)\sigma(n-Nm),$$
$$W_{a,b}(n) := \sum_{\substack{al+bm=n \\ l,m\in\mathbb{N}}} \sigma(l)\sigma(m)$$

where N, a, b positive integers. Some of them were calculated as early as 19th century. For example, $W_1(n)$ was evaluated by [7], [10], and [22]. The convolution sums $W_N(n)$ (for $1 \le N \le 24$ with a few exceptions) and $W_{a,b}(n)$ for $(a,b) \in \{(2,3),(3,4),(3,8),(2,9)\}$ have been evaluated by using either elementary methods or analytic methods (see [1–3, 5–10, 12, 16–18, 20, 22, 24, 25]).

A different important algebraic method by means of quasimodular forms was introduced by Royer in [23] and it was applied in [21], in order to get $W_{15,} W_{3,5}$. The quasimodular forms was defined in [13]. The number of representations of integers by certain quadratic forms (see [3, 4, 6, 12, 15, 21, 24, 25]) have been found by evaluation of these convolution sums. In this article, following the method of Royer [23], and using Magma for the calculations, we evaluate the following convolution sums

$$W_{27}(n), W_{3,9}(n), W_{40}(n), W_{5,8}(n),$$
$$W_{20}(n), W_{4,5}(n), W_{10}(n), W_{2,5}(n),$$
$$W_{55}(n) \text{ and } W_{5,11}(n),$$

by using the theory of quasimodular forms. The integers $27, 40, 20, 55$ are selected, as representatives of large classes, $p^3, q \cdot p^3, qp^2, qp$, respectively. Here p, q different primes.

Although, in general it may be a challenging problem to describe the integer solutions to a polynomial equation of several variables, by using the above convolution sums we will determine the formulas for the number of representations of integers of the following positive definite quadratic forms

$$Q_1 = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 9\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right),$$

$$\begin{aligned}Q_2 =& x_1^2+2x_2^2+x_3^2+2x_4^2+5\left(x_5^2+2x_6^2+x_7^2+2x_8^2\right),\\ Q_3 =& x_1^2+x_1x_2+3x_2^2+x_3^2+x_3x_4+3x_4^2\\ &+5\left(x_5^2+x_5x_6+3x_6^2+x_7^2+x_7x_8+3x_8^2\right).\end{aligned}$$

These quadratic forms are special cases of the following direct sum of binary quadratic forms Q

$$\begin{gathered}Q+Q+p^2(Q+Q), disc(Q)=-p,\\ Q+Q+q(Q+Q), disc(Q)=-p^3,\\ Q+Q+q(Q+Q), disc(Q)=-p.\end{gathered}$$

So our examples are representatives of large classes of octonary quadratic forms. One can also obtain the representation number formulas by generalized theta series without using the convolutions of divisor sums, see [14].

2. Evaluation of $W_{a,b}(n)$

In this section, we evaluate the convolution sums

$$\begin{gathered}W_{27}(n), W_{3,9}(n), W_{40}(n), W_{5,8}(n), W_{20}(n),\\ W_{4,5}(n), W_{10}(n), W_{2,5}(n), W_{55}(n) \text{ and } W_{5,11}(n),\end{gathered}$$

by using the lemma 1.17 in [23]

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(N)] = M_4[\Gamma_0(N)] \oplus DM_2[\Gamma_0(N)] \oplus \mathbb{C}DE_2,$$

about the structure of quasimodular forms, where the differential operator D is defined by $D := \frac{1}{2\pi i}\frac{d}{dz}$. As an application, we use these convolution sums together with the convolution sums

$$\begin{aligned}W_9(n) =& \frac{1}{216}\sigma_3(n)+\frac{1}{27}\sigma_3\left(\frac{n}{3}\right)+\frac{3}{8}\sigma_3\left(\frac{n}{9}\right)-\frac{1}{36}n\sigma_1(n)\\ &-\frac{1}{4}n\sigma_1\left(\frac{n}{9}\right)+\frac{1}{24}\sigma_1(n)+\frac{1}{24}\sigma_1\left(\frac{n}{9}\right)-\frac{1}{54}\tau_{4,9}(n).\\ W_3(n) =& \frac{1}{24}\sigma_3(n)+\frac{3}{8}\sigma_3\left(\frac{n}{3}\right)-\frac{1}{12}n\sigma(n)-\frac{1}{4}n\sigma\left(\frac{n}{3}\right)\\ &+\frac{1}{24}\sigma(n)+\frac{1}{24}\sigma\left(\frac{n}{3}\right),\\ W_5(n) =& \frac{5}{312}\sigma_3(n)+\frac{125}{132}\sigma_3\left(\frac{n}{5}\right)-\frac{1}{20}n\sigma(n)-\frac{1}{4}n\sigma\left(\frac{n}{5}\right)\\ &+\frac{1}{24}\sigma(n)+\frac{1}{24}\sigma\left(\frac{n}{5}\right)-\frac{1}{130}\tau_{4,5}(n),\end{aligned}$$

given in [23] to obtain a formula for the number of representations of a positive integer n by the quadratic forms.

2.1. Evaluation of $W_{27}(n)\ W_{3,9}(n)$

The vector space $M_4[\Gamma_0(27)]$ has dimension 12. The linearly independent Eisenstein forms are E_4, $E_4(3z), E_4(9z), E_4(27z)$,

$$E_4^{\psi,\psi} = \sum_{0<d|n} \sigma_3^{\psi,\psi}(n) q^n = \sum_{n=1}^{\infty} \psi(n)\sigma_3(n) q^n, E_4^{\psi,\psi}(3z)$$

where ψ is the nontrivial Dirichlet character mod 3. There are 4 newforms in $S_4\big(\Gamma_0(27)\big)$,

$$\begin{aligned}\Delta_{4,27,1} &= q + 3q^2 + \cdots + 0q^{27} + O(q^{28})\\ \Delta_{4,27,2} &= q - 3q^2 + \cdots + 0q^{27} + O(q^{28})\\ \Delta_{4,27,3} &= q + sq^2 + \cdots + 0q^{27} + O(q^{28})\end{aligned}$$

and $\Delta_{4,27,4}$. Here $\Delta_{4,27,4}$ is the Galois conjugate of $\Delta_{4,27,3}$ and s is the zero of $x^2 - 18$, i.e., $s = 3\sqrt{2}$. There are 2 old forms,

$$\Delta_{4,9} = (\Delta(3z))^{1/3} = q - 8q^4 + \cdots + 0q^{27} + O(q^{28}),$$

$\Delta_{4,9}(3z)$ in $S_4[\Gamma_0(27)]$.

$M_2\big(\Gamma_0(27)\big)$ 6 dimensional. There is only one newform

$$\Delta_{2,27} = q - 2q^4 + \cdots + 0q^{27} + O(q^{28})$$

and there are 5 linearly independent Eisenstein Series.

$$\Phi_{1,3}, E_2^{\psi,\psi}, \Phi_{1,9}, \Phi_{1,27}, E_2^{\psi,\psi}(3z),$$

where

$$\begin{aligned}\Phi_{a,b} &= \frac{1}{b-a}\left[bE_2(bz) - aE_2(az)\right]\\ E_2^{\psi,\psi}(z) &= \sum_{n=1}^{\infty} \psi(n)\sigma(n) q^n.\end{aligned}$$

Now

$$\begin{aligned}E_2(27z)E_2(z) &= 1 + \sum_{n=1}^{+\infty}\left(-24\left[\sigma(n) + \sigma\left(\frac{n}{27}\right)\right] + 576W_{27}(n)\right)q^n\\ W_{27}(n) &= -\frac{1}{108}n\sigma(n) + \frac{1}{24}\sigma(n) - \frac{1}{4}n\sigma\left(\frac{n}{27}\right) + \frac{1}{24}\sigma\left(\frac{n}{27}\right)\\ &\quad + \frac{1}{1944}\sigma_3(n) + \frac{1}{243}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{27}\sigma_3\left(\frac{n}{9}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{27}\right)\\ &\quad - \frac{1}{81}\tau_{4,27,1}(n) + \frac{1}{972}(-w-6)\tau_{4,27,3}(n) + \frac{1}{972}(w-6)\tau_{4,27,4}(n)\end{aligned}$$

$$-\frac{2}{243}\tau_{4,9}(n)-\frac{2}{27}\tau_{4,9}(\frac{n}{3}).$$

Similarly,

$$\begin{aligned}E_2(3z)E_2(9z)=&1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{9})+\sigma\left(\frac{n}{3}\right)\right]+576W_{3,9}(n)\right)q^n\\=&\frac{1}{10}E_{4,3}(z)+\frac{9}{10}E_{4,9}(z)+\frac{4}{9}D\Phi_{1,3}+\frac{16}{9}D\Phi_{1,9}-\frac{4}{9}DE_2.\end{aligned}$$

So,

$$\begin{aligned}W_{3,9}(n)=&-\frac{1}{36}n\sigma\left(\frac{n}{3}\right)+\frac{1}{24}\sigma\left(\frac{n}{3}\right)-\frac{1}{12}n\sigma\left(\frac{n}{9}\right)\\&+\frac{1}{24}\sigma\left(\frac{n}{9}\right)+\frac{1}{24}\sigma_3\left(\frac{n}{3}\right)+\frac{3}{8}\sigma_3\left(\frac{n}{9}\right).\end{aligned}$$

2.2. Evaluation of $W_{40}(n)$, $W_{5,8}(n)$, $W_{2,20}(n)$, $W_{4,10}(n)$

The vector space $M_4[\Gamma_0(40)]$ has dimension 22 and is spanned by the 8 linearly independent Eisenstein forms

$$E_4, E_{4,2}, E_{4,4}, E_{4,5}, E_{4,8}, E_{4,10}, E_{4,20}, E_{4,40},$$

11 old cusp forms

$$\begin{aligned}&\Delta_{4,8}=(\Delta(2z)\Delta(4z))^{1/6}=q-4q^3+\cdots+0q^{60}+O(q^{61}),\\&\Delta_{4,8}(5z),\\&\Delta_{4,10}(z)=q+2q^2+\cdots+(-160)q^{60}+O(q^{61}),\\&\Delta_{4,10}(2z),\Delta_{4,10}(4z),\\&\Delta_{4,20}(z)=q+4q^3+\cdots+0q^{60}+O(q^{61}),\\&\Delta_{4,20}(2z),\\&\Delta_{4,5}(z)=(\Delta(z)\Delta(5z))^{1/6}=q-4q^2+\cdots+(-80)q^{60}+O(q^{61}),\\&\Delta_{4,5}(2z),\Delta_{4,5}(4z),\Delta_{4,5}(8z),\end{aligned}$$

and three newforms

$$\begin{aligned}\Delta_{4,40,1}(z)=&q+4q^3+\cdots+0q^{60}+O(q^{61}),\\\Delta_{4,40,2}(z)=&q-6q^3+\cdots+0q^{60}+O(q^{61}),\\\Delta_{4,40,3}(z)=&q+10q^3+\cdots+0q^{60}+O(q^{61}).\end{aligned}$$

The vector space $M_2[\Gamma_0(40)]$ has dimension 10. It is spanned by 7 linearly independent Eisenstein series

$$\Phi_{1,5}, \Phi_{1,10}, \Phi_{1,20}, \Phi_{1,40}, \Phi_{1,2}, \Phi_{1,4}, \Phi_{1,8},$$

2 old forms

$$\Delta_{2,20} = q - 2q^3 + \cdots + 0q^{60} + O(q^{61}),$$

$\Delta_{2,20}(2z)$ and one newform

$$\Delta_{2,40} = q + q^5 + \cdots + 0q^{60} + O(q^{61}).$$

So,

$$\begin{aligned}
E_2(40z)E_2(z) =& 1 + \sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{40}) + \sigma(n)\right] + 576W_{40}(n)\right)q^n \\
=& \frac{1}{2080}E_4(z) + \frac{3}{2080}E_{4,2}(z) + \frac{3}{520}E_{4,4}(z) + \frac{5}{416}E_{4,5}(z) + \frac{2}{65}E_{4,8}(z) \\
&+ \frac{15}{416}E_{4,10}(z) + \frac{15}{104}E_{4,20} + \frac{10}{13}E_{4,40} - \frac{9}{2}\Delta_{4,40,1}(z) - \frac{18}{7}\Delta_{4,40,3}(z) \\
&- \frac{27}{14}\Delta_{4,8}(z) - \frac{675}{14}\Delta_{4,8}(5z) - \frac{6}{5}\Delta_{4,10}(z) - \frac{24}{5}\Delta_{4,10}(2z) \\
&- \frac{96}{5}\Delta_{4,10}(4z) - \frac{9}{2}\Delta_{4,20}(z) - 18\Delta_{4,20}(2z) - \frac{378}{65}\Delta_{4,5}(z) \\
&- \frac{648}{13}\Delta_{4,5}(2z) - \frac{2592}{13}\Delta_{4,5}(4z) - \frac{24192}{65}\Delta_{4,5}(8z) + \frac{117}{20}D\Phi_{1,40}(z) \\
&- \frac{3}{10}DE_2(z).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
W_{40}(n) =& -\frac{1}{160}n\sigma(n) + \frac{1}{24}\sigma(n) - \frac{1}{4}n\sigma\left(\frac{n}{40}\right) + \frac{1}{24}\sigma\left(\frac{n}{40}\right) \\
&+ \frac{1}{4992}\sigma_3(n) + \frac{1}{1664}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{416}\sigma_3\left(\frac{n}{4}\right) \\
&+ \frac{25}{4992}\sigma_3\left(\frac{n}{5}\right) + \frac{1}{78}\sigma_3\left(\frac{n}{8}\right) + \frac{25}{1664}\sigma_3\left(\frac{n}{10}\right) \\
&+ \frac{25}{416}\sigma_3\left(\frac{n}{20}\right) + \frac{25}{78}\sigma_3\left(\frac{n}{40}\right) - \frac{1}{128}\tau_{4,40,1}(n) - \frac{1}{224}\tau_{4,40,3}(n) \\
&- \frac{3}{896}\tau_{4,8}(n) - \frac{75}{896}\tau_{4,8}\left(\frac{n}{5}\right) - \frac{1}{480}\tau_{4,10}(n) - \frac{1}{120}\tau_{4,10}\left(\frac{n}{2}\right) \\
&- \frac{1}{30}\tau_{4,10}\left(\frac{n}{4}\right) - \frac{21}{2080}\tau_{4,5}(n) - \frac{9}{10}4\tau_{4,5}\left(\frac{n}{2}\right) \\
&- \frac{9}{26}\tau_{4,5}\left(\frac{n}{4}\right) - \frac{42}{65}\tau_{4,5}\left(\frac{n}{8}\right) - \frac{1}{128}\tau_{4,20}(n) - \frac{1}{32}\tau_{4,20}\left(\frac{n}{2}\right).
\end{aligned}$$

Similarly,

$$E_2(5z)E_2(8z) = 1 + \sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{8}) + \sigma\left(\frac{n}{5}\right)\right] + 576W_{5,8}(n)\right)q^n$$

$$
\begin{aligned}
&=\frac{1}{2080}E_4(z)+\frac{3}{2080}E_{4,2}(z)+\frac{3}{520}E_{4,4}(z)+\frac{5}{416}E_{4,5}(z)+\frac{2}{65}E_{4,8}(z)\\
&\quad+\frac{15}{416}E_{4,10}(z)+\frac{15}{104}E_{4,20}+\frac{10}{13}E_{4,40}+\frac{9}{2}\Delta_{4,40,1}(z)-\frac{18}{7}\Delta_{4,40,3}(z)\\
&\quad-\frac{27}{14}\Delta_{4,8}(z)-\frac{675}{14}\Delta_{4,8}(5z)+\frac{6}{5}\Delta_{4,10}(z)+\frac{24}{5}\Delta_{4,10}(2z)\\
&\quad+\frac{96}{5}\Delta_{4,10}(4z)+\frac{9}{2}\Delta_{4,20}(z)+18\Delta_{4,20}(2z)-\frac{378}{65}\Delta_{4,5}(z)\\
&\quad-\frac{648}{13}\Delta_{4,5}(2z)-\frac{2592}{13}\Delta_{4,5}(4z)+\frac{24192}{65}\Delta_{4,5}(8z)+\frac{3}{5}D\Phi_{1,5}(z)\\
&\quad+\frac{21}{20}D\Phi_{1,8}(z)+\frac{3}{10}DE_2(z).
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
W_{5,8}(n)=&-\frac{1}{32}n\sigma\left(\frac{n}{5}\right)+\frac{1}{24}\sigma\left(\frac{n}{5}\right)-\frac{1}{20}n\sigma\left(\frac{n}{8}\right)\\
&+\frac{1}{24}\sigma\left(\frac{n}{8}\right)+\frac{1}{4992}\sigma_3(n)+\frac{1}{1664}\sigma_3\left(\frac{n}{2}\right)+\frac{1}{416}\sigma_3\left(\frac{n}{4}\right)\\
&+\frac{25}{4992}\sigma_3\left(\frac{n}{5}\right)+\frac{1}{78}\sigma_3\left(\frac{n}{8}\right)+\frac{25}{1664}\sigma_3\left(\frac{n}{10}\right)\\
&+\frac{25}{416}\sigma_3\left(\frac{n}{20}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{40}\right)+\frac{1}{128}\tau_{4,40,1}(n)-\frac{1}{224}\tau_{4,40,3}(n)\\
&-\frac{3}{896}\tau_{4,8}(n)-\frac{75}{896}\tau_{4,8}\left(\frac{n}{5}\right)+\frac{1}{480}\tau_{4,10}(n)+\frac{1}{120}\tau_{4,10}\left(\frac{n}{2}\right)\\
&+\frac{1}{30}\tau_{4,10}\left(\frac{n}{4}\right)-\frac{21}{2080}\tau_{4,5}(n)-\frac{9}{104}\tau_{4,5}\left(\frac{n}{2}\right)-\frac{9}{26}\tau_{4,5}\left(\frac{n}{4}\right)\\
&-\frac{42}{65}\tau_{4,5}\left(\frac{n}{8}\right)+\frac{1}{128}\tau_{4,20}(n)+\frac{1}{32}\tau_{4,20}\left(\frac{n}{2}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E_2(20z)E_2(2z)=&1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{20})+\sigma\left(\frac{n}{2}\right)\right]+576W_{20}(n)\right)q^n\\
=&\frac{1}{130}E_{4,2}(z)+\frac{2}{65}E_{4,4}(z)+\frac{5}{26}E_{4,10}(z)\,\frac{10}{13}E_{4,20}-\frac{24}{5}\Delta_{4,10}(2z)\\
&-\frac{432}{65}\Delta_{4,5}(2z)-\frac{1728}{65}\Delta_{4,5}(4z)+\frac{3}{20}D\Phi_{1,2}+\frac{57}{20}D\Phi_{1,20}+\frac{3}{10}DE_2.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
W_{2,20}(n)=&-\frac{1}{80}n\sigma\left(\frac{n}{2}\right)+\frac{1}{24}\sigma\left(\frac{n}{2}\right)-\frac{1}{8}n\sigma\left(\frac{n}{20}\right)\\
&+\frac{1}{24}\sigma\left(\frac{n}{20}\right)+\frac{1}{312}\sigma_3\left(\frac{n}{2}\right)+\frac{1}{78}\sigma_3\left(\frac{n}{4}\right)
\end{aligned}
$$

$$+\frac{25}{312}\sigma_3\left(\frac{n}{10}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{20}\right)-\frac{1}{120}\tau_{4,10}\left(\frac{n}{2}\right)$$
$$-\frac{3}{260}\tau_{4,5}\left(\frac{n}{2}\right)-\frac{3}{65}\tau_{4,5}\left(\frac{n}{4}\right).$$

Now

$$\begin{aligned}E_2(10z)E_2(4z)=&1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{10})+\sigma\left(\frac{n}{4}\right)\right]+576W_{10,4}(n)\right)q^n\\=&\frac{1}{130}E_{4,2}(z)+\frac{2}{65}E_{4,4}(z)+\frac{5}{26}E_{4,10}(z)+\frac{10}{13}E_{4,20}(z)+\frac{24}{5}\Delta_{4,10}(2z)\\&-\frac{432}{65}\Delta_{4,5}(2z)-\frac{1728}{65}\Delta_{4,5}(4z)+\frac{9}{20}D\Phi_{1,4}+\frac{27}{20}D\Phi_{1,10}+\frac{3}{10}DE_2.\end{aligned}$$

So

$$\begin{aligned}W_{4,10}(n)=&-\frac{1}{40}n\sigma\left(\frac{n}{4}\right)+\frac{1}{24}\sigma\left(\frac{n}{4}\right)-\frac{1}{16}n\sigma\left(\frac{n}{10}\right)+\frac{1}{24}\sigma\left(\frac{n}{10}\right)\\&+\frac{1}{312}\sigma_3\left(\frac{n}{2}\right)+\frac{1}{78}\sigma_3\left(\frac{n}{4}\right)+\frac{25}{312}\sigma_3\left(\frac{n}{10}\right)\\&+\frac{25}{78}\sigma_3\left(\frac{n}{20}\right)+\frac{1}{120}\tau_{4,10}\left(\frac{n}{2}\right)-\frac{3}{260}\tau_{4,5}\left(\frac{n}{2}\right)-\frac{3}{65}\tau_{4,5}\left(\frac{n}{4}\right).\end{aligned}$$

2.3. Evaluation of $W_{20}(n)$, $W_{4,5}(n), W_{2,10}(n)$

The vector space $M_4[\Gamma_0(20)]$ has dimension 12 and is spanned by 6 linearly independent Eisenstein forms

$$E_4, E_{4,2}, E_{4,4}, E_{4,5}, E_{4,10}, E_{4,20},$$

5 old cusp forms

$$\Delta_{4,10}(z), \Delta_{4,10}(2z), \Delta_{4,5}(z), \Delta_{4,5}(2z), \Delta_{4,5}(4z),$$

and one newform

$$\Delta_{4,20}(z)=q+4q^3+\cdots+0q^{60}+O(q^{61}).$$

The vector space $M_2[\Gamma_0(20)]$ has dimension 6 and is spanned by 5 linearly independent Eisenstein forms $\Phi_{1,2}, \Phi_{1,4}, \Phi_{1,5}, \Phi_{1,10}, \Phi_{1,20}$, and one newform $\Delta_{2,20}$. So,

$$\begin{aligned}E_2(20z)E_2(z)=&1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{20})+\sigma(n)\right]+576W_{20}(n)\right)q^n\\=&\frac{1}{520}E_4(z)+\frac{3}{520}E_{4,2}(z)+\frac{2}{65}E_{4,4}(z)+\frac{5}{104}E_{4,5}(z)+\frac{15}{104}E_{4,10}(z)\\&+\frac{10}{13}E_{4,20}(z)-\frac{12}{5}\Delta_{4,10}(z)-\frac{48}{5}\Delta_{4,10}(2z)-6\Delta_{4,20}(z)\\&-\frac{576}{65}\Delta_{4,5}(z)-\frac{720}{13}\Delta_{4,5}(2z)-\frac{9216}{65}\Delta_{4,5}(4z)+\frac{57}{10}D\Phi_{1,20}(z)+\frac{3}{5}DE_2(z).\end{aligned}$$

Hence,

$$\begin{aligned}W_{20}(n)=&-\frac{1}{80}n\sigma(n)+\frac{1}{24}\sigma(n)-\frac{1}{4}n\sigma\left(\frac{n}{20}\right)+\frac{1}{24}\sigma\left(\frac{n}{20}\right)\\&+\frac{1}{1248}\sigma_3(n)+\frac{1}{416}\sigma_3\left(\frac{n}{2}\right)+\frac{1}{78}\sigma_3\left(\frac{n}{4}\right)\\&+\frac{25}{1248}\sigma_3\left(\frac{n}{5}\right)+\frac{25}{416}\sigma_3\left(\frac{n}{10}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{20}\right)\\&-\frac{1}{240}\tau_{4,10}(n)-\frac{1}{60}\tau_{4,10}\left(\frac{n}{2}\right)-\frac{1}{65}\tau_{4,5}(n)-\frac{5}{52}\tau_{4,5}\left(\frac{n}{2}\right)\\&-\frac{16}{65}\tau_{4,5}\left(\frac{n}{4}\right)-\frac{1}{96}\tau_{4,20}(n).\end{aligned}$$

Now,

$$\begin{aligned}E_2(10z)E_2(2z)=&1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{10})+\sigma\left(\frac{n}{2}\right)\right]+576W_{2,10}(n)\right)q^n\\=&\frac{1}{26}E_{4,2}+\frac{25}{26}E_{4,10}-\frac{288}{65}\Delta_{4,5}(2z)+\frac{3}{10}D\Phi_{1,2}(z)+\frac{27}{10}D\Phi_{1,10}(z)\\&+\frac{3}{5}DE_2(z),\end{aligned}$$

so

$$\begin{aligned}W_{2,10}=&-\frac{1}{40}n\sigma\left(\frac{n}{2}\right)+\frac{1}{24}\sigma\left(\frac{n}{2}\right)-\frac{1}{8}n\sigma\left(\frac{n}{10}\right)\\&+\frac{1}{24}\sigma\left(\frac{n}{10}\right)+\frac{5}{312}\sigma_3\left(\frac{n}{2}\right)+\frac{125}{312}\sigma_3\left(\frac{n}{10}\right)\\&-\frac{1}{130}\tau_{4,5}\left(\frac{n}{2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}E_2(4z)E_2(5z)=&1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{5})+\sigma\left(\frac{n}{4}\right)\right]+576W_{4,5}(n)\right)q^n\\=&\frac{1}{520}E_4(z)+\frac{3}{520}E_{4,2}(z)+\frac{2}{65}E_{4,4}(z)+\frac{5}{104}E_{4,5}(z)+\frac{15}{104}E_{4,10}(z)\\&+\frac{10}{13}E_{4,20}(z)+\frac{12}{5}\Delta_{4,10}(z)+\frac{48}{5}\Delta_{4,10}(2z)+6\Delta_{4,20}(z)\\&-\frac{576}{65}\Delta_{4,5}(z)-\frac{720}{13}\Delta_{4,5}(2z)-\frac{9216}{65}\Delta_{4,5}(4z)+\frac{6}{5}D\Phi_{1,5}(z)\\&+\frac{9}{10}D\Phi_{1,4}(z)+\frac{3}{5}DE_2(z).\end{aligned}$$

Therefore,

$$\begin{aligned}W_{4,5}(n) =& -\frac{1}{20}n\sigma\left(\frac{n}{4}\right)+\frac{1}{24}\sigma\left(\frac{n}{4}\right)-\frac{1}{16}n\sigma\left(\frac{n}{5}\right)\\&+\frac{1}{24}\sigma\left(\frac{n}{5}\right)+\frac{1}{1248}\sigma_3(n)+\frac{1}{416}\sigma_3\left(\frac{n}{2}\right)+\frac{1}{78}\sigma_3\left(\frac{n}{4}\right)\\&+\frac{25}{1248}\sigma_3\left(\frac{n}{5}\right)+\frac{25}{416}\sigma_3\left(\frac{n}{10}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{20}\right)\\&+\frac{1}{240}\tau_{4,10}(n)+\frac{1}{60}\tau_{4,10}\left(\frac{n}{2}\right)-\frac{1}{65}\tau_{4,5}(n)-\frac{5}{52}\tau_{4,5}\left(\frac{n}{2}\right)\\&-\frac{16}{65}\tau_{4,5}\left(\frac{n}{4}\right)+\frac{1}{96}\tau_{4,20}(n).\end{aligned}$$

2.4. Evaluation of $W_{10}(n)$ $W_{2,5}(n)$

The vector space $M_4[\Gamma_0(10)]$ has dimension 7 and is spanned by the linearly independent Eisenstein forms E_4, $E_{4,2}, E_{4,5}, E_{4,10}$, 2 old cusp forms $\Delta_{4,5}(z)$, $\Delta_{4,5}(2z)$, 1 newform $\Delta_{4,10}(z)$. The vector space $M_2[\Gamma_0(10)]$ has dimension 3, and spanned by $\Phi_{1,2}, \Phi_{1,5}, \Phi_{1,10}$. Now,

$$\begin{aligned}E_2(10z)E_2(z) =& 1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{10})+\sigma(n)\right]+576W_{10}(n)\right)q^n\\=&\frac{1}{130}E_4+\frac{2}{65}E_{4,2}+\frac{5}{26}E_{4,5}+\frac{10}{13}E_{4,10}-\frac{24}{5}\Delta_{4,10}(z)-\frac{432}{65}\Delta_{4,5}(z)\\&-\frac{1728}{65}\Delta_{4,5}(2z)+\frac{27}{5}D\Phi_{1,10}+\frac{6}{5}DE_2.\end{aligned}$$

So,

$$\begin{aligned}W_{10}(n) =& -\frac{1}{40}n\sigma(n)+\frac{1}{24}\sigma(n)-\frac{1}{4}n\sigma\left(\frac{n}{10}\right)+\frac{1}{24}\sigma\left(\frac{n}{10}\right)\\&+\frac{1}{312}\sigma_3(n)+\frac{1}{78}\sigma_3\left(\frac{n}{2}\right)+\frac{25}{312}\sigma_3\left(\frac{n}{5}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{10}\right)\\&-\frac{1}{120}\tau_{4,10}(n)-\frac{3}{260}\tau_{4,5}(n)-\frac{3}{65}\tau_{4,5}(\frac{n}{2}).\end{aligned}$$

Similarly,

$$\begin{aligned}E_2(2z)E_2(5z) =& 1+\sum_{n=1}^{+\infty}(-24\left[\sigma(\frac{n}{5})+\sigma\left(\frac{n}{2}\right)\right]+576W_{2,5}(n))q^n\\=&\frac{1}{130}E_4+\frac{2}{65}E_{4,2}+\frac{5}{26}E_{4,5}+\frac{10}{13}\Delta_{4,5}(z)+\frac{24}{5}\Delta_{4,10}(z)-\frac{432}{65}\Delta_{4,5}(z)\\&-\frac{1728}{65}\Delta_{4,5}(2z)+\frac{12}{5}D\Phi_{1,5}+\frac{3}{5}\Phi_{1,2}+\frac{6}{5}DE_2.\end{aligned}$$

So,

$$\begin{aligned}W_{2,5}(n) = &-\frac{1}{20}n\sigma(\frac{n}{2})+\frac{1}{24}\sigma(\frac{n}{2})-\frac{1}{8}n\sigma(\frac{n}{5})+\frac{1}{24}\sigma(\frac{n}{5})+\frac{1}{312}\sigma_3(n)\\&+\frac{1}{78}\sigma_3(\frac{n}{2})+\frac{25}{312}\sigma_3(\frac{n}{5})+\frac{25}{78}\sigma_3(\frac{n}{10})+\frac{1}{120}\tau_{4,10}(n)\\&-\frac{3}{260}\tau_{4,5}(n)-\frac{3}{65}\tau_{4,5}\left(\frac{n}{2}\right).\end{aligned}$$

2.5. Evaluation of $W_{55}(n)$ $W_{5,11}(n)$

The vector space $M_4[\Gamma_0(55)]$ has dimension 20. There are 4 linearly independent Eisenstein forms E_4, $E_{4,5}, E_{4,11}, E_{4,55}$, 6 old cusp forms

$$\Delta_{4,5}, \Delta_{4,5}(11z), \Delta_{4,11,2}(z)$$

(the Galois conjugate of

$$\Delta_{4,11,1}(z) = q + sq^2 + \cdots + (-88s+77)q^{55} + O(q^{56})$$

with polynomial $x^2-2x-2, s = 1+\sqrt{3}$),

$$\Delta_{4,11,1}(5z), \Delta_{4,11,2}(5z)$$

and 10 newforms

$$\begin{aligned}\Delta_{4,55,1} = q + q^2 + \cdots + (-55)q^{55} + O(q^{56}),\\ \Delta_{4,55,2}, \Delta_{4,55,3}, \Delta_{4,55,4}, \Delta_{4,55,5}, \Delta_{4,55,6},\\ \Delta_{4,55,7}, \Delta_{4,55,8}, \Delta_{4,55,9}, \Delta_{4,55,10}.\end{aligned}$$

Here

$$\Delta_{4,55,2} = q + rq^2 + \cdots + (-55)q^{55} + O(q^{56}), r = \frac{-7+\sqrt{17}}{2},$$

and $\Delta_{4,55,3}$ is the Galois conjugate of $\Delta_{4,55,2}$ by x^2+7x+8. $\Delta_{4,55,5}$, $\Delta_{4,55,6}$, are the Galois conjugates of

$$\Delta_{4,55,4} = q + uq^2 + \cdots + 55q^{55} + O(q^{56})$$

by $x^3-5x^2-11x+59$, the roots are $u, p, 5-u-p$.

$$\Delta_{4,55,8}, \Delta_{4,55,9}, \Delta_{4,55,10}$$

are the Galois conjugates of

$$\Delta_{4,55,7} = q + wq^2 + \cdots + 55q^{55} + O(q^{56})$$

by

$$x^4 - x^3 - 25x^2 + 9x + 96.$$

The roots are w, g, h and $1-w-g-h$.

The vector space $M_2[\Gamma_0(55)]$ has dimension 8 and is spanned by three new forms $\Delta_{2,55,1}$, $\Delta_{2,55,2}$, $\Delta_{2,55,3}(z)$ (the Galois conjugate of $\Delta_{2,55,2}(z)$ by $x^2-2x-1, t=1+\sqrt{2}$). Moreover, there are two old forms

$$\Delta_{2,11} = q - 2q^2 + \cdots + q^{55} + O\left(q^{56}\right)$$

(the lifting of unique newform in $S_2[\Gamma_0(11)]$), $\Delta_{2,11}(5z)$ and three Eisenstein forms $\Phi_{1,5}, \Phi_{1,11}$, $\Phi_{1,55}$.

Consequently, we get

$$\begin{aligned}
E_2(z)E_2(55z) =& 1 + \sum_{n=1}^{+\infty}(-24\left[\sigma(n)+\sigma\left(\frac{n}{55}\right)\right] + 576\sum_{n=1}^{+\infty}W_{55}(n))q^n \\
=& \frac{1}{3172}E_4(z) + \frac{25}{3172}E_{4,5}(z) + \frac{121}{3172}E_{4,11}(z) + \frac{3025}{3172}E_{4,55}(z) + \frac{324}{55}D\Phi_{1,55} \\
&+ \frac{12}{55}DE_2 - \frac{864}{2665}\Delta_{4,5}(z) - \frac{104544}{2665}\Delta_{4,5}(11z) \\
&+ (\frac{243072}{515999}s - \frac{1074240}{515999})\Delta_{4,11,1}(z) + (\frac{6076800}{515999}s - \frac{26856000}{515999})\Delta_{4,11,1}(5z) \\
&- (\frac{243072}{515999}s + \frac{588096}{515999})\Delta_{4,11,2}(z) - (\frac{6076800}{515999}s + \frac{14702400}{515999})\Delta_{4,11,2}(5z) \\
&+ (\frac{275220}{2238559}u^2 - \frac{1271088}{2238559}u - \frac{7581348}{2238559})\Delta_{4,55,4}(z) \\
&+ ((\frac{-275220}{2238559}u + \frac{105012}{2238559})p - \frac{275220}{2238559}u^2 + \frac{1376100}{2238559}u - \frac{4553928}{2238559})\Delta_{4,55,5}(z) \\
&+ ((\frac{275220}{2238559}u - \frac{105012}{2238559})p - \frac{105012}{2238559}u - \frac{4028868}{2238559})\Delta_{4,55,6}(z) \\
&+ (\frac{67959}{1411190}w^3 + \frac{17223}{705595}w^2 + \frac{1963209}{1411190}w + \frac{2276304}{705595})\Delta_{4,55,7}(z) \\
&+ ((\frac{-67959}{1411190}w + \frac{20481}{282238})g^2 + (-\frac{67959}{1411190}w^2 + \frac{67959}{1411190}w - \frac{132117}{705595})g \\
&- \frac{67959}{1411190}w^3 + \frac{67959}{1411190}w^2 + \frac{339795}{282238}w - \frac{5164239}{1411190})\Delta_{4,55,8}(z) \\
&+ (((\frac{67959}{1411190}w - \frac{20481}{282238})g + (-\frac{20481}{282238}w - \frac{161829}{1411190}))h + (\frac{67959}{1411190}w - \frac{20481}{282238})g^2 \\
&+ (\frac{67959}{1411190}w^2 - \frac{85182}{705595}w + \frac{20481}{282238})g - \frac{20481}{282238}w^2 + \frac{20481}{282238}w - \frac{1302057}{705595})\Delta_{4,55,9}(z) \\
&+ (((\frac{-67959}{1411190}w + \frac{20481}{282238})g + (\frac{20481}{282238}w + \frac{161829}{1411190}))h \\
&+ (\frac{20481}{282238}w + \frac{161829}{1411190})g + \frac{161829}{1411190}w - \frac{2765943}{1411190})\Delta_{4,55,10}(z).
\end{aligned}$$

So,

$$W_{55}(n) = -\frac{1}{220}n\sigma(n) + \frac{1}{24}\sigma(n) - \frac{1}{4}n\sigma\left(\frac{n}{55}\right) + \frac{1}{24}\sigma\left(\frac{n}{55}\right)$$

$$+\frac{5}{38064}\sigma_3(n)+\frac{125}{38064}\sigma_3\left(\frac{n}{5}\right)+\frac{605}{38064}\sigma_3\left(\frac{n}{11}\right)$$
$$+\frac{15125}{38064}\sigma_3\left(\frac{n}{55}\right)+(\frac{7645}{35816944}u^2-\frac{8827}{8954236}u-\frac{210593}{35816944})\tau_{4,55,4}(n)$$
$$+((-\frac{7645}{35816944}u+\frac{2917}{35816944})p+(-\frac{7645}{35816944}u^2+\frac{38225}{35816944}u-\frac{63249}{17908472}))\tau_{4,55,5}(n)$$
$$+((\frac{7645}{35816944}u-\frac{2917}{35816944})p+(-\frac{2917}{35816944}u-\frac{111913}{35816944}))\tau_{4,55,6}(n)$$
$$+(\frac{7551}{90316160}w^3+\frac{5741}{135474240}w^2-\frac{654403}{270948480}w-\frac{47423}{8467140})\tau_{4,55,7}(n)$$
$$+((-\frac{7551}{90316160}w+\frac{6827}{54189696})g^2+(-\frac{7551}{90316160}w^2-\frac{44039}{135474240})g$$
$$+(-\frac{7551}{90316160}w^3+\frac{7551}{90316160}w^2+\frac{37755}{18063232}w-\frac{1721413}{270948480}))\tau_{4,55,8}(n)$$
$$+(((\frac{7551}{90316160}w-\frac{6827}{54189696})g+(-\frac{6827}{54189696}w-\frac{17981}{90316160}))h$$
$$+((\frac{7551}{90316160}w-\frac{6827}{54189696})g^2+(\frac{7551}{90316160}w^2-\frac{14197}{67737120}w+\frac{6827}{54189696})g$$
$$+(-\frac{6827}{54189696}w^2+\frac{6827}{54189696}6w-\frac{144673}{45158080})))\tau_{4,55,9}(n)$$
$$+(((-\frac{7551}{90316160}w+\frac{6827}{54189696})g+(\frac{6827}{54189696}w+\frac{17981}{90316160}))h$$
$$+((\frac{6827}{54189696}w+\frac{17981}{90316160})g+(\frac{17981}{90316160}w-\frac{307327}{90316160})))\tau_{4,55,10}(n)$$
$$+(\frac{422}{515999}s-\frac{1865}{515999})\tau_{4,11,1}(n)+(\frac{10550}{515999}s-\frac{46625}{515999})\tau_{4,11,1}\left(\frac{n}{5}\right)$$
$$+(-\frac{422}{515999}s-\frac{1021}{515999})\tau_{4,11,2}(n)+(-\frac{10550}{515999}s-\frac{25525}{515999})\tau_{4,11,2}\left(\frac{n}{5}\right)$$
$$-\frac{3}{5330}\tau_{4,5}(n)-\frac{363}{5330}\tau_{4,5}\left(\frac{n}{11}\right).$$

Similarly,

$$E_2(5z)E_2(11z)=1+\sum_{n=1}^{+\infty}\left(-24\left[\sigma(\frac{n}{11})+\sigma\left(\frac{n}{5}\right)\right]+576W_{5,11}(n)\right)q^n$$
$$=\frac{1}{3172}E_4(z)+\frac{25}{3172}E_{4,5}(z)+\frac{121}{3172}E_{4,11}(z)+\frac{3025}{3172}E_{4,55}(z)$$
$$+\frac{24}{55}D\Phi_{1,5}+\frac{12}{11}D\Phi_{1,11}+\frac{12}{55}DE_2-\frac{864}{2665}\Delta_{4,5}(z)-\frac{104544}{2665}\Delta_{4,5}(11z)$$
$$+(\frac{243072}{515999}s-\frac{1074240}{515999})\Delta_{4,11,1}(z)+(\frac{6076800}{515999}s-\frac{26856000}{515999})\Delta_{4,11,1}(5z)$$
$$-(\frac{243072}{515999}s+\frac{588096}{515999})\Delta_{4,11,2}(z)-(\frac{6076800}{515999}s+\frac{14702400}{515999})\Delta_{4,11,2}(5z)$$

$$+\left(\frac{275220}{2238559}u^2+\frac{1271088}{2238559}u-\frac{7581348}{2238559}\right)\Delta_{4,55,4}(z)$$
$$+\left(\left(-\frac{275220}{2238559}u+\frac{105012}{2238559}\right)p-\frac{275220}{2238559}u^2+\frac{1376100}{2238559}u-\frac{4553928}{2238559}\right)\Delta_{4,55,5}(z)$$
$$+\left(\left(\frac{275220}{2238559}u-\frac{105012}{2238559}\right)p-\frac{105012}{2238559}u-\frac{4028868}{2238559}\right)\Delta_{4,55,6}(z)$$
$$+\left(-\frac{67959}{1411190}w^3-\frac{17223}{705595}w^2+\frac{1963209}{1411190}w+\frac{2276304}{705595}\right)\Delta_{4,55,7}(z)$$
$$+\left(\left(\frac{67959}{1411190}w-\frac{20481}{282238}\right)g^2+\left(\frac{67959}{1411190}w^2-\frac{67959}{1411190}w+\frac{132117}{705595}\right)g\right.$$
$$\left.+\frac{67959}{1411190}w^3-\frac{67959}{1411190}w^2-\frac{339795}{282238}w+\frac{5164239}{1411190}\right)\Delta_{4,55,8}(z)$$
$$+\left(\left(\left(-\frac{67959}{1411190}w+\frac{20481}{282238}\right)g+\left(\frac{20481}{282238}w+\frac{161829}{1411190}\right)\right)h\right.$$
$$-\left(\frac{67959}{1411190}w-\frac{20481}{282238}\right)g^2-\left(\frac{67959}{1411190}w^2-\frac{85182}{705595}w+\frac{20481}{282238}\right)g$$
$$\left.+\frac{20481}{282238}w^2-\frac{20481}{282238}w+\frac{1302057}{705595}\right)\Delta_{4,55,9}(z)$$
$$+\left(\left(\left(\frac{67959}{1411190}w-\frac{20481}{282238}\right)g-\left(\frac{20481}{282238}w+\frac{161829}{1411190}\right)\right)h\right.$$
$$\left.-\left(\frac{20481}{282238}w+\frac{161829}{1411190}\right)g-\frac{161829}{1411190}w+\frac{2765943}{1411190}\right)\Delta_{4,55,10}(z).$$

So,

$$W_{5,11}(n)=-\frac{1}{44}n\sigma\left(\frac{n}{5}\right)+\frac{1}{24}\sigma\left(\frac{n}{5}\right)-\frac{1}{20}n\sigma\left(\frac{n}{11}\right)$$
$$+\frac{1}{24}\sigma\left(\frac{n}{11}\right)+\frac{5}{38064}\sigma_3(n)+\frac{125}{38064}\sigma_3\left(\frac{n}{5}\right)$$
$$+\frac{605}{38064}\sigma_3\left(\frac{n}{11}\right)+\frac{15125}{38064}\sigma_3\left(\frac{n}{55}\right)$$
$$+\left(\frac{7645}{35816944}u^2-\frac{8827}{8954236}u-\frac{210593}{35816944}\right)\tau_{4,55,4}(n)$$
$$+\left(\left(-\frac{7645}{35816944}u+\frac{2917}{35816944}\right)p+\left(-\frac{7645}{35816944}u^2+\frac{38225}{35816944}u-\frac{63249}{17908472}\right)\right)\tau_{4,55,5}(n)$$
$$+\left(\left(\frac{7645}{35816944}u-\frac{2917}{35816944}\right)p+\left(-\frac{2917}{35816944}u-\frac{111913}{35816944}\right)\right)\tau_{4,55,6}(n)$$
$$+\left(-\frac{7551}{90316160}w^3-\frac{5741}{135474240}w^2+\frac{654403}{270948480}w+\frac{47423}{8467140}\right)\tau_{4,55,7}(n)$$
$$+\left(\left(\frac{7551}{90316160}w-\frac{6827}{54189696}\right)g^2+\left(\frac{7551}{90316160}w^2-\frac{7551}{90316160}w+\frac{44039}{135474240}\right)g\right.$$
$$\left.+\left(\frac{7551}{90316160}w^3-\frac{7551}{90316160}w^2-\frac{37755}{18063232}w+\frac{1721413}{270948480}\right)\right)\tau_{4,55,8}(n)$$
$$+\left(\left(\left(-\frac{7551}{90316160}w+\frac{6827}{54189696}\right)g+\left(\frac{6827}{54189696}w+\frac{17981}{90316160}\right)\right)h\right.$$

$$+((-\frac{7551}{90316160}w+\frac{6827}{54189696})g^2+(-\frac{7551}{90316160}w^2+\frac{14197}{67737120}w-\frac{6827}{54189696})g$$
$$+(\frac{6827}{54189696}w^2+\frac{6827}{54189696}w+\frac{144673}{45158080})))\tau_{4,55,9}(n)$$
$$+(((\frac{7551}{90316160}w-\frac{6827}{54189696})g+(-\frac{6827}{54189696}w-\frac{17981}{90316160}))h$$
$$+((-\frac{6827}{54189696}w-\frac{17981}{90316160})g+(-\frac{17981}{90316160}w+\frac{307327}{90316160})))\tau_{4,55,10}(n)$$
$$+(\frac{422}{515999}s-\frac{1865}{515999})\tau_{4,11,1}(n)$$
$$+(\frac{10550}{515999}s-\frac{46625}{515999})\tau_{4,11,1}\left(\frac{n}{5}\right)+(-\frac{422}{515999}s-\frac{1021}{515999})\tau_{4,11,2}(n)$$
$$+(-\frac{10550}{515999}+s\frac{25525}{515999})\tau_{4,11,2}\left(\frac{n}{5}\right)-\frac{3}{5330}\tau_{4,5}(n)-\frac{363}{5330}\tau_{4,5}\left(\frac{n}{11}\right).$$

By the same method, we calculate

$$W_{\sigma a,\tau b}(n)=\sum_{\substack{k,m\in\mathbb{Z}\\ ak+bm=n}}\sigma(k)\,\tau_{2,11}(m)\ \text{ for some } a \text{ and } b.$$

$$W_{\sigma,\tau 5}(n)=-\frac{1}{20}\tau_{4,55,1}(n)+(-\frac{15}{2006}r-\frac{275}{4012})\tau_{4,55,2}(n)+(\frac{15}{2006}r-\frac{65}{4012})\tau_{4,55,3}(n)$$
$$+(\frac{1695}{4477118}u^2-\frac{73771}{17908472}u+\frac{253919}{17908472})\tau_{4,55,4}(n)$$
$$+((-\frac{1695}{4477118}u-\frac{39871}{17908472})p+(-\frac{1695}{4477118}u^2+\frac{8475}{4477118}u+\frac{328499}{17908472}))\tau_{4,55,5}(n)$$
$$+((\frac{1695}{4477118}u+\frac{39871}{17908472})p+(\frac{39871}{17908472}u+\frac{16143}{2238559}))\tau_{4,55,6}(n)$$
$$+(\frac{1513}{2463168}w^3-\frac{5999}{3078960}w^2-\frac{162281}{12315840}w+\frac{11703}{256580})\tau_{4,55,7}(n)$$
$$+((-\frac{1513}{2463168}w-\frac{5477}{4105280})g^2+(-\frac{1513}{2463168}w^2+\frac{1513}{2463168}w+\frac{2237}{1026320})g$$
$$+(-\frac{1513}{2463168}w^3+\frac{1513}{2463168}w^2+\frac{37825}{2463168}w+\frac{164553}{4105280}))\tau_{4,55,8}(n)$$
$$+(((\frac{1513}{2463168}w+\frac{5477}{4105280})g+(\frac{5477}{4105280}w+\frac{3471}{4105280}))h$$
$$+((\frac{1513}{2463168}w+\frac{5477}{4105280})g^2+(\frac{1513}{2463168}w^2+\frac{4433}{6157920}w-\frac{5477}{4105280})g$$
$$+(\frac{5477}{4105280}w^2-\frac{5477}{4105280}w+\frac{6907}{1026320})))\tau_{4,55,9}(n)$$
$$+(((-\frac{1513}{2463168}w-\frac{5477}{4105280})g+(-\frac{5477}{4105280}w-\frac{3471}{4105280}))h$$
$$+((-\frac{5477}{4105280}w-\frac{3471}{4105280})g+(-\frac{3471}{4105280}w+\frac{31099}{4105280})))\tau_{4,55,10}(n)$$

$$+(-\frac{691}{90742}s+\frac{675}{45371})\tau_{4,11,1}(n)+(-\frac{318475}{2177808}s+\frac{885625}{1088904})\tau_{4,11,1}\left(\frac{n}{5}\right)$$
$$+(\frac{691}{90742}s-\frac{16}{45371})\tau_{4,11,2}(n)+(\frac{318475}{2177808}s+\frac{94525}{181484})\tau_{4,11,2}\left(\frac{n}{5}\right)$$
$$-\frac{4}{205}\tau_{4,5}(n)+\frac{3993}{410}\tau_{4,5}\left(\frac{n}{11}\right)-\frac{1}{4}n\tau_{2,11}\left(\frac{n}{5}\right)+\frac{1}{24}\tau_{2,11}\left(\frac{n}{5}\right),$$

$$W_{\sigma 5,\tau}(n)=-\frac{1}{20}\tau_{4,55,1}(n)+(\frac{15}{2006}r+\frac{275}{4012})\tau_{4,55,2}(n)+(-\frac{15}{2006}r+\frac{65}{4012})\tau_{4,55,3}(n)$$
$$+(\frac{1695}{4477118}u^2-\frac{73771}{17908472}u+\frac{253919}{17908472})\tau_{4,55,4}(n)$$
$$+((-\frac{1695}{4477118}u-\frac{39871}{17908472})p+(-\frac{1695}{4477118}u^2+\frac{8475}{4477118}u+\frac{328499}{17908472}))\tau_{4,55,5}(n)$$
$$+((\frac{1695}{4477118}u+\frac{39871}{17908472})p+(\frac{39871}{17908472}u+\frac{16143}{2238559}))\tau_{4,55,6}(n)$$
$$+(-\frac{1513}{2463168}w^3+\frac{5999}{3078960}w^2+\frac{162281}{12315840}w-\frac{11703}{256580})\tau_{4,55,7}(n)$$
$$+((\frac{1513}{2463168}w+\frac{5477}{4105280})g^2+(\frac{1513}{2463168}w^2-\frac{1513}{2463168}w-\frac{2237}{1026320})g$$
$$+(\frac{1513}{2463168}w^3-\frac{1513}{2463168}w^2-\frac{37825}{2463168}w-\frac{164553}{4105280}))\tau_{4,55,8}(n)$$
$$+(((-\frac{1513}{2463168}w-\frac{5477}{4105280})g+(-\frac{5477}{4105280}w-\frac{3471}{4105280}))h$$
$$+((-\frac{1513}{2463168}w-\frac{5477}{4105280})g^2+(-\frac{1513}{2463168}w^2-\frac{4433}{6157920}w+\frac{5477}{4105280})g$$
$$+(-\frac{5477}{4105280}w^2+\frac{5477}{4105280}w-\frac{6907}{1026320})))\tau_{4,55,9}(n)$$
$$+(((\frac{1513}{2463168}w+\frac{5477}{4105280})g+(\frac{5477}{4105280}w+\frac{3471}{4105280}))h$$
$$+((\frac{5477}{4105280}w+\frac{3471}{4105280})g+(\frac{3471}{4105280}w-\frac{31099}{4105280})))\tau_{4,55,10}(n)$$
$$+(-\frac{12739}{2177808}s+\frac{35425}{1088904})\tau_{4,11,1}(n)+(-\frac{17275}{90742}s+\frac{16875}{45371})\tau_{4,11,1}\left(\frac{n}{5}\right)$$
$$+(\frac{12739}{2177808}s+\frac{3781}{181484})\tau_{4,11,2}(n)+(\frac{17275}{90742}s-\frac{400}{45371})\tau_{4,11,2}\left(\frac{n}{5}\right)$$
$$-\frac{4}{205}\tau_{4,5}(n)+\frac{3993}{410}\tau_{4,5}\left(\frac{n}{11}\right)-\frac{1}{20}n\tau_{2,11}(n)+\frac{1}{24}\tau_{2,11}(n)$$

$$W_{\sigma 55,\tau}(n)=-\frac{1}{220}\tau_{4,55,1}(n)+(-\frac{15}{22066}r-\frac{25}{4012})\tau_{4,55,2}(n)+(\frac{15}{22066}r-\frac{65}{44132})\tau_{4,55,3}(n)$$
$$+(-\frac{1695}{49248298}u^2+\frac{73771}{196993192}u-\frac{253919}{196993192})\tau_{4,55,4}(n)$$

$$
\begin{aligned}
&+((\frac{1695}{49248298}u+\frac{39871}{196993192})p+(\frac{1695}{49248298}u^2-\frac{8475}{49248298}u-\frac{328499}{196993192}))\tau_{4,55,5}(n)\\
&+((-\frac{1695}{49248298}u-\frac{39871}{196993192})p+(-\frac{39871}{196993192}u-\frac{16143}{24624149}))\tau_{4,55,6}(n)\\
&+(-\frac{1513}{27094848}w^3+\frac{5999}{33868560}w^2+\frac{162281}{135474240}w-\frac{11703}{2822380})\tau_{4,55,7}(n)\\
&+((\frac{1513}{27094848}w+\frac{5477}{45158080})g^2+(\frac{1513}{27094848}w^2-\frac{1513}{27094848}w-\frac{2237}{11289520})g\\
&+(\frac{1513}{27094848}w^3-\frac{1513}{27094848}w^2-\frac{37825}{27094848}w-\frac{164553}{45158080}))\tau_{4,55,8}(n)\\
&+(((-\frac{1513}{27094848}w-\frac{5477}{45158080})g\\
&+(-\frac{5477}{45158080}w-\frac{3471}{45158080}))h+((-\frac{1513}{27094848}w-\frac{5477}{45158080})g^2\\
&+(-\frac{1513}{27094848}w^2-\frac{403}{6157920}w+\frac{5477}{45158080})g\\
&+(-\frac{5477}{45158080}w^2+\frac{5477}{45158080}w-\frac{6907}{11289520})))\tau_{4,55,9}(n)\\
&+(((\frac{1513}{27094848}w+\frac{5477}{45158080})g+(\frac{5477}{45158080}w+\frac{3471}{45158080}))h\\
&+((\frac{5477}{45158080}w+\frac{3471}{45158080})g+(\frac{3471}{45158080}w-\frac{31099}{45158080})))\tau_{4,55,10}(n)\\
&+(\frac{12739}{23955888}s-\frac{35425}{11977944})\tau_{4,11,1}(n)+(\frac{17275}{998162}s-\frac{16875}{499081})\tau_{4,11,1}\left(\frac{n}{5}\right)\\
&+(-\frac{12739}{23955888}s-\frac{3781}{1996324})\tau_{4,11,2}(n)+(-\frac{17275}{998162}s+\frac{400}{499081})\tau_{4,11,2}\left(\frac{n}{5}\right)\\
&-\frac{3}{410}\tau_{4,5}(n)+\frac{44}{205}\tau_{4,5}\left(\frac{n}{11}\right)-\frac{1}{220}n\tau_{2,11}(n)+\frac{1}{24}\tau_{2,11}(n),
\end{aligned}
$$

$$
\begin{aligned}
W_{\sigma 11,\tau 5}(n)=&-\frac{1}{220}\tau_{4,55,1}(n)+(\frac{15}{22066}r+\frac{25}{4012})\tau_{4,55,2}(n)+(-\frac{15}{22066}r+\frac{65}{44132})\tau_{4,55,3}(n)\\
&+(-\frac{1695}{49248298}u^2+\frac{73771}{196993192}u-\frac{253919}{196993192})\tau_{4,55,4}(n)\\
&+((\frac{1695}{49248298}u+\frac{39871}{196993192})p+(\frac{1695}{49248298}u^2-\frac{8475}{49248298}u-\frac{328499}{196993192}))\tau_{4,55,5}(n)\\
&+((-\frac{1695}{49248298}u-\frac{39871}{196993192})p+(-\frac{39871}{196993192}u-\frac{16143}{24624149}))\tau_{4,55,6}(n)\\
&+(\frac{1513}{27094848}w^3-\frac{5999}{33868560}w^2-\frac{162281}{135474240}w+\frac{11703}{2822380})\tau_{4,55,7}(n)\\
&+((-\frac{1513}{27094848}w-\frac{5477}{45158080})g^2+(-\frac{1513}{27094848}w^2+\frac{1513}{27094848}w+\frac{2237}{11289520})g\\
&+(-\frac{1513}{27094848}w^3+\frac{1513}{27094848}w^2+\frac{37825}{27094848}w+\frac{164553}{45158080}))\tau_{4,55,8}(n)
\end{aligned}
$$

$$
\begin{aligned}
&+(((\frac{1513}{27094848}w+\frac{5477}{45158080})g+(\frac{5477}{45158080}w+\frac{3471}{45158080}))h\\
&+((\frac{1513}{27094848}w+\frac{5477}{45158080})g^2+(\frac{1513}{27094848}w^2+\frac{403}{6157920}w-\frac{5477}{45158080})g\\
&+(\frac{5477}{45158080}w^2-\frac{5477}{45158080}w+\frac{6907}{11289520})))\tau_{4,55,9}(n)\\
&+(((-\frac{1513}{27094848}w-\frac{5477}{45158080})g+(-\frac{5477}{45158080}w-\frac{3471}{45158080}))h\\
&+((-\frac{5477}{45158080}w-\frac{3471}{45158080})g+(-\frac{3471}{45158080}w+\frac{31099}{45158080})))\tau_{4,55,10}(n)\\
&+(\frac{691}{998162}s-\frac{675}{499081})\tau_{4,11,1}(n)+(\frac{318475}{23955888}s-\frac{885625}{11977944})\tau_{4,11,1}\left(\frac{n}{5}\right)\\
&+(-\frac{691}{998162}s+\frac{16}{499081})\tau_{4,11,2}(n)+(-\frac{318475}{23955888}s-\frac{94525}{1996324})\tau_{4,11,2}\left(\frac{n}{5}\right)\\
&-\frac{3}{410}\tau_{4,5}(n)+\frac{44}{205}\tau_{4,5}\left(\frac{n}{11}\right)-\frac{1}{44}n\tau_{2,11}\left(\frac{n}{5}\right)+\frac{1}{24}\tau_{2,11}\left(\frac{n}{5}\right),
\end{aligned}
$$

$$
\begin{aligned}
W_{\tau,\tau 5}(n)=&(-\frac{445}{54599}u^2+\frac{1329}{218396}u+\frac{26429}{218396})\tau_{4,55,4}(n)\\
&+((\frac{445}{54599}u-\frac{7571}{218396})p+(\frac{445}{54599}u^2-\frac{2225}{54599}u+\frac{6849}{218396}))\tau_{4,55,5}(n)\\
&+((-\frac{445}{54599}u+\frac{7571}{218396})p+(\frac{7571}{218396}u-\frac{15503}{109198}))\tau_{4,55,6}(n)\\
&+(\frac{31}{4614}s-\frac{55}{4614})\tau_{4,11,1}(n)+(\frac{775}{4614}s-\frac{1375}{4614})\tau_{4,11,1}\left(\frac{n}{5}\right)\\
&+(-\frac{31}{4614}s+\frac{7}{4614})\tau_{4,11,2}(n)+(-\frac{775}{4614}s+\frac{175}{4614})\tau_{4,11,2}\left(\frac{n}{5}\right).
\end{aligned}
$$

3. Application to the Number of Representations

Theorem 1. *The number $N_i(n)$ of representations of a positive integer n by the quadratic forms*

$$
\begin{aligned}
Q_1=&x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2+9\left(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2\right),\\
Q_2=&x_1^2+2x_2^2+x_3^2+2x_4^2+5\left(x_5^2+2x_6^2+x_7^2+2x_8^2\right),\\
Q_3=&x_1^2+x_1x_2+3x_2^2+x_3^2+x_3x_4+3x_4^2+5\left(x_5^2+x_5x_6+3x_6^2+x_7^2+x_7x_8+3x_8^2\right),
\end{aligned}
$$

are equal to

$$
\begin{aligned}
N_1(n)=&\frac{4}{9}\sigma_3(n)-\frac{76}{9}\sigma_3(\frac{n}{3})-76\sigma_3(\frac{n}{9})+324\sigma_3(\frac{n}{27})+\frac{16}{3}\tau_{4,27,1}(n)\\
&+\frac{1}{9}(4w+24)\tau_{4,27,3}(n)+\frac{1}{9}(-4w+24)\tau_{4,27,4}(n)+\frac{8}{9}\tau_{4,9}(n)+8\tau_{4,27,1}\left(\frac{n}{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
N_2(n) =& \frac{24}{5} n\sigma\left(\frac{n}{8}\right) - 32\sigma\left(\frac{n}{8}\right) + 24n\sigma\left(\frac{n}{40}\right) - 32\sigma\left(\frac{n}{40}\right) \\
&+ \frac{2}{13}\sigma_3(n) - \frac{2}{13}\sigma_3\left(\frac{n}{2}\right) - \frac{8}{13}\sigma_3\left(\frac{n}{4}\right) + \frac{50}{13}\sigma_3\left(\frac{n}{5}\right) \\
&- \frac{32}{13}\sigma_3\left(\frac{n}{8}\right) - \frac{50}{13}\sigma_3\left(\frac{n}{10}\right) - \frac{200}{13}\sigma_3\left(\frac{n}{20}\right) \\
&- \frac{800}{13}\sigma_3\left(\frac{n}{40}\right) + \frac{8}{7}\tau_{4,40,3}(n) + \frac{6}{7}\tau_{4,8}(n) + \frac{150}{7}\tau_{4,8}\left(\frac{n}{5}\right) \\
&+ \frac{24}{13}\tau_{4,5}(n) + \frac{184}{13}\tau_{4,5}\left(\frac{n}{2}\right) + \frac{736}{13}\tau_{4,5}\left(\frac{n}{4}\right) + \frac{8064}{65}\tau_{4,5}\left(\frac{n}{8}\right),
\end{aligned}
$$

$$
\begin{aligned}
N_3(n) =& \frac{60}{793}\sigma_3(n) + \frac{1500}{793}\sigma_3\left(\frac{n}{5}\right) + \frac{7260}{793}\sigma_3\left(\frac{n}{11}\right) \\
&+ \frac{181500}{793}\sigma_3\left(\frac{n}{55}\right) + \left(-\frac{94238}{2238559}u^2 + \frac{172872}{2238559}u + \frac{2848918}{2238559}\right)\tau_{4,55,4}(n) \\
&+ \left(\left(\frac{94238}{2238559}u - \frac{298318}{2238559}\right)p + \left(\frac{94238}{2238559}u^2 - \frac{471190}{2238559}u + \frac{1812300}{2238559}\right)\right)\tau_{4,55,5}(n) \\
&+ \left(\left(-\frac{94238}{2238559}u + \frac{298318}{2238559}\right)p + \left(\frac{298318}{2238559}u + \frac{320710}{2238559}\right)\right)\tau_{4,55,6}(n) + \left(-\frac{26716}{140727}s \right.\\
&\left.+ \frac{111400}{140727}\right)\tau_{4,11,1}(n) + \left(-\frac{667900}{140727}s + \frac{2785000}{140727}\right)\tau_{4,11,1}\left(\frac{n}{5}\right) \\
&+ \left(\frac{26716/140727}{140727}s + \frac{57968}{140727}\right)\tau_{4,11,2}(n) + \left(\frac{667900}{140727}s + \frac{1449200}{140727}\right)\tau_{4,11,2}\left(\frac{n}{5}\right) \\
&+ \frac{264}{533}\tau_{4,5}(n) + \frac{31944}{533}\tau_{4,5}\left(\frac{n}{11}\right)
\end{aligned}
$$

respectively.

Proof. Let

$$
r_1(l) = \#\left\{ \begin{array}{c} (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : \\ x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 = l \end{array} \right\}
$$

for $l \in \{0\} \cup \mathbb{N}$. Since $M_2(\Gamma_0(3))$ is spanned by $\Phi_{1,3}$, it follows that

$$
r_1(l) = 12\left(\sigma(l) - 3\sigma\left(\frac{l}{3}\right)\right).
$$

Now

$$
\begin{aligned}
N_1(n) =& r_1(0)\, r_1\left(\frac{n}{9}\right) + r_1(n)\, r_1(0) + \sum_{\substack{l,m\in\mathbb{N} \\ l+9m=n}} r_1(l)\, r_1(m) \\
=& 12\sigma\left(\frac{n}{9}\right) - 36\sigma\left(\frac{n}{27}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right)
\end{aligned}
$$

$$+\sum_{\substack{l,m\in\mathbb{N}\\ l+9m=n}}\left(12\sigma(l)-36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m)-36\sigma\left(\frac{m}{3}\right)\right)$$

$$=12\sigma\left(\frac{n}{9}\right)-36\sigma\left(\frac{n}{27}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)$$

$$+144\sum_{\substack{l,m\in\mathbb{N}\\ l+9m=n}}\sigma(l)\sigma(m)-432\sum_{\substack{l,m\in\mathbb{N}\\ l+9m=n}}\sigma(l)\sigma\left(\frac{m}{3}\right)$$

$$-432\sum_{\substack{l,m\in\mathbb{N}\\ l+9m=n}}\sigma\left(\frac{l}{3}\right)\sigma(m)+1296\sum_{\substack{l,m\in\mathbb{N}\\ l+9m=n}}\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right)$$

$$=12\sigma\left(\frac{n}{9}\right)-36\sigma\left(\frac{n}{27}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)$$

$$+144W_9(n)-432W_{27}(n)-432W_{3,9}(n)+1296W_9\left(\frac{n}{3}\right)$$

$$=12\sigma\left(\frac{n}{9}\right)-36\sigma\left(\frac{n}{27}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)$$

$$+\frac{2}{3}\sigma_3(n)+\frac{16}{3}\sigma_3\left(\frac{n}{3}\right)+54\sigma_3\left(\frac{n}{9}\right).$$

Let

$$r_2(l)=\#\left\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4: x_1^2+2x_2^2+x_3^2+2x_4^2=l\right\}$$

for $l\in\{0\}\cup\mathbb{N}$.

$M_2(\Gamma_0(8))$ is spanned by three linearly independent Eisenstein series

$$\Phi_{1,2},\Phi_{1,4}\text{ and }\Phi_{1,8}.$$

The theta series of $x_1^2+2x_2^2+x_3^2+2x_4^2$ is

$$1+4q+8q^2+16q^3+24q^4+24q^5+\dots$$

$$=\frac{1}{12}(1+\sum_{n=1}^{+\infty}\left(24\sigma(n)-48\sigma\left(\frac{n}{2}\right)\right)q^n)$$

$$-\frac{1}{4}(1+\sum_{n=1}^{+\infty}\left(8\sigma(n)-32\sigma\left(\frac{n}{4}\right)\right)q^n)$$

$$+\frac{7}{6}(1+\sum_{n=1}^{+\infty}\left(\frac{24}{7}\sigma(n)-\frac{8\cdot 24}{7}\cdot\sigma\left(\frac{n}{8}\right)\right)q^n),$$

so,

$$r_2(l)=4\sigma(l)-4\sigma\left(\frac{l}{2}\right)+8\sigma\left(\frac{l}{4}\right)-32\sigma\left(\frac{l}{8}\right).$$

Now

$$N_2(n)=r_2(0)r_2\left(\frac{n}{5}\right)+r_2(n)r_2(0)+\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}}r_2(l)r_2(m)$$

$$=4\sigma\left(\frac{n}{5}\right)-4\sigma\left(\frac{n}{10}\right)+8\sigma\left(\frac{n}{20}\right)-32\sigma\left(\frac{n}{40}\right)$$
$$+4\sigma(n)-4\sigma\left(\frac{n}{2}\right)+8\sigma\left(\frac{n}{4}\right)-32\sigma\left(\frac{n}{8}\right)+$$
$$\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}}\left(4\sigma(l)-4\sigma\left(\frac{l}{2}\right)+8\sigma\left(\frac{l}{4}\right)-32\sigma\left(\frac{l}{8}\right)\right)$$
$$\left(4\sigma(m)-4\sigma\left(\frac{m}{2}\right)+8\sigma\left(\frac{m}{4}\right)-32\sigma\left(\frac{m}{8}\right)\right).$$

So,

$$N_2(n)=4\sigma(n)-4\sigma\left(\frac{n}{2}\right)+8\sigma\left(\frac{n}{4}\right)+4\sigma\left(\frac{n}{5}\right)-32\sigma\left(\frac{n}{8}\right)$$
$$-4\sigma\left(\frac{n}{10}\right)+8\sigma\left(\frac{n}{20}\right)-32\sigma\left(\frac{n}{40}\right)+16W_5(n)$$
$$-16W_{10}(n)+32W_{20}(n)-128W_{40}(n)-16W_{2,5}(n)$$
$$+16W_5\left(\frac{n}{2}\right)-32W_{10}\left(\frac{n}{2}\right)+128W_{20}\left(\frac{n}{2}\right)$$
$$+32W_{4,5}(n)-32W_{2,5}\left(\frac{n}{2}\right)+64W_5\left(\frac{n}{4}\right)$$
$$-32\cdot 8W_{10}\left(\frac{n}{4}\right)-32\cdot 4W_{5,8}(n)+32\cdot 4W_{4,5}\left(\frac{n}{2}\right)$$
$$-32\cdot 8W_{2,5}\left(\frac{n}{4}\right)+32\cdot 8W_5\left(\frac{n}{8}\right).$$

Let

$$r_3(l)=\#\left\{\begin{array}{c}(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4:\\ x_1^2+x_1x_2+3x_2^2+x_3^2+x_3x_4+3x_4^2=l\end{array}\right\}$$

for $l\in\{0\}\cup\mathbb{N}$. Since $\mathrm{M}_2\left(\Gamma_0(11)\right)$ is spanned by $\Phi_{1,11}$ and

$$\Delta_{2,11}(z)=\sum_{n=1}^{\infty}\tau_{2,11}(n)q^n=\left(\Delta(q)\Delta\left(q^{11}\right)\right)^{1/12}$$
$$=q\prod_{n=1}^{\infty}\left(1-q^n\right)^2\prod_{n=1}^{\infty}\left(1-q^{11n}\right)^2,$$

we have

$$r_3(l)=\frac{12}{5}\sigma(l)-\frac{132}{5}\sigma\left(\frac{l}{11}\right)+\frac{8}{5}\tau_{2,11}(l).$$

So,

$$N_3(n)=r_3(0)r_3\left(\frac{n}{5}\right)+r_3(n)r_3(0)+\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}}r_3(l)r_3(m)$$

$$
\begin{aligned}
&=\frac{12}{5}\sigma(n)-\frac{132}{5}\sigma\left(\frac{n}{11}\right)+\frac{8}{5}\tau_{2,11}(n)+\frac{12}{5}\sigma\left(\frac{n}{5}\right)\\
&-\frac{132}{5}\sigma\left(\frac{n}{55}\right)+\frac{8}{5}\tau_{2,11}\left(\frac{n}{5}\right)+\frac{144}{25}W_5(n)\\
&-\frac{1584}{25}W_{55}(n)-\frac{1584}{25}W_{5,11}(n)+\frac{17\,424}{25}W_5\left(\frac{n}{11}\right)\\
&+\frac{96}{25}(W_{\sigma,\tau 5}+W_{\sigma 5,\tau})-\frac{1056}{25}(W_{\sigma 11,\tau 5}+W_{\sigma 55,\tau})+\frac{64}{25}W_{\tau,\tau 5}
\end{aligned}
$$

where

$$
\begin{aligned}
W_{\sigma,\tau 5}&=\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}}\sigma(l)\,\tau_{2,11}(m),\quad W_{\sigma 5,\tau}=\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}}\tau_{2,11}(l)\,\sigma(m)\\
W_{\sigma 11,\tau 5}&=\sum_{\substack{l,m\in\mathbb{N}\\ 11l+5m=n}}\sigma(l)\,\tau_{2,11}(m),\quad W_{\sigma 55,\tau}=\sum_{\substack{l,m\in\mathbb{N}\\ l+55m=n}}\tau_{2,11}(l)\,\sigma(m)\\
W_{\tau,\tau 5}&=\sum_{\substack{l,m\in\mathbb{N}\\ l+5m=n}}\tau_{2,11}(l)\,\tau_{2,11}(m).
\end{aligned}
$$

□

3.1. Numerical Examples

In Tables 1 through 6, we show the calculations which have been done by the formulas obtained in this article.

Table 1: Some Convolution Sums and the Representation Numbers N_1, N_2 for $n = 1 - 25$.

n	$W_{27}(n)$	$W_{3,9}(n)$	$N_1(n)$	$W_{40}(n)$	$W_{5,8}(n)$	$N_2(n)$
1	0	0	12	0	0	4
2	0	0	36	0	0	8
3	0	0	12	0	0	16
4	0	0	84	0	0	24
5	0	0	72	0	0	28
6	0	0	36	0	0	48
7	0	0	96	0	0	64
8	0	0	180	0	0	88
9	0	0	24	0	0	148
10	0	0	360	0	0	152
11	0	0	576	0	0	208
12	0	1	228	0	0	288
13	0	0	1176	0	1	280
14	0	0	1152	0	0	464
15	0	3	504	0	0	496
16	0	0	1524	0	0	536
17	0	0	2376	0	0	840
18	0	4	216	0	3	776
19	0	0	3264	0	0	1136
20	0	0	3528	0	0	1320
21	0	10	1536	0	3	1216
22	0	0	5472	0	0	1856
23	0	0	6336	0	4	1728
24	0	15	2340	0	0	2336
25	0	0	8292	0	0	2836

Table 2: The Same Convolution Sums and the Representation Numbers N_1, N_2 for $n = 26 - 50$.

n	$W_{27}(n)$	$W_{3,9}(n)$	$N_1(n)$	$W_{40}(n)$	$W_{5,8}(n)$	$N_2(n)$
26	0	0	9576	0	9	2448
27	0	24	888	0	0	3616
28	1	0	11472	0	7	3264
29	3	0	12024	0	4	3992
30	4	33	4536	0	0	5136
31	7	0	12624	0	12	4032
32	6	0	15444	0	0	5912
33	12	45	5328	0	6	5824
34	8	0	19656	0	12	6224
35	15	0	16560	0	0	8512
36	13	65	1752	0	21	6648
37	18	0	20760	0	7	8376
38	12	0	27936	0	12	9120
39	28	77	8232	0	16	9184
40	14	0	30312	0	0	12520
41	24	0	29448	1	18	10536
42	24	102	12384	3	21	11936
43	31	0	34224	4	8	13168
44	18	0	43776	7	28	12768
45	39	143	3024	6	6	17164
46	20	0	52416	12	36	14448
47	42	0	42912	8	28	15520
48	32	155	19236	15	15	17824
49	36	0	52812	13	24	17764
50	24	0	67212	18	18	23432

Table 3: Some Convolution Sums for $n = 1 - 25$.

n	$W_{4,10}(n)$	$W_{2,20}(n)$	W_{20}	$W_{4,5}$	$W_{2,10}$	$W_{55}(n)$	$W_{5,11}(n)$
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
9	0	0	0	1	0	0	0
10	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0
12	0	0	0	0	1	0	0
13	0	0	0	3	0	0	0
14	1	0	0	3	3	0	0
15	0	0	0	0	0	0	0
16	0	0	0	0	4	0	1
17	0	0	0	4	0	0	0
18	3	0	0	9	7	0	0
19	0	0	0	4	0	0	0
20	0	0	0	0	6	0	0
21	0	0	1	7	0	0	3
22	4	0	3	12	15	0	0
23	0	1	4	12	0	0	0
24	3	0	7	7	17	0	0
25	0	3	6	6	0	0	0

Table 4: The Same Convolution Sums for $n = 26 - 50$.

n	$W_{4,10}(n)$	$W_{2,20}(n)$	W_{20}	$W_{4,5}$	$W_{2,10}$	$W_{55}(n)$	$W_{5,11}(n)$
26	7	0	12	21	27	0	4
27	0	4	8	16	0	0	3
28	9	0	15	21	34	0	0
29	0	7	13	18	0	0	0
30	6	0	18	18	36	0	0
31	0	6	12	28	0	0	7
32	12	0	28	28	52	0	9
33	0	12	14	28	0	0	0
34	16	0	24	26	64	0	0
35	0	8	24	48	0	0	0
36	21	0	31	24	75	0	6
37	0	15	18	49	0	0	12
38	20	0	39	39	91	0	4
39	0	13	20	60	0	0	0
40	18	0	42	56	102	0	0
41	0	18	35	42	0	0	12
42	31	0	45	55	122	0	21
43	0	15	36	93	0	0	12
44	43	0	81	56	155	0	0
45	0	37	49	99	0	0	0
46	41	0	78	54	169	0	8
47	0	26	64	123	0	0	18
48	45	0	101	92	193	0	16
49	0	45	69	101	0	0	7
50	42	0	126	77	228	0	0

Table 5: Some Convolution Sums and Representation Numbers N_3 for $n = 1 - 25$.

n	$W_{\sigma,\tau 5}(n)$	$W_{\sigma 5,\tau}(n)$	$W_{\sigma 55,\tau}(n)$	$W_{\sigma 11,\tau 5}(n)$	$W_{\tau,\tau 5}(n)$	$N_3(n)$
1	0	0	0	0	0	4
2	0	0	0	0	0	4
3	0	0	0	0	0	8
4	0	0	0	0	0	20
5	0	0	0	0	0	20
6	1	1	0	0	1	48
7	3	-2	0	0	-2	32
8	4	-1	0	0	-1	68
9	7	2	0	0	2	108
10	6	1	0	0	1	108
11	10	5	0	0	0	148
12	2	-8	0	0	2	144
13	7	-3	0	0	2	216
14	-1	4	0	0	-6	256
15	6	1	0	0	-4	288
16	-13	11	0	1	-4	244
17	9	-16	0	0	4	392
18	-20	0	0	0	5	468
19	-9	6	0	0	6	576
20	-18	-3	0	0	2	636
21	-3	14	0	-2	-6	704
22	-40	-30	0	0	0	628
23	4	9	0	0	-6	1064
24	-27	18	0	0	0	1216
25	-12	-2	0	0	-6	1364

Table 6: The Same Convolution Sums and Representation Numbers N_3 for $n = 26 - 50$.

n	$W_{\sigma,\tau 5}(n)$	$W_{\sigma 5,\tau}(n)$	$W_{\sigma 55,\tau}(n)$	$W_{\sigma 11,\tau 5}(n)$	$W_{\tau,\tau 5}(n)$	$N_3(n)$
26	-17	14	0	-1	-2	1416
27	-9	-42	0	3	8	1176
28	-34	21	0	0	14	1920
29	29	14	0	0	-2	2232
30	-35	-10	0	0	-14	2400
31	-15	13	0	2	-6	2024
32	20	-59	0	-6	-10	2580
33	20	25	0	0	-2	3176
34	-35	40	0	0	1	3752
35	46	-9	0	0	0	4208
36	12	1	0	1	10	4068
37	21	-51	0	-3	6	3600
38	26	27	0	4	16	5056
39	10	20	0	0	-6	5312
40	5	-20	0	0	22	6460
41	80	18	0	2	0	5464
42	2	-49	0	6	-5	5264
43	-7	31	0	-8	-12	6592
44	80	70	0	0	-24	8516
45	38	-27	0	0	16	8964
46	-20	-33	0	-2	0	8624
47	47	-36	0	3	-2	7856
48	54	23	0	-4	-33	10144
49	16	34	0	7	-6	9684
50	49	-21	0	0	-12	13492

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