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A Novel Normality Test Using an Identity Transformation of the Gaussian Function

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Abstract. Normality is the most frequently required assumption for statistical techniques. Thus, evaluation of the normality assumption is the first step of many statistical analyses. Although there are many normality tests in the literature, none dominate for all conditions. This paper introduces a novel normality test, and its performance is compared with some of the other normality tests via a Monte Carlo simulation study. Tests are evaluated according to the Type I error and Power.

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1. Introduction

For a given sample dataset, testing whether it follows a normal distribution is a common starting point for many statistical analysis techniques. The literature has many different normality tests using one or more characteristics of the normal distribution function such, as the mean, variance, skewness, kurtosis etc. The Shapiro-Wilk Test [3], Jarqua-Bera [2], and Anderson and Darling Test [1] are some of the most familiar.

In this study, we aim to simultaneously handle all the characteristics mentioned. Logically, the Gaussian function is the unique and most appropriate tool having these characteristics; thus, we have built our test on the Gaussian density function. In section 2, we define a continuous random variable by transforming data using the Gaussian PDF. Then we derive some statistical characteristics like mean, variance, standard deviation and cumulative distribution for this variable. These characteristics are then used to construct a novel normality test, testing the null hypothesis that a given sample data comes from the normal distribution. We show and discuss results from simulation studies in section 3, then finish with some concluding remarks.

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2. Test For Normality

Let X be a continuous random variable from a normal distribution with mean μ and variance $\delta^2, X \sim N(\mu, \delta^2)$. It is known that X's density function, called the Gaussian function, and distribution function are shown in (1) and (2), respectively.

$$f(x) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\delta)^2} \tag{1}$$

$$F_X(t) = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}((x-\mu)/\delta)^2} dx \tag{2}$$

If we use a transformation to define a new random variable as a function of our data $Y = f(X)$, the mean and variance of Y are obtained by the following process.

$$\begin{aligned} E[Y] &= E[f(X)] = \int_{-\infty}^{\infty} \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\delta)^2} \cdot \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\delta)^2} dx \\ &= \frac{1}{2\pi\delta^2} \int_{-\infty}^{\infty} e^{-((x-\mu)/\delta)^2} dx = \frac{1}{2\pi\delta^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\sqrt{2}(x-\mu)/\delta)^2} dx \\ & \boxed{\text{let } u = \sqrt{2}(x-\mu)/\delta, \text{ so } du = \tfrac{\sqrt{2}}{\delta} dx \longrightarrow} \\ &= \frac{1}{2\pi\delta^2} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \frac{\delta}{\sqrt{2}} du = \frac{\sqrt{2\pi}}{2\sqrt{2}\pi\delta} = \frac{1}{2\sqrt{\pi}\delta} \end{aligned} \tag{3}$$

$$\begin{aligned} E[Y^2] &= E[f(X)^2] = \int_{-\infty}^{\infty} \{\frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\delta)^2}\}^2 \cdot \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\delta)^2} dx \\ &= \frac{1}{\delta^3 2\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{3}{2}((x-\mu)/\delta)^2} dx = \frac{1}{\delta^3 2\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\sqrt{3}(x-\mu)/\delta)^2} dx \\ & \boxed{\text{let } u = \sqrt{3}(x-\mu)/\delta, \text{ so } du = \tfrac{\sqrt{3}}{\delta} dx \longrightarrow} \\ &= \frac{1}{\delta^3 2\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \frac{\delta}{\sqrt{3}} du = \frac{1}{\delta^3 2\pi\sqrt{(2\pi)}} \cdot \frac{\delta\sqrt{2\pi}}{\sqrt{3}} = \frac{1}{2\sqrt{3}\pi\delta^2} \end{aligned} \tag{4}$$

$$Var[Y] = E[(Y - E[Y])^2] = E[Y^2] - E[Y]^2 = \frac{1}{2\sqrt{3}\pi\delta^2} - (\frac{1}{2\sqrt{\pi}\delta})^2 = \frac{2-\sqrt{3}}{4\sqrt{3}\pi\delta^2} \tag{5}$$

Thus, we have $\mu_Y = 1/(2\sqrt{\pi}\delta)$, and $\delta_Y^2 = (2-\sqrt{3})/(4\sqrt{3}\pi\delta^2)$ for a random variable Y. Further, we may extract the distribution function of Y as shown here, using the CDF transformation method for a random variable. Here we make use of the common standardization notation $z = (x-\mu)/\delta$.

$$F(y) \;=\; P\left(Y \le y\right) = P\left(\frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\delta)^2} \le y\right) = P\left(e^{-\frac{1}{2}((x-\mu)/\delta)^2} \le \delta\sqrt{2\pi}y\right)$$

$$
\begin{aligned}
&= P\left(\ln e^{-\frac{1}{2}((x-\mu)/\delta)^2} \le \ln \delta\sqrt{2\pi}y\right) = P\left(((x-\mu)/\delta)^2 \ge -2\ln\delta\sqrt{2\pi}y\right) \\
&= P\left((x-\mu)/\delta \le -\sqrt{-2\ln\delta\sqrt{2\pi}y}\right) + P\left((x-\mu)/\delta \ge \sqrt{-2\ln\delta\sqrt{2\pi}y}\right) \\
&= 2P\left((x-\mu)/\delta \ge \sqrt{-2\ln\delta\sqrt{2\pi}y}\right) = 2\left[1 - P\left((x-\mu)/\delta \le \sqrt{-2\ln\delta\sqrt{2\pi}y}\right)\right] \\
&= 2\left[1 - P\left(z \le \sqrt{-2\ln\delta\sqrt{2\pi}y}\right)\right] = 2\left[1 - \Phi\left(\sqrt{-2\ln\delta\sqrt{2\pi}y}\right)\right] \qquad (6)
\end{aligned}
$$

The complete CDF is

$$
F(y) = \begin{cases} 0 & y \le 0 \\ 2\left[1 - \Phi\left(\sqrt{-2\ln\delta\sqrt{2\pi}y}\right)\right] & 0 < y < \frac{1}{\delta\sqrt{2\pi}} \\ 1 & y \ge \frac{1}{\delta\sqrt{2\pi}} \end{cases}. \qquad (7)
$$

For the original data, we have $X \in \mathbb{R}$. With the Gaussian density function, we know that $\underset{x\to\pm\infty}{lim} f(X) = 0$, and

$$
f(\mu) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2}((\mu-\mu)/\delta)^2} = \frac{1}{\delta\sqrt{2\pi}}.
$$

Thus, our random variable Y has support $(0, 1/(\delta\sqrt{2\pi})]$. The following calculations show that some features of density functions are satisfied by (7).

$$
\begin{aligned}
F(0) &= 2\left[1 - \Phi\left(\sqrt{-2\ln\delta\sqrt{2\pi}\cdot 0}\right)\right] = 2\left[1 - \Phi\left(\sqrt{-2\ln 0}\right)\right] \\
&= 2\left[1 - \Phi\left(\sqrt{-2\cdot -\infty}\right)\right] = 2\left[1 - \Phi(\infty)\right] = 2[1-1] = 0
\end{aligned}
$$

$$
\begin{aligned}
F\left(\frac{1}{\delta\sqrt{2\pi}}\right) &= 2\left[1 - \Phi\left(\sqrt{-2\ln\delta\sqrt{2\pi}\cdot\frac{1}{\delta\sqrt{2\pi}}}\right)\right] = 2\left[1 - \Phi\left(\sqrt{-2\ln 1}\right)\right] \\
&= 2\left[1 - \Phi(\sqrt{-2\cdot 0})\right] = 2[1 - \Phi(0)] = 2[1 - 0.5] = 1
\end{aligned}
$$

We can use the random variable Y to test if a given sample follows a normal distribution. After transformation with the Gaussian density function, any data generated from a normal distribution should have mean $1/(2\sqrt{\pi}\delta)$ and variance $(2-\sqrt{3})/(4\sqrt{3}\pi\delta^2)$. These facts are the basis of our proposed normality test. In our derivation here, we have relied upon the transformation of a data sample, using unknown population parameters μ and δ. While these unknown parameters cancel out in the calculation of our critical value, the sample test statistic does rely on the conversion to Y. Here, we muse use the sample statistics $\overline{X}$ and S.

Our hypotheses

H_0: Data are normally distributed,
vs.
H_1: Data are not normally distributed,

can be rewritten in a more precise representation that lends itself to a test the relies on both (3) and (5). These hypotheses and the one-sided test A, are as follows.

H_0: $\mu_{sample} = 1/(2\sqrt{\pi}\delta)$
vs
H_1: $\mu_{sample} \neq 1/(2\sqrt{\pi}\delta)$

$$A = \frac{\mu_{sample} - \mu_0}{\delta_0} = \frac{\mu_{sample} - 1/(2\sqrt{\pi}\delta)}{\sqrt{(2-\sqrt{3})/(4\sqrt{3}\pi\delta^2)}} \tag{8}$$

The upper bound of the $(1-\alpha)\%$ confidence interval is found below with respect to $0 < 1/(2\sqrt{\pi}\delta) < y_{up}$.

$$\begin{aligned} P(0 \leq y \leq y_{up}) = 1-\alpha \rightarrow F(y_{up}) - F(0) = 1-\alpha \\ &\rightarrow 2\left[1 - \Phi\left(\sqrt{-2\ln\delta\sqrt{2\pi}y_{up}}\right)\right] - 0 = 1-\alpha \\ &\rightarrow \Phi\left(\sqrt{-2\ln\delta\sqrt{2\pi}y_{up}}\right) = \frac{1+\alpha}{2} \\ &\rightarrow \sqrt{-2\ln\delta\sqrt{2\pi}y_{up}} = \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \\ &\rightarrow y_{up} = \frac{1}{\delta\sqrt{2\pi}}\exp\left[-\frac{1}{2}\Phi^{-2}\left(\frac{1+\alpha}{2}\right)\right] \end{aligned} \tag{9}$$

Consequently, the 90% confidence interval, for example, is expressed as $(0, 0.3958/\delta)$. These boundaries are used to determine the comparison criteria of the test statistic, as shown in (10).

$$A_0 = \frac{\exp\left[-\frac{1}{2}\Phi^{-2}\left(\frac{1+\alpha}{2}\right)\right] / \left(\delta\sqrt{2\pi}\right) - 1/(2\sqrt{\pi}\delta)}{\sqrt{(2-\sqrt{3})/(4\sqrt{3}\pi\delta^2)}} = \frac{\exp\left[-\frac{1}{2}\Phi^{-2}\left(\frac{1+\alpha}{2}\right)\right] / \sqrt{2\pi} - 1/(2\sqrt{\pi})}{\sqrt{(2-\sqrt{3})/(4\sqrt{3}\pi)}} \tag{10}$$

For $\alpha = 0.10$, we find this quantity to be $A_0 = 1.025$. Note that the critical value for our test is hence independent of the values μ and δ, which we must estimate from our data sample. For a sample of size n, the test statistic is then given in (11), where $\overline{Y}$ and S_y indicate the sample mean and standard deviation of the transformed data.

$$A_{sample} = \frac{\sqrt{n}\left(\overline{Y} - 1/(2\sqrt{\pi}S_y)\right)}{S_y} \tag{11}$$

So, we can say that we are unable to reject the null hypothesis, saying given data could be normally distributed, if $A_{sample} \leq A_0$. Similarly, we reject the null hypothesis if $A_{sample} > A_0$; the data are not from a Gaussian distribution.

3. Simulation Studies

The efficiency and efficacy of a hypothesis test is characterized by Type I and Type II errors. The *Type I error* is the probability of falsely rejecting the null hypothesis - saying the given data is not from normal distribution when it really is. On the other hand, a *Type II error* is to falsely accept the null hypothesis when the given data is actually non-normal. The *Power* indicates the probability with which a test can correctly reject the null hypothesis. The power is equal to the Type II error subtracted from unity. We used two sets of Monte Carlo simulation studies to compare the performance of our test with the other common normality tests already mentioned; in both cases, we used $\alpha = 0.10$.

First, we ran 18 sets of $5,000$ simulations to evaluate performance with respect to Type I errors. We generated data from two normal distributions: $N(0,1)$ and $N(50,5)$, using sample sizes

$$n = [5, 20, 30, 50, 100, 250, 300, 500, 1000].$$

Secondly, in order to evaluate the power of the tests (and hence, Type II error rate), we generated data from four other distributions: $Uniform(50,100)$, $Gamma(5;3)$, $Exponential(5)$, and $Student(15)$, using the same sample sizes. Results from the studies are reported in Table 1 and Table 2. Table 1 shows the Type I error percentages of: our proposed test (**pt**), Jarqua-Bera Test (**jb**), Shapiro-Wilk Test (**ws**), Anderson and Darling Test (**and**). These results suggest our test is superior to the others. For almost all sample sizes evaluated, the false negative rate was much lower than the other tests. Even more interesting is the relative consistency demonstrated by our proposed test.

Table 1: Type I Error Probabilities of Compared Normality Tests.

	$N(0;1)$				$N(50;5)$			
n	**pt**	**jb**	**ws**	**and**	**pt**	**jb**	**ws**	**and**
5	0.0	0.0	9.8	0.0	0.0	0.0	9.4	0.0
20	1.1	2.0	12.5	10.1	1.6	2.1	12.3	10.0
30	1.7	3.4	12.3	10.3	1.8	3.7	12.4	10.0
50	2.0	4.7	12.0	10.0	1.8	4.4	12.4	10.5
100	1.9	6.0	12.3	10.3	2.0	6.1	12.1	9.9
250	1.9	7.6	11.2	10.0	2.0	7.6	12.0	10.1
300	1.7	7.9	12.0	10.2	1.8	7.6	11.5	9.3
500	2.1	9.1	12.2	9.7	1.8	8.2	11.4	9.8
1000	1.9	9.6	12.0	10.3	1.8	9.2	12.1	10.0

In Table 2, we see that the proposed test does not seem to be more powerful than the other three, which exhibit high power rates for sample sizes greater than 100 for the first three distributions. For the two symmetric distributions - uniform and Student's t, - the performance of our test is similar to that of the Anderson and Darling test. As would be expected, we see that increasing the sample size increases the power. For the uniform and exponential distributions, a sample size larger than 100 is enough to detect non-normality approximately perfectly; much larger samples are required for the gamma distribution. Not surprisingly, when data were generated from the Student's t distribution, none of the compared tests were sufficient to detect non-normality except for extremely large samples.

4. Concluding Remarks

In this study we proposed a novel normality test using the density function to transform data before testing. Of course, we have used very simple calculus methods. However, the simplicity of the calculus employed does not negate the value of our proposed test. Simulation studies show that the proposed test gives approximately perfect results for all sample sizes according to Type I error. However, according to Power, we can not say that our proposed test works better than the others. It is also seen that the Type I error rate seems invariant with respect to sample size, while the Type II error decreases with higher sample sizes. The logic underlying our test could be readily adapted to specific tests for other probability distributions. This could be a promising avenue of further research.

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Table 2: Power Probabilities of Compared Normality Tests.

	Uniform(50;100)				Gamma(5;3)				Exp(5)				t(15)			
n	**pt**	**jb**	**ws**	**and**	**pt**	**jb**	**ws**	**and**	**pt**	**jb**	**ws**	**and**	**pt**	**Jb**	**ws**	**and**
5	0.0	0.0	13.0	0.0	0.0	0.0	11.8	0.0	0.0	0.0	26.2	0.0	0.0	0.0	9.4	0.0
20	7.8	0.0	35.8	29.2	6.1	13.6	37.3	31.6	26.8	50.3	90.8	85.8	2.9	4.7	16.1	12.3
30	20.6	0.1	59.1	45.4	8.9	24.3	48.8	40.0	37.7	77.7	98.7	96.8	4.7	8.4	18.4	13.3
50	55.2	8.6	88.7	74.1	12.7	47.7	70.7	61.0	55.4	98.3	100.0	99.8	6.8	13.6	22.0	15.8
100	95.7	94.9	99.9	98.2	16.9	86.2	94.4	87.6	79.4	100.0	100.0	100.0	9.4	20.5	26.6	17.2
250	100.0	100.0	100.0	100.0	29.9	100.0	100.0	99.9	98.6	100.0	100.0	100.0	18.0	35.7	38.4	22.8
300	100.0	100.0	100.0	100.0	32.0	100.0	100.0	100.0	99.4	100.0	100.0	100.0	21.7	41.1	43.7	25.7
500	100.0	100.0	100.0	100.0	45.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	31.7	53.2	53.9	34.1
1000	100.0	100.0	100.0	100.0	69.5	100.0	100.0	100.0	100.0	100.0	100.0	100.0	56.7	77.0	75.6	52.7