EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 3, No. 6, 2010, 1006-1031
ISSN 1307-5543 – www.ejpam.com

SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY

Asymptotic Expansion of n-dimensional Faxén-type Integrals

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Abstract. The asymptotic expansion of n-dimensional extensions of Faxén's integral $I_n(z)$ are derived for large complex values of the variable z. The theory relies on the asymptotics of the generalised hypergeometric, or Wright, function. The coefficients in the exponential expansion are obtained by means of an algorithm applicable for arbitrary n. Numerical examples are given to illustrate the accuracy of the expansions.

2000 Mathematics Subject Classifications: 30E15, 33C70, 41A60, 41A63

Key Words and Phrases: Faxén's integral, Asymptotic expansion, Wright function, generalised hypergeometric functions

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1. Introduction

We consider the n-dimensional integral

$$I_n(z) = \lambda_n \int_0^\infty \cdots \int_0^\infty x_1^{\nu_1-1} \ldots x_n^{\nu_n-1} e^{-f(x_1,\ldots,x_n;z)}\, dx_1 \ldots dx_n, \tag{1}$$

where

$$f(x_1,\ldots,x_n;z) = \sum_{j=1}^n x_j^{\mu_j} - z x_1^{m_1} \ldots x_n^{m_n}, \qquad \lambda_n = \prod_{j=1}^n \mu_j, \tag{2}$$

and the factor λ_n has been added for later convenience. We suppose that the exponents (not necessarily integers) satisfy $\mu_j > m_j > 0$, $\mathrm{Re}(\nu_j) > 0$ $(1 \le j \le n)$ and that z denotes a complex

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variable. For convergence of $I_n(z)$ we require that μ_j and m_j be further restricted so that the parameter κ, defined by

$$\kappa = 1 - \sum_{j=1}^{n} \frac{m_j}{\mu_j}, \tag{3}$$

should satisfy $0 < \kappa < 1$. The geometrical interpretation of this condition results from consideration of the Newton diagram associated with the phase function f. In the two-dimensional case $n = 2$, the Newton diagram is given by the boundary of the convex hull formed by the point (m_1, m_2) and the points $(\mu_1, 0)$ and $(0, \mu_2)$ situated on the coordinate axes, with the line joining these last two points being termed the back face. Extension to $n \geq 3$ dimensions is straightforward with the back face being a hyperplane in n dimensions passing through the points μ_j $(1 \leq j \leq n)$ on the coordinate axes. The condition $0 < \kappa < 1$ then corresponds to the internal point $(m_1, \ldots, m_n)$ being situated in front of the back face of the Newton diagram.

In the case $n = 1$, we have (dropping the subscript 1 on the parameters)

$$I_1(z) = \mu \int_0^\infty x^{\nu-1} e^{-x^\mu + z x^m} dx = \int_0^\infty \tau^{(\nu/\mu)-1} e^{-\tau + z\tau^{m/\mu}} d\tau,$$

where $\kappa = 1 - (m/\mu) < 1$. This integral can be expressed as $\mathrm{Fi}(m/\mu, \nu/\mu; z)$, where Fi denotes Faxén's integral defined by [9, p. 332]

$$\mathrm{Fi}(a, b; z) = \int_0^\infty \tau^{b-1} e^{-\tau + z\tau^a} d\tau \qquad (0 \leq \mathrm{Re}(a) < 1,\ \mathrm{Re}(b) > 0).$$

Consequently, (1) can be considered as an n-dimensional extension of Faxén's integral. A different extension of Faxén's integral as a one-dimensional integral with more than one internal point in the phase function f has been considered in [7]. Another integral that is related to (1), but with different domains of integration, is

$$J_n(z) = \lambda_n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{\nu_1 - 1} \ldots x_n^{\nu_n - 1} e^{-f(x_1, \ldots, x_n; z)} dx_1 \ldots dx_n, \tag{4}$$

in which the μ_j are now all restricted to be positive even integers. When the exponents ν_j are nonintegers, the integral $J_n(z)$ is specified by taking the integration paths along the negative x_j-axes to be along the upper side of the branch cuts on these axes. Variants of the integral $J_n(z)$ can also be considered in which $p < n$ of the integrals in (4) are evaluated over the interval $(-\infty, \infty)$, with the remainder over the interval $[0, \infty)$.

Special cases of the integral $I_n(z)$ when $n = 1$ and $\nu = 1$ were first studied asymptotically in [4, 5], and more generally in [1], using the method of steepest descents. In [13] the asymptotic expansion of $I_n(z)$ for large complex z was obtained by application of the asymptotic theory of the generalised hypergeometric, or Wright, function ${}_p\Psi_q(z)$ defined in (5). The expansion was found to consist of an exponential expansion, which is dominant in the sector $|\arg z| < \frac{1}{2}\pi\kappa$, together with an algebraic expansion dominant in the rest of the z-plane. An application of the integral $I_n(z)$ in the particular case $n = 2$ has been given in [8] in the discussion of two-dimensional Laplace integrals with more general phase and amplitude functions.

More recently, Breen and Wood [3] have discussed an application of $I_n(z)$ as a representation of the solutions of certain high-order linear differential equations. One of these equations has the form

$$y^{(n)}(z) - \sum_{r=0}^{p} \alpha_r z^r y^{(r)}(z) = 0 \qquad (n > p > 0),$$

where the α_r are arbitrary coefficients. This equation has a basis of solutions given by

$$y(z;s) = \int_0^\infty \ldots \int_0^\infty x_1^{\nu_1 - 1} \ldots x_p^{\nu_p - 1} \exp\{szx_1 \ldots x_p - (x_1^n + \ldots + x_p^n)/n\}\, dx_1 \ldots dx_p,$$

where $s^n = \nu_p$ and the exponents ν_r are related to the coefficients α_r in a manner that we do not specify here. This integral is clearly related to $I_p(sz)$ in (1) with the parameters $\mu_j = n$ and $m_j = 1$ $(1 \le j \le p)$. The integral representation of solutions of the above differential equation when there are two lower-order derivatives $(p = 2)$ was first given by Spitzer [18], with the general case of $p < n$ lower-order derivatives being considered in [16]. These results are described in [14, pp. 130–133]. In [3] the asymptotics of the solutions $y(z;s)$ were obtained using the theory developed in [13].

In this paper, we review the asymptotic expansion of the integral $I_n(z)$ in (1) using the asymptotic theory of the Wright function. A recent account of the asymptotic theory of the latter function has been presented in [11] and a discussion of the properties of ${}_0\Psi_1(z)$ (the generalised Bessel function), together with its application to the solution of fractional diffusion-wave equations, can be found in [6]. It is shown how the expansion of $I_n(z)$ may be employed to determine the asymptotic structure of the integral $J_n(z)$ and its variants when some of the integrals in (4) are taken over $[0, \infty)$.

2. The Expansion of the Wright Function ${}_p\Psi_q(z)$ for $|z| \to \infty$

The asymptotic expansion of the integrals $I_n(z)$ and $J_n(z)$ will be obtained by utilising the asymptotic theory of the Wright (or generalised hypergeometric) function which we present in this section. The Wright function ${}_p\Psi_q(z)$ is defined by

$${}_p\Psi_q(z) \equiv {}_p\Psi_q\left(\begin{array}{c} (\alpha_1, a_1), \ldots, (\alpha_p, a_p) \\ (\beta_1, b_1), \ldots, (\beta_q, b_q) \end{array} ; z \right) = \sum_{k=0}^\infty g(k) \frac{z^k}{k!}, \qquad g(k) := \frac{\prod_{r=1}^p \Gamma(\alpha_r k + a_r)}{\prod_{r=1}^q \Gamma(\beta_r k + b_r)}, \tag{5}$$

where p and q are nonnegative integers, the parameters α_r and β_r are real and positive and a_r and b_r are arbitrary complex numbers. In addition, it is assumed that the α_r and a_r are subject to the restriction

$$\alpha_r k + a_r \ne 0, -1, -2, \ldots \qquad (k = 0, 1, 2, \ldots \;;\; 1 \le r \le p) \tag{6}$$

so that no gamma function in the numerator of (5) is singular. In the special case $\alpha_r = \beta_r = 1$, the function ${}_p\Psi_q(z)$ reduces to a multiple of the generalised hypergeometric function ${}_pF_q((a_p); (b_q); z)$; see, for example, [17, p. 40].

We summarise the asymptotic expansion of the Wright function ${}_p\Psi_q(z)$ for $|z| \to \infty$ given in Wright [20, 21] and Braaksma [2]; for a summary, see also [12, §2.3] and [11]. We first introduce the parameters associated with $g(k)$ given by

$$\kappa = 1 + \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{p} \alpha_r, \qquad h = \prod_{r=1}^{p} \alpha_r^{\alpha_r} \prod_{r=1}^{q} \beta_r^{-\beta_r},$$

$$\vartheta = \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r + \tfrac{1}{2}(q-p), \qquad \vartheta' = 1 - \vartheta, \tag{7}$$

where, as usual, an empty product has unit value. If it is supposed that α_r and β_r are such that $\kappa > 0$, then ${}_p\Psi_q(z)$ is uniformly and absolutely convergent for all finite z. It is clear that ${}_p\Psi_q(z)$ is an entire function of z in this case. If $\kappa = 0$, the sum in (1) has a finite radius of convergence equal to h^{-1}, whereas for $\kappa < 0$ the sum is divergent for all nonzero values of z. The parameter κ will be found to play a critical role in the asymptotic theory of ${}_p\Psi_q(z)$ by determining the sectors in the z-plane in which its behaviour is either exponentially large, algebraic or exponentially small in character as $|z| \to \infty$.

The exponential expansion $E_{p,q}(z)$ is given by the formal asymptotic sum

$$E_{p,q}(z) = Z^{\vartheta} e^{Z} \sum_{j=0}^{\infty} A_j Z^{-j}, \qquad Z = \kappa(hz)^{1/\kappa}, \tag{8}$$

where the coefficients A_j are those appearing in the inverse factorial expansion of $g(s)/s!$ in the form

$$\frac{g(s)}{\Gamma(s+1)} = \kappa(h\kappa^{\kappa})^{s} \left\{ \sum_{j=0}^{M-1} \frac{A_j}{\Gamma(\kappa s + \vartheta' + j)} + \frac{O(1)}{\Gamma(\kappa s + \vartheta' + M)} \right\} \tag{9}$$

for $|s| \to \infty$ uniformly in $|\arg s| \le \pi - \epsilon$, $\epsilon > 0$ and arbitrary positive integer M. The leading coefficient A_0 is specified by

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)} \kappa^{-\frac{1}{2}-\vartheta} \prod_{r=1}^{p} \alpha_r^{a_r - \frac{1}{2}} \prod_{r=1}^{q} \beta_r^{\frac{1}{2} - b_r}. \tag{10}$$

The coefficients A_j are independent of s and depend only on the parameters p, q, α_r, β_r, a_r and b_r. An algorithm for their evaluation in specific cases when $\alpha_r > 0$, $\beta_r > 0$ is described in Appendix A.

The algebraic expansion $H_{p,q}(z)$ follows from the Mellin-Barnes integral representation [12, §2.3]

$${}_p\Psi_q(z) = \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \Gamma(s) g(-s) (z e^{\mp \pi i})^{-s} ds, \qquad |\arg(-z)| < \tfrac{1}{2}\pi(2-\kappa), \tag{11}$$

where the upper or lower sign is chosen according as $\arg z > 0$ or $\arg z < 0$, respectively. The path of integration in (11) is indented near $s = 0$ to separate* the poles of $\Gamma(s)$ situated at

*This is always possible when the condition (6) is satisfied.

$s = 0, -1, -2, \ldots$ from those of $g(-s)$ at

$$s_{k,r} = (a_r + k)/\alpha_r, \qquad k = 0, 1, 2, \ldots \ (1 \le r \le p). \tag{12}$$

In general there will be p such sequences of simple poles though, depending on the values of α_r and a_r, some of these poles could be multiple poles or even ordinary points if any of the $\Gamma(\beta_r s + b_r)$ are singular there. Displacement of the integration contour in (11) to the right over the poles of $g(-s)$ followed by evaluation of the residues then generates the algebraic expansion of ${}_p\Psi_q(z)$ valid as $|z| \to \infty$ in the sector in (11).

If it is assumed that the parameters are such that the poles in (12) are all simple, we obtain the algebraic expansion given by $H_{p,q}(ze^{\mp\pi i})$, where

$$H_{p,q}(z) = \sum_{j=1}^{p} \alpha_j^{-1} z^{-a_j/\alpha_j} S_{p,q}(z; j) \tag{13}$$

and $S_{p,q}(z; j)$ denotes the formal asymptotic sum

$$S_{p,q}(z; j) = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \Gamma\left(\frac{k + a_j}{\alpha_j}\right) \frac{\prod_{r=1}^{\prime p} \Gamma(a_r - \alpha_r s_{k,j})}{\prod_{r=1}^{q} \Gamma(b_r - \beta_r s_{k,j})} z^{-k/\alpha_j}, \tag{14}$$

with the prime indicating the omission of the term corresponding to $r = j$ in the product. This expression consists of p expansions with the leading behaviour z^{-a_j/α_j} $(1 \le j \le p)$. When the parameters α_r and a_r are such that some of the poles are of higher order, the expansion (13) is invalid and the residues must then be evaluated according to the multiplicity of the poles concerned; this will lead to terms involving $\log z$ in the algebraic expansion.

We present the asymptotic expansion of ${}_p\Psi_q(z)$ for large $|z|$ only in the case when $0 < \kappa \le 2$, since the value of this parameter associated with the integrals (1) and (4) must satisfy $0 < \kappa < 1$. A fuller list of expansion theorems is given in [2, 20, 21]; see also [11]. We have the following theorems, where throughout we let ϵ denote an arbitrarily small positive quantity.

Theorem 1. *If* $0 < \kappa < 2$*, then*

$$ {}_p\Psi_q(z) \sim \begin{cases} E_{p,q}(z) + H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \le \frac{1}{2}\pi\kappa \\ H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg(-z)| \le \frac{1}{2}\pi(2-\kappa) - \epsilon \end{cases} \tag{15}$$

as $|z| \to \infty$*. The upper or lower sign in* $H_{p,q}(ze^{\mp\pi i})$ *is chosen according as* z *lies in the upper or lower half-plane, respectively.*

It is seen that the z-plane is divided into two sectors, with a common vertex at $z = 0$, by the rays (the anti-Stokes lines) $\arg z = \pm\frac{1}{2}\pi\kappa$. In the sector $|\arg z| < \frac{1}{2}\pi\kappa$, the asymptotic character of ${}_p\Psi_q(z)$ is exponentially large, whereas in the complementary sector $|\arg(-z)| < \frac{1}{2}\pi(2-\kappa)$, ${}_p\Psi_q(z)$ is algebraic in character. The choice of signs in $H_{p,q}(ze^{\mp\pi i})$ results from the fact that the positive real axis $\arg z = 0$ is a Stokes line, where the algebraic expansion is maximally subdominant. Since ${}_p\Psi_q(z)$ is an entire function of z, we may write ${}_p\Psi_q(z) = {}_p\Psi_q(ze^{-2\pi i})$. Then, when $\pi \le \arg z < 2\pi$ the algebraic expansion is (with the lower sign) $H_{p,q}(ze^{-2\pi i}e^{\pi i}) = H_{p,q}(ze^{-\pi i})$, and so has the same form as when $0 < \arg z \le \pi$. Hence the algebraic expansion associated with ${}_p\Psi_q(z)$ can be written alternatively as

$$H_{p,q}(ze^{-\pi i}) \text{ in } \epsilon \le \arg z \le 2\pi - \epsilon. \tag{16}$$

The above theorem does not take into account the presence of an exponentially small contribution beyond the sector $|\arg z| \le \frac{1}{2}\pi\kappa$. This is covered by the more precise result in the following theorem [2, p. 331], [20, 21].

Theorem 2. *If* $\frac{2}{3} \le \kappa \le 2$, *then*

$${}_p\Psi_q(z) \sim E_{p,q}(z) + E_{p,q}(ze^{\mp 2\pi i}) + H_{p,q}(ze^{\mp\pi i}) \qquad (|\arg z| \le \pi) \tag{17}$$

as $|z| \to \infty$. *When* $0 < \kappa < \frac{2}{3}$, *we have*

$${}_p\Psi_q(z) \sim \begin{cases} E_{p,q}(z) + H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \le \frac{3}{2}\pi\kappa - \epsilon \\ H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad \frac{3}{2}\pi\kappa + \epsilon \le |\arg z| \le \pi \end{cases} \tag{18}$$

as $|z| \to \infty$. *The upper or lower signs are chosen according as* $\arg z > 0$ *or* $\arg z < 0$, *respectively.*

Since, when $\frac{2}{3} \le \kappa \le 2$, $E_{p,q}(z)$ is exponentially small in $\frac{1}{2}\pi\kappa < |\arg z| \le \pi$ then in the sense of Poincaré the expansion $E_{p,q}(z)$ can be neglected. Similarly, $E(ze^{-2\pi i})$ is exponentially small compared to $E_{p,q}(z)$ in $0 \le \arg z < \pi$ and consequently there is no inconsistency between (17) and the second expansion in (15). However, in the neighbourhood of $\arg z = \pi$, $E_{p,q}(z)$ and $E_{p,q}(ze^{\mp 2\pi i})$ are of comparable magnitude and, for real parameters, they combine to generate a real result on $\arg z = \pi$. A similar remark applies to the expansion $E(ze^{2\pi i})$ in $-\pi < \arg z \le 0$.

When $\kappa < \frac{2}{3}$, $E_{p,q}(z)$ is exponentially small in the sectors $\frac{1}{2}\pi\kappa < |\arg z| < \frac{3}{2}\pi\kappa$ and the behaviour of ${}_p\Psi_q(z)$ in the complementary sector $\frac{3}{2}\pi\kappa < |\arg z| \le \pi$ is then algebraic. An even more precise result can be given by recognising that the rays $\arg z = \pm\pi\kappa$ are also Stokes lines, where $E_{p,q}(z)$ is maximally subdominant with respect to $H_{p,q}(ze^{\mp\pi i})$. This will result in the expansion $E_{p,q}(z)$ switching off (as $|\arg z|$ increases) across the Stokes lines $\arg z = \pm\pi\kappa$. Thus, when $0 < \kappa < 1$, (17) and (18) can be replaced by

$${}_p\Psi_q(z) \sim \begin{cases} E_{p,q}(z) + H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \le \pi\kappa - \epsilon \\ H_{p,q}(ze^{\mp\pi i}) & \text{in} \quad \pi\kappa + \epsilon \le |\arg z| \le \pi \end{cases} \tag{19}$$

as $|z| \to \infty$. In Appendix B we present a numerical example for ${}_2\Psi_0(z)$ which demonstrates the truth of this assertion; a fuller discussion is given in [10]. Although the expansions in (15) and (18) are valid asymptotic descriptions, more accurate evaluation will result from using (19) which takes into account the Stokes phenomenon.[†] In the application to the integrals $I_n(z)$ and $J_n(z)$ we shall employ the expansion of ${}_p\Psi_q(z)$ in the form given in (19).

[†]The expansion in the neighbourhood of the Stokes lines $\arg z = 0$ and $\arg z = \pm\pi\kappa$ would necessitate a detailed treatment of the Stokes phenomenon that we do not consider here.

3. The Expansion of $I_n(z)$ for $|z| \to \infty$

The series representation of the integral $I_n(z)$ can be obtained by use of the Maclaurin expansion of the factor $\exp\{zx_1^{m_1}\ldots x_n^{m_n}\}$ in (1) followed by termwise integration to yield

$$\begin{aligned} I_n(z) &= \lambda_n \sum_{k=0}^{\infty} \frac{z^k}{k!} \prod_{r=1}^{n} \int_0^\infty x_r^{\nu_r+m_rk-1} \exp\{-x_r^{\mu_r}\}\, dx_r \\ &= \sum_{k=0}^{\infty} \prod_{r=1}^{n} \Gamma\left(\frac{\nu_r+m_rk}{\mu_r}\right) \frac{z^k}{k!}. \end{aligned} \tag{20}$$

Comparison with (5) shows that the above series is a particular case of the Wright (or generalised hypergeometric) function given by

$$I_n(z) = {}_n\Psi_0\left(\begin{matrix} (\alpha_1, a_1), \ldots, (\alpha_n, a_n) \\ - \end{matrix} ; z \right) \equiv {}_n\Psi_0(z), \tag{21}$$

where the parameters

$$\alpha_r = \frac{m_r}{\mu_r}, \quad a_r = \frac{\nu_r}{\mu_r} \qquad (1 \le r \le n) \tag{22}$$

and the dash denotes the omission of a parameter sequence.

The asymptotic expansion of $I_n(z)$ for $|z| \to \infty$ then follows from (19). With κ defined in (3) (which follows from the definition in (7)) we therefore have

$$I_n(z) \sim \begin{cases} E_{n,0}(z) + H_{n,0}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \le \pi\kappa - \epsilon \\ H_{n,0}(ze^{\mp\pi i}) & \text{in} \quad \pi\kappa + \epsilon \le |\arg z| \le \pi, \end{cases} \tag{23}$$

where the exponential expansion $E_{n,0}(z)$ and the algebraic expansion $H_{n,0}(ze^{\mp\pi i})$ are obtained from (8), (13) and (14) with $p = n$, $q = 0$, and the upper or lower signs are chosen according as $\arg z > 0$ or $\arg z < 0$, respectively. The leading coefficient A_0 in $E_{n,0}(z)$ is, from (10), given by

$$A_0 = (2\pi)^{n/2} \kappa^{-\frac{1}{2}-\vartheta} \prod_{r=1}^{n} \left(\frac{m_r}{\mu_r}\right)^{(\nu_r/\mu_r)-\frac{1}{2}}. \tag{24}$$

The large $|z|$ behaviour of $I_n(z)$ is consequently exponentially large in the sector $|\arg z| < \frac{1}{2}\pi\kappa$. Outside of this sector the behaviour is dominated by an algebraic expansion, with a subdominant exponentially small contribution being present in the sectors $\frac{1}{2}\pi\kappa < |\arg z| < \pi\kappa$.

We now give an example of the expansion of $I_n(z)$ when $n = 3$; other numerical examples can be found in [13]. Consider the three-dimensional integral

$$I_3(z) = 36 \int_0^\infty \int_0^\infty \int_0^\infty (x_3/x_2)^{\frac{1}{2}} \exp\{-(x_1^3 + x_2^3 + x_3^4 - z(x_1x_2)^{\frac{1}{2}}x_3)\}\, dx_1 dx_2 dx_3,$$

which is associated with the parameters $\mu_1 = \mu_2 = 3$, $\mu_3 = 4$, $m_1 = m_2 = \frac{1}{2}$, $m_3 = 1$ and $\nu_1 = 1$, $\nu_2 = \frac{1}{2}$, $\nu_3 = \frac{3}{2}$. From (21) we therefore have

$$I_3(z) = {}_3\Psi_0\left(\begin{array}{c}(\frac{1}{6},\frac{1}{3}),(\frac{1}{6},\frac{1}{6}),(\frac{1}{4},\frac{3}{8}) \\ - \end{array};z\right) \equiv {}_3\Psi_0(z). \tag{25}$$

From (3), (7) and (24) we obtain the parameters

$$\kappa = \tfrac{5}{12}, \quad h = 2^{-5/6}3^{-1/3}, \quad \vartheta = -\tfrac{5}{8}, \quad A_0 = 4\pi^{3/2}3^{5/8}5^{1/8}.$$

Then, from (8),

$$E_{3,0}(z) = Z^{-5/8}e^Z\sum_{j=0}^{\infty}A_jZ^{-j}, \qquad Z = \tfrac{5}{48}(\tfrac{1}{3}z^3)^{4/5}, \tag{26}$$

where, by use of the algorithm described in Appendix A, the first few normalised coefficients $c_j \equiv A_j/A_0$ are found to be

$$c_1 = \tfrac{67}{144}, \quad c_2 = \tfrac{23785}{41472}, \quad c_3 = \tfrac{106119923}{89579520}, \quad c_4 = \tfrac{181613304677}{51597803520}, \quad c_5 = \tfrac{102937183723339}{7430083706880}, \ \ldots .$$

The poles in (12) are situated at

$$s_{k,1} = 2+6k, \quad s_{k,2} = 1+6k, \quad s_{k,3} = \tfrac{3}{2}+4k \qquad (k = 0,1,2,\ldots)$$

and so are all simple poles. Hence, from (13) and (14), we find the algebraic expansion given by

$$H_{3,0}(z) = \sum_{j=1}^{3}\frac{\mu_j}{m_j}z^{-\nu_j/m_j}S_{3,0}(z;j), \tag{27}$$

where

$$S_{3,0}(z;j) = \sum_{k=0}^{\infty}\frac{(-)^k}{k!}\Gamma\left(\frac{\mu_jk+\nu_j}{m_j}\right)\prod_{r=1}^{3}{}'\,\Gamma\left(\frac{\nu_r - m_rs_{k,j}}{\mu_r}\right)z^{-\mu_jk/m_j} \tag{28}$$

with the prime denoting the omission of the gamma function factor corresponding to $r = j$.

Then, from (23) we obtain the expansion

$$I_3(z) \sim \begin{cases} E_{3,0}(z) + H_{3,0}(ze^{\mp\pi i}) & \text{in} \quad |\arg z| \le \frac{5}{12}\pi - \epsilon \\ H_{3,0}(ze^{\mp\pi i}) & \text{in} \quad \frac{5}{12}\pi + \epsilon \le |\arg z| \le \pi \end{cases} \tag{29}$$

as $|z| \to \infty$. It follows that $I_3(z)$ is exponentially large in the sector $|\arg z| < \frac{5}{24}\pi$ with the dominant expansion being algebraic in the rest of the z-plane. In the sectors $\frac{5}{12}\pi < |\arg z| < \frac{5}{24}\pi$ the exponential expansion $E_{3,0}(z)$ is subdominant and switches off (as $|\arg z|$ increases) across the Stokes lines $\arg z = \pm\frac{5}{12}\pi$. We show in Table 1 the values of the absolute relative error in the computation of $I_3(z)$ as a function of $\theta = \arg z$ when $|z| = 15$ using the optimally truncated asymptotic expansions (that is, truncated at or near the least term in modulus) in (29).

Table 1: Values of the absolute relative error in the computation of $I_3(z)$ in (25) when $|z| = 15$ as a function of $\theta = \arg z$ using an optimal truncation of the expansions in (29).

θ/π	|Rel. Error|	θ/π	|Rel. Error|
0	5.816×10^{-13}	0.625	9.259×10^{-14}
0.125	9.262×10^{-14}	0.750	2.744×10^{-13}
0.250	1.612×10^{-13}	0.875	9.841×10^{-14}
0.375	3.534×10^{-14}	1.000	1.727×10^{-13}
0.500	1.177×10^{-13}		

4. The integral $J_n(z)$

We can apply a similar treatment to the integral

$$\begin{aligned} J_n(z) &= \lambda_n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{\nu_1 - 1} \dots x_n^{\nu_n - 1} e^{-f(x_1, \dots, x_n; z)} \, dx_1 \dots dx_n \\ &= \lambda_n \sum_{k=0}^{\infty} \frac{z^k}{k!} \prod_{r=1}^{n} \int_{-\infty}^{\infty} x_r^{\nu_r + m_r k - 1} \exp\{-x_r^{\mu_r}\} \, dx_r, \end{aligned}$$

where the phase function f and the factor λ_n are defined in (2) and it is supposed that the parameters μ_j $(1 \le j \le n)$ appearing in f are positive even integers. In the evaluation of the above integrals when $x_r < 0$, we shall write $x_r = |x_r| e^{\pi i}$.

We now introduce the notation $e(x)$ for brevity in this section and define the quantities $B_r(k)$ by

$$e(x) := e^{\pi i x}, \qquad B_r(k) := 1 - e(\nu_r + m_r k). \tag{30}$$

Then we find

$$\begin{aligned} \mu_r \int_{-\infty}^{\infty} x_r^{\nu_r + m_r k - 1} \exp\{-x_r^{\mu_r}\} \, dx_r &= \mu_r B_r(k) \int_{0}^{\infty} x_r^{\nu_r + m_r k - 1} \exp\{-x_r^{\mu_r}\} \, dx_r \\ &= B_r(k) \Gamma\left(\frac{\nu_r + m_r k}{\mu_r}\right) \end{aligned}$$

and hence that

$$J_n(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \prod_{r=1}^{n} \left\{ B_r(k) \Gamma\left(\frac{\nu_r + m_r k}{\mu_r}\right) \right\}. \tag{31}$$

4.1. The Representation of $J_n(z)$ in Terms of ${}_n\Psi_0(z)$ Functions

Expansion of the product of exponential factors $\prod_{r=1}^{n} B_r(k)$ in (31) can be achieved by making use of the standard expansion

$$(1 - z_1)(1 - z_2) \dots (1 - z_n) = 1 - \sum_{i=1}^{n} z_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_i z_j$$

$$-\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} z_i z_j z_k + \ldots + (-)^n z_1 z_2 \ldots z_n.$$

Substitution of this expansion into (31) with $z_r = e(\nu_r + m_r k)$ then shows that $J_n(z)$ may be expressed as a linear combination of ${}_n\Psi_0(z)$ with rotated argument in the form

$$\begin{aligned} J_n(z) &= {}_n\Psi_0(z) - \sum_{i=1}^{n} e(\nu_i)\,{}_n\Psi_0(ze(m_i)) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} e(\nu_i+\nu_j)\,{}_n\Psi_0(ze(m_i+m_j)) \\ &- \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} e(\nu_i+\nu_j+\nu_k)\,{}_n\Psi_0(ze(m_i+m_j+m_k)) + \ldots \\ &+ (-)^n e(N)\,{}_n\Psi_0(ze(M)), \end{aligned} \tag{32}$$

where we have defined

$$N := \nu_1 + \ldots + \nu_n, \qquad M := m_1 + \ldots + m_n. \tag{33}$$

The parameters appearing in each ${}_n\Psi_0$ function are those given in (22).

The asymptotic expansion (19), or equivalently (23), can then be employed to deal with each ${}_n\Psi_0(ze^{\pi i\omega})$ with argument rotated by ω, it being remembered that ${}_n\Psi_0(z)$ is an integral function of z with arg z evaluated modulo 2π. Rather than attempt to present a complicated general result, we indicate how to proceed with the asymptotic expansion of $J_n(z)$ as $|z| \to \infty$ in specific cases in Section 5.

4.2. The Algebraic Contribution to the Expansion of $J_n(z)$

We consider the contribution to the asymptotic expansion of the integral $J_n(z)$ as $|z| \to \infty$ that results from the algebraic expansions associated with each ${}_n\Psi_0$ function of rotated argument in (32). It will be shown that this combination of algebraic expansions cancels in the sector

$$\epsilon \le \arg z \le (2-M)\pi - \epsilon, \tag{34}$$

where M is defined in (33), and that consequently the expansion of $J_n(z)$ in this sector is purely exponential in character.

We shall suppose in this section that all the poles $s_{k,j}$ in (12) are simple; a case when there are multiple poles present is discussed in Appendix C. Then, the algebraic expansion of ${}_n\Psi_0(z)$ associated with the parameters in (22) is, from (13), (14) and (16), given by

$$H_{n,0}(ze^{-\pi i}) = \sum_{j=1}^{n} \frac{\mu_j}{m_j} h_j(z)$$

valid as $|z| \to \infty$ in the sector $\epsilon \le \arg z \le 2\pi - \epsilon$, where

$$h_j(z) := (ze^{-\pi i})^{-\nu_j/m_j} S_{n,0}(ze^{-\pi i}; j) \qquad (1 \le j \le n) \tag{35}$$

and

$$S_{n,0}(ze^{-\pi i};j)=\sum_{k=0}^{\infty}\frac{(-)^k}{k!}\Gamma\left(\frac{\mu_j k+\nu_j}{m_j}\right)\prod_{r=1}^{n}{}'\,\Gamma\left(\frac{\nu_r-m_r s_{k,j}}{\mu_r}\right)(ze^{-\pi i})^{-\mu_j k/m_j}.$$

From the representation of $J_n(z)$ in (32) as a finite sum of ${}_n\Psi_0$ functions of rotated argument, the contribution from the different algebraic expansions when arg z lies in the common sector (34) can then be written in the form

$$\sum_{j=1}^{n}\frac{\mu_j}{m_j}\sum_{k=0}^{n}(-)^k T_{k,j}(z), \tag{36}$$

where

$$\begin{aligned} T_{0,j}(z) &= h_j(z), \qquad T_{1,j}(z)=\sum_{j_1=1}^{n}e(\nu_{j_1})h_j(ze(m_{j_1})),\\ T_{2,j}(z) &= \sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}e(\nu_{j_1}+\nu_{j_2})h_j(ze(m_{j_1}+m_{j_2})),\\ T_{3,j}(z) &= \sum_{j_1=1}^{n-2}\sum_{j_2=j_1+1}^{n-1}\sum_{j_3=j_2+1}^{n}e(\nu_{j_1}+\nu_{j_2}+\nu_{j_3})h_j(ze(m_{j_1}+m_{j_2}+m_{j_3})),\ \ldots \end{aligned}$$

and in general, for $1\le k\le n-1$,

$$T_{k,j}(z)=\sum_{j_1=1}^{n-k+1}\sum_{j_2=j_1+1}^{n-k+2}\cdots\sum_{j_k=j_{k-1}+1}^{n}e(\nu_{j_1}+\ldots+\nu_{j_k})h_j(ze(m_{j_1}+\ldots+m_{j_k}))$$

with

$$T_{n,j}(z)=e(N)h_j(ze(M)). \tag{37}$$

It will be sufficient to consider just one value of j and accordingly we choose $j=1$. Since the μ_j $(1\le j\le n)$ are even integers it follows from (35) that

$$e(\nu_j)h_j(ze(m_j))=h_j(z). \tag{38}$$

We now separate off the term corresponding to $j_1=1$ in the sums $T_{k,j}(z)$ and make repeated use of (38) to find

$$T_{1,1}(z)=h_1(z)+\sum_{j_1=2}^{n}e(\nu_{j_1})h_1(ze(m_{j_1})),$$

$$T_{2,1}(z)=\sum_{j_2=2}^{n}e(\nu_{j_2})h_1(ze(m_{j_2}))+\sum_{j_1=2}^{n-1}\sum_{j_2=j_1+1}^{n}e(\nu_{j_1}+\nu_{j_2})h_1(ze(m_{j_1}+m_{j_2})),$$

$$T_{3,1}(z) = \sum_{j_2=2}^{n-1}\sum_{j_3=j_2+1}^{n}e(\nu_{j_2}+\nu_{j_3})h_1(ze(m_{j_2}+m_{j_3}))$$

$$+\sum_{j_1=2}^{n-2}\sum_{j_2=j_1+1}^{n-1}\sum_{j_3=j_2+1}^{n} e(\nu_{j_1}+\nu_{j_2}+\nu_{j_3})h_1(ze(m_{j_1}+m_{j_2}+m_{j_3}))$$

and so on. An obvious relabelling of the summation indices then shows that in the inner sum in (36), taken over $0 \le k \le n-1$, all the terms cancel except the last to yield

$$\sum_{k=0}^{n-1}(-)^k T_{k,1}(z) = (-)^{n-1}\sum_{j_1=2}^{2}\sum_{j_2=j_1+1}^{3}\cdots\sum_{j_n=j_{n-1}+1}^{n} e(\nu_{j_1}+\ldots+\nu_{j_n})h_1(ze(m_{j_1}+\ldots+m_{j_n}))$$

$$= (-)^{n-1}e(\nu_2+\ldots+\nu_n)h_1(ze(m_2+\ldots+m_n)).$$

$$= (-)^{n-1}e(N)h_1(ze(M)) = (-)^{n-1}T_{n,1}(z)$$

upon application of (38). It therefore follows that $\sum_{k=1}^{n}(-)^k T_{k,1}(z) \equiv 0$. An analogous procedure applies to other values of $j \le n$ and so we obtain in the simple-pole case

$$\sum_{k=0}^{n}(-)^k T_{k,j}(z) \equiv 0 \qquad (1 \le j \le n) \tag{39}$$

in the sector (34). The asymptotic expansion of $J_n(z)$ in this sector is therefore purely exponential in character.

4.3. The Integral $K_{n,p}(z)$

The procedure described above can be applied to the variant of the integral $J_n(z)$ obtained by taking $p < n$ integrals evaluated over $(-\infty,\infty)$ with the rest being evaluated over $[0,\infty)$. Thus, if we define

$$K_{n,p}(z) = \lambda_n \int_0^\infty \cdots \int_0^\infty \left(\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty x_1^{\nu_1-1}\ldots x_n^{\nu_n-1}e^{-f(x_1,\ldots,x_n;z)}\,dx_1\ldots dx_p\right)dx_{p+1}\ldots dx_n, \tag{40}$$

where it is now supposed that μ_r $(1 \le r \le p)$ are even integers and $\mu_r > 0$ $(p+1 \le r \le n)$, then we obtain following the procedure described in Section 4 the series expansion

$$K_{n,p}(z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}\prod_{r=1}^{n}\Gamma\left(\frac{\nu_r+m_r k}{\mu_r}\right)\prod_{r=1}^{p}B_r(k). \tag{41}$$

In the evaluation of the integrals when $x_r < 0$ $(1 \le r \le p)$ we have again taken $x_r = |x_r|e^{\pi i}$.

The product $\prod_{r=1}^{p}B_r(k)$ may be expanded as a sum of exponentials so that $K_{n,p}(z)$ can be written as a finite sum of ${}_n\Psi_0(z)$ functions of rotated argument and parameters given in (22) in an analogous manner to that in (32). The asymptotic expansion of ${}_n\Psi_0(z)$ in (19) can then be employed to obtain the expansion of $K_{n,p}(z)$ as $|z| \to \infty$. We give an example of the asymptotic structure of $K_{n,p}(z)$ in Section 5.

5. Numerical Examples

In this section we give some numerical examples to illustrate the application of the expansion (19) to the construction of the asymptotic structure of the integrals $J_n(z)$ and $K_{n,p}(z)$ defined in (4) and (40).

5.1. Example 1

Let us consider the two-dimensional integral

$$J_2(z) = \mu_1\mu_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^{\nu_1-1}x_2^{\nu_2-1}\exp\{-x_1^{\mu_1}-x_2^{\mu_2}+zx_1^{m_1}x_2^{m_2}\}\,dx_1dx_2, \tag{42}$$

where $\nu_1, \nu_2 > 0$ and μ_1, μ_2 are positive even integers. From (32) and (21), this integral can be expressed in terms of the Wright function ${}_2\Psi_0(z)$ of rotated argument, where

$${}_2\Psi_0(z) \equiv {}_2\Psi_0\left(\begin{array}{c}(\frac{m_1}{\mu_1},\frac{\nu_1}{\mu_1}),(\frac{m_2}{\mu_2},\frac{\nu_2}{\mu_2})\\ -\end{array};z\right).$$

For simplicity in presentation we only consider the case when $m_1 = m_2 = m$. Then we find from (32)

$$J_2(z) = {}_2\Psi_0(z) - B_1\,{}_2\Psi_0(ze^{\pi im}) + B_2\,{}_2\Psi_0(ze^{2\pi im}), \tag{43}$$

where

$$B_1 = (e^{\pi i\nu_1}+e^{\pi i\nu_2}), \qquad B_2 = e^{\pi i(\nu_1+\nu_2)}. \tag{44}$$

Provided ν_1 and ν_2 are real, it is readily shown that $J_2(z)$ possesses a basic symmetry about the half-rays $\arg z = \omega$ and $\arg z = \omega - \pi$, where $\omega = \pi(1-m)$. For, upon recalling that the argument of ${}_p\Psi_q(z)$ can be written modulo 2π, we find[‡] from (43) with the above definition of ω that

$$\begin{aligned}
J_2(\overline{z}e^{i\omega}) &= e^{\pi i(\nu_1+\nu_2)}\Big\{{}_2\Psi_0(\overline{z}e^{i\omega+2\pi im}) - (e^{-\pi i\nu_1}+e^{-\pi i\nu_2})\,{}_2\Psi_0(\overline{z}e^{i\omega+\pi im}) \\
&\qquad\qquad + e^{-\pi i(\nu_1+\nu_2)}\,{}_2\Psi_0(\overline{z}e^{i\omega})\Big\} \\
&= e^{\pi i(\nu_1+\nu_2)}\Big\{{}_2\Psi_0(\overline{z}e^{-i\omega}) - (e^{-\pi i\nu_1}+e^{-\pi i\nu_2})\,{}_2\Psi_0(\overline{z}e^{-\pi i}) \\
&\qquad\qquad + e^{-\pi i(\nu_1+\nu_2)}\,{}_2\Psi_0(\overline{z}e^{i\omega-2\pi i})\Big\} \\
&= e^{\pi i(\nu_1+\nu_2)}\overline{J_2(ze^{i\omega})},
\end{aligned} \tag{45}$$

where the bar denotes the complex conjugate. Hence it is sufficient in this case to restrict our attention to an appropriate half-plane.

We display in Fig. 1 the large-$|z|$ sectorial behaviour of $J_2(z)$ in the case $\mu_1 = \mu_2 = 4$ for different values of m, where we suppose that $B_1 \neq 0$. In Fig. 1(a), $m = \frac{1}{4}$ ($\kappa = \frac{7}{8}$) so that the symmetry line is $\arg z = \frac{3}{4}\pi$, $-\frac{1}{4}\pi$ and, from (34), the algebraic expansions cancel in the

[‡]When $m_1 \neq m_2$, it can be shown that the symmetry relation (45) still holds with $\omega = \pi - \frac{1}{2}\pi(m_1+m_2)$.

sector $(0, \frac{3}{2}\pi)$. There are three overlapping exponentially large sectors $|\arg(ze^{\pi ir/4})| < \frac{7}{16}\pi$ $(r = 0, 1, 2)$, with the expansion in the sector $(\frac{7}{16}\pi, \frac{17}{16}\pi)$ being exponentially small. It can be seen that there are sectors in which $J_2(z)$ consists of either one, two or three exponentially large expansions. In Fig. 1(b), $m = \frac{1}{2}$ $(\kappa = \frac{3}{4})$ so that the symmetry line is the imaginary z-axis and the algebraic expansions cancel in the upper half-plane. There are again three exponentially large sectors $|\arg(ze^{\pi ir/2})| < \frac{3}{8}\pi$ $(r = 0, 1, 2)$, which overlap to produce sectors containing either one or two exponentially large expansions, with the exponentially small sector now being $(\frac{3}{8}\pi, \frac{5}{8}\pi)$. In Fig. 1(c), $m = \frac{3}{4}$ $(\kappa = \frac{5}{8})$ so that the symmetry line is $\arg z = \frac{1}{4}\pi$, $-\frac{3}{4}\pi$ and the algebraic expansions cancel in the sector $(0, \frac{1}{2}\pi)$. The exponentially large sectors $|\arg(ze^{3\pi ir/4})| < \frac{5}{16}\pi$ $(r = 0, 1, 2)$ only overlap in the sector $(\frac{3}{16}\pi, \frac{5}{16}\pi)$, with the behaviour in the sectors $(\frac{13}{16}\pi, \frac{15}{16}\pi)$ and $(-\frac{7}{16}\pi, -\frac{5}{16}\pi)$ consisting of algebraic and exponentially small expansions. Finally in Fig. 1(d), $m = 1$ $(\kappa = \frac{1}{2})$ so that the symmetry line is the real z-axis and, by (34), there is no longer a sector in which the algebraic expansions cancel. In this case, there are two non-overlapping exponentially large sectors given by $|\arg(ze^{\pi ir})| < \frac{1}{4}\pi$ $(r = 0, 1)$, with the behaviour of $J_2(z)$ in the sectors $(\frac{1}{4}\pi, \frac{3}{4}\pi)$ and $(-\frac{3}{4}\pi, -\frac{1}{4}\pi)$ consisting of algebraic and exponentially small expansions. If the parameters ν_1 and ν_2 are such that $B_1 = 0$, then the number of exponentially large sectors decreases by one.

We now consider the asymptotic expansion of $J_2(z)$ in (42) when $\mu_1 = \mu_2 = 4$ in some specific cases in more detail. We first take $m = \frac{1}{2}$ $(\kappa = \frac{3}{4})$ so that from (43)

$$J_2(z) = {}_2\Psi_0(z) - B_1\, {}_2\Psi_0(ze^{\frac{1}{2}\pi i}) + B_2\, {}_2\Psi_0(ze^{\mp\pi i}), \tag{46}$$

where we choose the upper or lower sign according as $\arg z > 0$ or $\arg z < 0$, respectively. Recalling from (19) that the Stokes lines for the exponential expansion $E_{p,q}(z)$ are given by $\arg z = \pm\pi\kappa$, we see that the Stokes lines associated with $E_{2,0}(ze^{\frac{1}{2}\pi ir})$ $(r = 0, 1, 2)$ are the rays $\arg z = \pm\frac{3}{4}\pi, \pm\frac{1}{4}\pi$; see Fig. 1(b). Taking into account these Stokes lines and the fact that the algebraic expansions all cancel in the upper half-plane by (34), we find that the exponential expansion of $J_2(z)$ in (46) is then given by

$$\begin{array}{ll} E_{2,0}(z) + B_2\, E_{2,0}(ze^{-\pi i}) & \text{in } (\frac{1}{4}\pi, \frac{1}{2}\pi] \\ E_{2,0}(z) - B_1\, E_{2,0}(ze^{\frac{1}{2}\pi i}) & \text{in } (-\frac{1}{4}\pi, \frac{1}{4}\pi) \\ E_{2,0}(z) - B_1\, E_{2,0}(ze^{\frac{1}{2}\pi i}) + B_2\, E_{2,0}(ze^{\pi i}) & \text{in } [-\frac{1}{2}\pi, -\frac{1}{4}\pi), \end{array}$$

as $|z| \to \infty$ in the right-half plane. The expansion $E_{2,0}(z)$ is obtained from (8) with $Z = \frac{3}{8}z^{4/3}$ and $\vartheta = \frac{1}{4}(\nu_1 + \nu_2) - 1$. The coefficient A_0 is specified by (24) with the coefficients A_j $(j \geq 1)$ being determined in specific cases by the algorithm described in Appendix A. The expansion of $J_2(z)$ is exponentially small in the sector $(\frac{3}{8}\pi, \frac{5}{8}\pi)$. Although there is an algebraic expansion present in the lower half-plane, we do not consider its contribution here as it is subdominant throughout this domain. The exponential expansion in the left-hand half-plane can be obtained via (45).

For our second case, we take $m = \frac{3}{4}$ $(\kappa = \frac{5}{8})$ and $\nu_1 = \frac{1}{2}$, $\nu_2 = \frac{3}{2}$, so that from (44) we

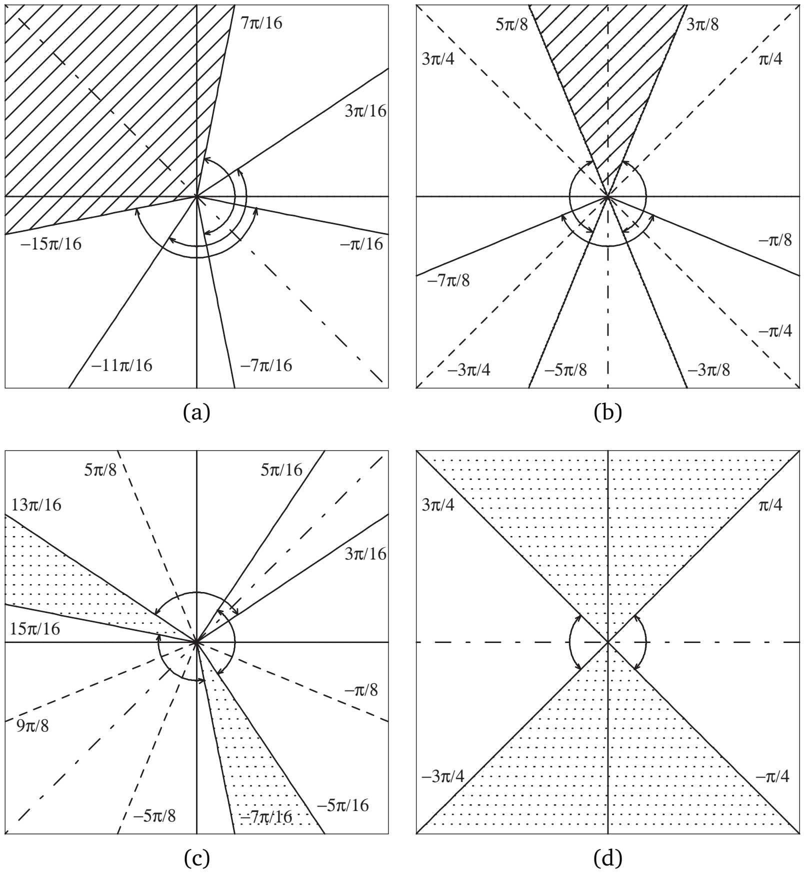

Figure 1: The sectorial behaviour of $J_2(z)$ for $\mu_1 = \mu_2 = 4$ and $m_1 = m_2 = m$ when it is supposed that $B_1 \neq 0$: (a) $m = \frac{1}{4}$, (b) $m = \frac{1}{2}$, (c) $m = \frac{3}{4}$ and (d) $m = 1$. The sectors marked with a circular arc with arrows denote exponentially large sectors. The hatched regions denote exponentially small behaviour and the shaded regions denote mixed algebraic and exponentially small behaviour. The dashed lines are Stokes lines and the dash-dot line is the axis of basic symmetry.

have $B_1 = 0$, $B_2 = 1$ and

$$J_2(z) = {}_2\Psi_0(z) + {}_2\Psi_0(ze^{\frac{3}{2}\pi i}) = {}_2\Psi_0(z) + {}_2\Psi_0(ze^{-\frac{1}{2}\pi i}). \tag{47}$$

The sector in which the algebraic expansions associated with $J_2(z)$ cancel is $(0, \frac{1}{2}\pi)$. Referring to Fig. 2(a), we see that the expansion of $J_2(z)$ as $|z| \to \infty$ is exponentially large in the sector $(-\frac{5}{16}\pi, \frac{13}{16}\pi)$, where $E_{2,0}(z)$ is given by (8) with $Z = \frac{5}{64}(6^{3/5}z^{8/5})$ and $\vartheta = -\frac{1}{2}$. In the sectors $(\frac{13}{16}\pi, \frac{9}{8}\pi)$ and $(-\frac{5}{8}\pi, -\frac{5}{16}\pi)$ the expansion of $J_2(z)$ is mixed algebraic and exponentially small, whereas due to the presence of the Stokes lines associated with the exponential expansions on $\arg z = -\frac{5}{8}\pi$ and $\arg z = \frac{9}{8}\pi$ the expansion in the sector $|\arg(ze^{-3\pi i/4})| < \frac{1}{8}\pi$ is purely algebraic.

From (12) and (13), the algebraic expansion $H_{2,0}(z)$ is controlled by the poles situated at

$$s_{k,1}=\tfrac{2}{3}+\tfrac{16}{3}k, \qquad s_{k,2}=2+\tfrac{16}{3}k \qquad (k=0,1,2,\ldots)$$

which are all simple, and consequently

$$H_{2,0}(z)=\frac{16}{3}\sum_{j=1}^{2}\sum_{k=0}^{\infty}\frac{(-)^k}{k!}\Gamma(s_{k,j})\Gamma(\gamma_j-\tfrac{3}{16}s_{k,j})z^{-s_{k,j}}$$

with $\gamma_1=\frac{3}{8}$ and $\gamma_2=\frac{1}{8}$. Taking into account the Stokes lines on $\arg z=0$ and $\arg z=\frac{1}{2}\pi$ for the algebraic expansions associated with ${}_2\Psi_0(z)$ and ${}_2\Psi_0(ze^{-\pi i/2})$, respectively, we see from (15) and (16) that the algebraic expansion $H(z)$ of $J_2(z)$ is given by

$$H(z)=\begin{cases} H_{2,0}(ze^{-\pi i})+H_{2,0}(ze^{\frac{1}{2}\pi i})\equiv 0 & \text{in } (0,\frac{1}{2}\pi)\\ H_{2,0}(ze^{-\pi i})+H_{2,0}(ze^{-\frac{3}{2}\pi i}) & \text{in } (\frac{1}{2}\pi,2\pi)\end{cases}$$

It is easily verified with the above form of $H_{2,0}(z)$ that $H(z)\equiv 0$ in the sector $(0,\frac{1}{2}\pi)$, in accordance with (34). Then the asymptotic expansion of $J_2(z)$ in (47) has the form

$$J_2(z)\sim\begin{cases} E_{2,0}(z)+E_{2,0}(ze^{-\frac{1}{2}\pi i}) & \text{in } (0,\frac{1}{2}\pi)\\ E_{2,0}(z)+E_{2,0}(ze^{-\frac{1}{2}\pi i})+H(z) & \text{in } (\frac{1}{2}\pi,\frac{5}{8}\pi)\\ E_{2,0}(ze^{-\frac{1}{2}\pi i})+H(z) & \text{in } (\frac{5}{8}\pi,\frac{9}{8}\pi)\\ H(z) & \text{in } (\frac{9}{8}\pi,\frac{5}{4}\pi]\end{cases}$$

as $|z|\to\infty$, with that in the remainder of the plane being determined by the symmetry relation (45).

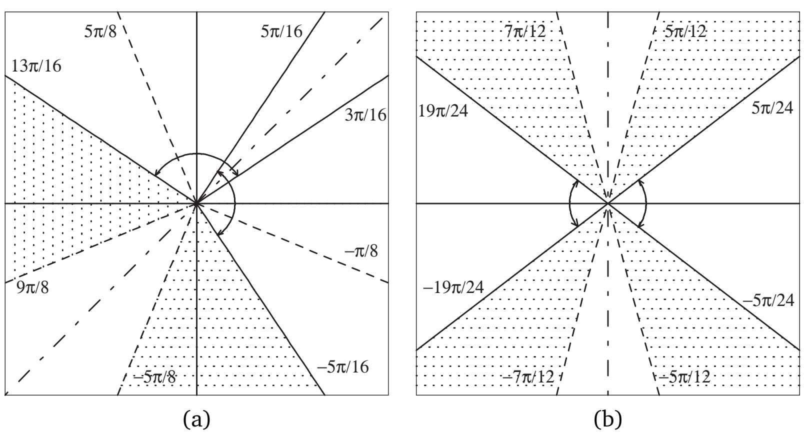

Figure 2: The sectorial behaviour of (a) $J_2(z)$ for $\mu_1=\mu_2=4$ and $m_1=m_2=\frac{3}{4}$ when it is supposed that $B_1=0$ and (b) of $K_{3,1}(z)$ for $\mu_1=\mu_2=3$, $\mu_3=4$ and $m_1=m_2=\frac{1}{2}$, $m_3=1$. The sectors marked with a circular arc with arrows denote exponentially large sectors. The shaded regions denote mixed algebraic and exponentially small behaviour. The dashed lines are Stokes lines and the dash-dot line is the axis of basic symmetry.

Finally, we consider $m = 1$ ($\kappa = \frac{1}{2}$) and $\nu_1 = \nu_2 = \frac{1}{4}$, so that $B_1 = 2e^{\pi i/4}$ and $B_2 = i$ and

$$J_2(z) = (1+i)\,{}_2\Psi_0(z) - 2e^{\frac{1}{4}\pi i}\,{}_2\Psi_0(ze^{-\pi i}), \tag{48}$$

where we have replaced the argument of the second Wright function by $ze^{-\pi i}$. From Fig. 1(d), the Stokes lines coincide with the imaginary axis and the symmetry axis is the real axis. The exponential expansion $E_{2,0}(z)$ in (8) has $Z = \frac{1}{8}z^2$ and $\vartheta = -\frac{7}{8}$. The poles in (12) are all double situated at $s_k = 4k + \frac{1}{4}$ $(k = 0, 1, 2, \ldots)$ and, from (60), we obtain the algebraic expansion in this case given by

$$H_{2,0}(z) = -16\sum_{k=0}^{\infty}\frac{\Gamma(4k+\frac{1}{4})}{(k!)^2}\{\psi(4k+\tfrac{1}{4}) - \tfrac{1}{2}\psi(k+1) - \log z\}z^{-4k-\frac{1}{4}}.$$

Since, by (34), there is no sector in which the algebraic expansions cancel when $m = 1$, we obtain the expansion of $J_2(z)$ in (48) given by

$$J_2(z) \sim \begin{cases} (1+i)\{E_{2,0}(z) + H_{2,0}(ze^{-\pi i})\} - 2e^{\frac{1}{4}\pi i}H_{2,0}(z) & \text{in } [0, \frac{1}{2}\pi) \\ \\ (1+i)H_{2,0}(ze^{-\pi i}) - 2e^{\frac{1}{4}\pi i}\{E(ze^{-\pi i}) + H_{2,0}(z)\} & \text{in } (\frac{1}{2}\pi, \pi] \end{cases}$$

as $|z| \to \infty$. The expansion in the lower half-plane can be obtained by (45). The rays $\arg z = 0, \pi$ are Stokes lines for the algebraic expansions $H_{2,0}(ze^{-\pi i})$ and $H_{2,0}(z)$, respectively and the rays $\arg z = \pm\frac{1}{2}\pi$ are Stokes lines for the exponential expansions.

In Table 1 we present the absolute relative errors in the asymptotic expansion of $J_2(z)$ in (46), (47) and (48) for a given value of $|z|$ and varying $\theta = \arg z$. In each case the exponential and algebraic expansions have been optimally truncated, with the exact value of $J_2(z)$ being computed both by evaluation of the Wright functions and also high-precision numerical quadrature of the integral in (42). We remark that in the three cases considered, an accurate determination of the subdominant expansions on the Stokes lines would require a detailed treatment of the Stokes phenomenon.

5.2. Example 2

We consider an example of the integral $K_{n,p}(z)$ defined in (40) with $n = 3$, $p = 1$, namely

$$K_{3,1}(z) = 36\int_{-\infty}^{\infty}\left(\int_0^{\infty}\int_0^{\infty}(x_3/x_2)^{\frac{1}{2}}\exp\{-(x_1^3 + x_2^3 + x_3^4 - z(x_1x_2)^{\frac{1}{2}}x_3\}\,dx_1dx_2\right)dx_3,$$

which is associated with the parameters $\mu_1 = \mu_2 = 3$, $\mu_3 = 4$, $m_1 = m_2 = \frac{1}{2}$, $m_3 = 1$ and $\nu_1 = 1$, $\nu_2 = \frac{1}{2}$, $\nu_3 = \frac{3}{2}$. From (41), we therefore find

$$K_{3,1}(z) \;=\; \sum_{k=0}^{\infty}\frac{z^k}{k!}\prod_{r=1}^{3}\Gamma\left(\frac{\nu_r + m_r k}{\mu_r}\right)(1 - e(\nu_3 + m_3 k))$$

Table 2: Values of the absolute relative error in the computation of $J_2(z)$ when $\mu_1 = \mu_2 = 4$ and $m_1 = m_2 = m$ as a function of $\theta = \arg z$.

$m=\frac{1}{2},\ \nu_1=\frac{1}{2},\ \nu_2=1$, $\lvert z\rvert=20$		$m=\frac{3}{4},\ \nu_1=\frac{1}{2},\ \nu_2=\frac{3}{2}$, $\lvert z\rvert=20$		$m=1,\ \nu_1=\nu_2=\frac{1}{4}$, $\lvert z\rvert=15$	
θ/π	|Rel. Error|	θ/π	|Rel. Error|	θ/π	|Rel. Error|
0	2.172×10^{-10}	0.250	8.081×10^{-14}	0	4.869×10^{-11}
0.125	3.326×10^{-10}	0.375	2.528×10^{-13}	0.125	9.220×10^{-12}
0.250	1.228×10^{-9}	0.500	1.104×10^{-12}	0.250	4.427×10^{-14}
0.375	3.939×10^{-8}	0.625	2.528×10^{-13}	0.375	1.025×10^{-15}
0.500	1.756×10^{-4}	0.750	1.414×10^{-13}	0.500	5.061×10^{-15}
−0.125	3.326×10^{-10}	0.875	7.550×10^{-14}	0.625	1.046×10^{-15}
−0.250	3.684×10^{-10}	1.000	7.014×10^{-14}	0.750	5.306×10^{-14}
−0.375	3.326×10^{-10}	1.125	2.584×10^{-13}	0.875	9.220×10^{-12}
−0.500	2.172×10^{-10}	1.250	1.058×10^{-13}	1.000	4.869×10^{-11}

$$= \ {}_3\Psi_0(z) + i\,{}_3\Psi_0(ze^{\mp\pi i}),$$

where the ${}_3\Psi_0(z)$ function is that defined in (25) with the same parameter values.

It is easily shown that $K_{3,1}(\bar{z}e^{\pm\pi i}) = i\overline{K_{3,1}(z)}$, where the bar denotes the complex conjugate, so that there is a basic symmetry about the imaginary axis. The sectorial behaviour of $K_{3,1}(z)$ is shown in Fig. 2(b) with $\kappa = \frac{5}{12}$. There are two exponentially large sectors, four sectors with mixed algebraic and exponentially small behaviour and two sectors straddling the imaginary axis of angular width $\frac{1}{6}\pi$ (bounded by the Stokes lines) in which the large-$|z|$ behaviour is algebraic. From the asymptotic expansion of ${}_3\Psi_0(z)$ given in (29) we then obtain

$$K_{3,1}(z) \sim \begin{cases} E_{3,0}(z) + H_{3,0}(ze^{-\pi i}) + iH_{3,0}(z) & \text{in } (0, \frac{5}{12}\pi) \\ H_{3,0}(ze^{-\pi i}) + iH_{3,0}(z) & \text{in } (\frac{5}{12}\pi, \frac{1}{2}\pi] \\ E_{3,0}(z) + H_{3,0}(ze^{\pi i}) + iH_{3,0}(z) & \text{in } (-\frac{5}{12}\pi, 0) \\ H_{3,0}(ze^{\pi i}) + iH_{3,0}(z) & \text{in } [-\frac{1}{2}\pi, -\frac{5}{12}\pi) \end{cases}$$

as $|z| \to \infty$, where the expansions $E_{3,0}(z)$ and $H_{3,0}(z)$ are given in (26), (27) and (28). The expansion of $K_{3,1}(z)$ in the left-hand half-plane is described by the above symmetry relation.

6. Concluding Remarks

We have shown how the n-dimensional analogues of Faxén's integral in (1), (4) and (40) can be expressed in terms of either a single Wright function ${}_n\Psi_0(z)$, or a linear combination of such functions with rotated arguments. Knowledge of the asymptotic expansion of ${}_n\Psi_0(z)$ for $|z| \to \infty$ then enables the asymptotic structure of these integrals to be determined. Not surprisingly, this asymptotic structure becomes more complicated the larger the value of n.

The asymptotic behaviour of ${}_n\Psi_0(z)$ for large $|z|$ consists of an exponential expansion and an algebraic expansion. The formal sum $E_{n,0}(z)$ is a compact representation of the exponential expansion in the n-dimensional case. The evaluation of the coefficients in this expansion can be easily carried out in specific cases for low values of n, although the computational effort involved in their calculation rapidly increases with the dimension of the integrals. The algebraic expansion $H_{n,0}(z)$ consists, in general, of n different expansions each with its own asymptotic scale. The occurrence of terms in $\log z$ (when $n \geq 2$) depends to a considerable degree on the symmetry in the associated Newton diagram of the phase function $f(x_1, \ldots, x_n; z)$ defined in (2).

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Appendix A. An algorithm for the computation of the coefficients $c_j = A_j/A_0$

We describe an algorithm for the computation of the normalised coefficients A_j/A_0 appearing in the exponential expansion $E_{p,q}(z)$ in (8). Methods of computing these coefficients by recursion in the case when $\alpha_r = \beta_r = 1$ have been given by Riney [15] and Wright [22]. Here we describe an algebraic method valid for arbitrary $\alpha_r > 0$ and $\beta_r > 0$; see also [12, pp. 46–49].

We rewrite the inverse factorial expansion (9) in the form

$$\frac{g(s)\Gamma(\kappa s + \vartheta')}{\Gamma(s+1)} = \kappa A_0 (h\kappa^\kappa)^s \left\{ \sum_{j=0}^{M-1} \frac{c_j}{(\kappa s + \vartheta')_j} + \frac{O(1)}{(\kappa s + \vartheta')_M} \right\}, \tag{49}$$

for $|s| \to \infty$ uniformly in $|\arg s| \le \pi - \epsilon$, where $g(s)$ is the ratio of gamma functions defined in (5), $(a)_j = \Gamma(a+j)/\Gamma(a)$ and $c_j = A_j/A_0$. Introduction of the scaled gamma function $\Gamma^*(z)$ defined by

$$\Gamma^*(z) := \Gamma(z)(2\pi)^{-\frac{1}{2}} e^z z^{\frac{1}{2}-z}$$

leads to the representation

$$\Gamma(\alpha s + a) = \Gamma^*(\alpha s + a)(2\pi)^{\frac{1}{2}} e^{-\alpha s}(\alpha s)^{\alpha s + a - \frac{1}{2}} e(\alpha s; a),$$

where

$$e(\alpha s; a) = \exp\left\{(\alpha s + a - \tfrac{1}{2}) \log\left(1 + \frac{a}{\alpha s}\right) - a\right\}.$$

Some straightforward algebra then shows that the left-hand side of (49) becomes

$$\frac{g(s)\Gamma(\kappa s + \vartheta')}{\Gamma(s+1)} = \kappa A_0 (h\kappa^\kappa)^s R(s)\Upsilon(s), \tag{50}$$

where

$$\Upsilon(s) = \frac{\prod_{r=1}^{p} \Gamma^*(\alpha_r s + a_r)}{\prod_{r=1}^{q} \Gamma^*(\beta_r s + b_r)} \frac{\Gamma^*(\kappa s + \vartheta')}{\Gamma^*(s+1)}$$

and

$$R(s) = \frac{\prod_{r=1}^{p} e(\alpha_r s; a_r)}{\prod_{r=1}^{q} e(\beta_r s; b_r)} \frac{e(\kappa s; \vartheta')}{e(s; 1)}.$$

Substitution of (50) into (49) finally produces

$$R(s)\Upsilon(s) = \sum_{j=0}^{M-1} \frac{c_j}{(\kappa s + \vartheta')_j} + \frac{O(1)}{(\kappa s + \vartheta')_M} \tag{51}$$

as $|s| \to \infty$ in $|\arg s| \le \pi - \epsilon$.

Now let $\chi = (\kappa s)^{-1}$ and expand $R(s)$ and $\Upsilon(s)$ for $\chi \to 0$ making use of the well-known expansion [19, p. 71], [12, p. 32]

$$\Gamma^*(z) \sim \sum_{k=0}^{\infty} (-)^k \gamma_k z^{-k} \qquad (|z| \to \infty;\ |\arg z| \le \pi - \epsilon),$$

where γ_k are the Stirling coefficients. The first few coefficients are given by $\gamma_0 = 1$, $\gamma_1 = -\frac{1}{12}$, $\gamma_2 = \frac{1}{288}$, $\gamma_3 = \frac{139}{51840}, \ldots$. Some routine algebra then yields

$$\Gamma^*(\alpha s + a) = 1 - \frac{\gamma_1 \kappa \chi}{\alpha} + O(\chi^2), \qquad e(\alpha s; a) = 1 + \frac{\kappa \chi}{2\alpha} a(a-1) + O(\chi^2),$$

whence

$$R(s) = 1 + \frac{\kappa\chi}{2}\left\{\sum_{r=1}^{p} \frac{a_r(a_r - 1)}{\alpha_r} - \sum_{r=1}^{q} \frac{b_r(b_r - 1)}{\beta_r} - \frac{\vartheta}{\kappa}(1 - \vartheta)\right\} + O(\chi^2),$$

$$\Upsilon(s) = 1 + \frac{\kappa\chi}{12}\left\{\sum_{r=1}^{p} \frac{1}{\alpha_r} - \sum_{r=1}^{q} \frac{1}{\beta_r} + \frac{1}{\kappa} - 1\right\} + O(\chi^2).$$

Upon equating coefficients of χ in (51) we obtain

$$c_1 = \tfrac{1}{2}\kappa(\mathscr{A} + \tfrac{1}{6}\mathscr{B}), \tag{52}$$

where

$$\mathscr{A} = \sum_{r=1}^{p} \frac{a_r(a_r-1)}{\alpha_r} - \sum_{r=1}^{q} \frac{b_r(b_r-1)}{\beta_r} - \frac{\vartheta}{\kappa}(1-\vartheta),$$

$$\mathscr{B} = \sum_{r=1}^{p} \frac{1}{\alpha_r} - \sum_{r=1}^{q} \frac{1}{\beta_r} + \frac{1}{\kappa} - 1.$$

The higher coefficients are then obtained by continuation of this expansion process applied to $R(s)$ and $\Upsilon(s)$ in (51) with the help of *Mathematica*. In specific cases (i.e., with numerical values for the various parameters) it is possible to generate the coefficients in this manner quite easily. In our computations we have used up to a maximum of 40 coefficients.

Appendix B. The Asymptotic Expansion of ${}_2\Psi_0(z)$

We demonstrate the validity of the assertion in (19) concerning the asymptotic expansion of the Wright function ${}_p\Psi_q(z)$ as $|z| \to \infty$ by considering a particular case. Let us take $p = 2$, $q = 0$ with the parameter values $\alpha_1 = \alpha_2 = \frac{1}{4}$ and $\nu_1 = \nu_2 = \frac{1}{8}$; that is, we consider the function

$${}_2\Psi_0(z) \equiv {}_2\Psi_0\left(\begin{array}{c} (\frac{1}{4}, \frac{1}{8}), (\frac{1}{4}, \frac{1}{8}) \\ - \end{array} ; z \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \Gamma^2(\tfrac{1}{4}k + \tfrac{1}{8}). \tag{53}$$

From (7), this function is associated with the parameters $\kappa = \frac{1}{2}$, $h = \frac{1}{2}$ and $\vartheta = -\frac{3}{4}$. The exponential expansion is, from (8) and (10), then given by

$$E_{2,0}(z) = Z^{-3/4} e^Z \sum_{j=0}^{\infty} A_j Z^{-j}, \qquad Z = \tfrac{1}{8} z^2, \tag{54}$$

where $A_0 = 2^{9/4}\pi$; we have employed coefficients with $j \le 28$ in our computations (see Appendix A). The first ten coefficients $c_j \equiv A_j/A_0$ for ${}_2\Psi_0(z)$ in (53) are listed in Table 3.

Table 3: The coefficients c_j for $1 \le j \le 10$ associated with the function in (53).

j	c_j	j	c_j
1	$\frac{13}{16}$	2	$\frac{729}{512}$
3	$\frac{31575}{8192}$	4	$\frac{7432635}{524288}$
5	$\frac{554191155}{8388608}$	6	$\frac{100179200205}{268435456}$
7	$\frac{10645956497295}{4294967296}$	8	$\frac{10406881110208275}{549755813888}$
9	$\frac{1437596137005803775}{8796093022208}$	10	$\frac{44306301734958080317 5}{281474976710656}$

From (11), the Mellin-Barnes integral representation for ${}_2\Psi_0(z)$ in (53) is given by

$$ {}_2\Psi_0(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(s)\,\Gamma^2(\tfrac{1}{8}-\tfrac{1}{4}s)(ze^{\mp\pi i})^{-s}ds \qquad (|\arg(-z)|<\tfrac{3}{4}\pi), $$

where the upper or lower sign is chosen according as $\arg z > 0$ or $\arg z < 0$, respectively and the integration path separates the poles of $\Gamma(s)$ from the sequence of double poles of $\Gamma^2(\frac{1}{8}-\frac{1}{4}s)$ situated at $s = 4k+\frac{1}{2}$, $k = 0,1,2,\ldots$. Displacement of the integration path over the sequence of double poles and evaluation of the residues leads to the algebraic expansion

$$ H_{2,0}(ze^{\mp\pi i}) = -16\sum_{k=0}^{\infty}\frac{\Gamma(4k+\frac{1}{2})}{(k!)^2}(ze^{\mp\pi i})^{-4k-\frac{1}{2}}\left\{\psi(4k+\tfrac{1}{2})-\tfrac{1}{2}\psi(k+1)-\log(ze^{\mp\pi i})\right\}, \quad (55) $$

where ψ denotes the logarithmic derivative of the gamma function; compare (60).

Theorems 1 and 2 show that ${}_2\Psi_0(z)$ is exponentially large given by (54) in the sector $|\arg z| < \frac{1}{4}\pi$, with the dominant expansion in the rest of the z-plane being the algebraic expansion in (55). The exponential expansion $E_{2,0}(z)$, which is subdominant in the sectors $\frac{1}{4}\pi < |\arg z| < \frac{3}{4}\pi$, is maximally subdominant on the rays (the Stokes lines) $\arg z = \pm\frac{1}{2}\pi$. Accordingly, as $|\arg z|$ increases, the expansion $E_{2,0}(z)$ should undergo a Stokes phenomenon and switch off smoothly across the rays $\arg z = \pm\frac{1}{2}\pi$, to leave the algebraic expansion $H_{2,0}(ze^{\mp\pi i})$ in the sectors $\frac{1}{2}\pi < |\arg z| \le \pi$, as stated in (19); that is

$$ {}_2\Psi_0(z) \sim \begin{cases} E_{2,0}(z) + H_{2,0}(ze^{\mp\pi i}) & \text{in } \ |\arg z| \le \frac{1}{2}\pi - \epsilon \\ H_{2,0}(ze^{\mp\pi i}) & \text{in } \ \frac{1}{2}\pi + \epsilon \le |\arg z| \le \pi \end{cases} \quad (56) $$

as $|z| \to \infty$.

To demonstrate this, we set $z = |z|e^{i\theta}$ and define the Stokes multiplier $S(\theta)$ (at fixed $|z|$) for the Stokes line $\arg z = \frac{1}{2}\pi$ by

$$ {}_2\Psi_0(z) = H_{2,0}^{\mathrm{opt}}(ze^{\mp\pi i}) + A_0 Z^{-3/4} e^Z S(\theta), $$

where the superscript ‘opt’ denotes that the algebraic expansion is truncated at its optimal truncation point and Z is defined in (54). In the first half of Table 4 we show the values[§] of $\mathrm{Re}(S)$ for varying θ in the neighbourhood of $\theta = \frac{1}{2}\pi$ when $|z| = 15$, where the value of ${}_2\Psi_0(z)$ has been computed by high-precision summation of (53). The second half of Table 4 displays the absolute error in the computation of ${}_2\Psi_0(z)$ using the asymptotic expansion in (56) in the sector $\frac{1}{2}\pi \le \theta \le \frac{3}{4}\pi$ with the same value of $|z|$. The values in the column labelled (a) were obtained using the first expansion in (56), that is with the exponential expansion retained in the sector $\frac{1}{2}\pi < \theta \le \frac{3}{4}\pi$, whereas those in the column labelled (b) were obtained using the second expansion in (56). Both asymptotic series were truncated at their respective optimal truncation points. The first half of Table 4 confirms that the exponential expansion $E_{2,0}(z)$ switches off (as $\arg z$ increases) across the Stokes line $\arg z = \frac{1}{2}\pi$ to leave the algebraic

[§]The Stokes multiplier $S(\theta)$ has a small imaginary part that we do not show.

Table 4: The variation of the real part of the Stokes multiplier $S(\theta)$ and the absolute error in the computation of ${}_2\Psi_0(z)$ for different θ in the sector $\frac{1}{2}\pi \le \theta \le \frac{3}{4}\pi$ when $|z| = 15$: (a) with $E_{2,0}(z)$ and (b) without $E_{2,0}(z)$.

θ/π	Re(S)	θ/π	Re(S)	θ/π	|Error| (a)	|Error| (b)
0.40	0.97776	0.51	0.32448	0.50	4.010×10^{-13}	3.662×10^{-13}
0.45	0.94397	0.52	0.20592	0.55	2.819×10^{-12}	1.324×10^{-13}
0.47	0.83063	0.53	0.11579	0.60	1.571×10^{-10}	7.241×10^{-14}
0.48	0.72932	0.54	0.05599	0.65	7.955×10^{-8}	4.808×10^{-14}
0.49	0.60226	0.55	0.02191	0.70	2.036×10^{-4}	3.569×10^{-14}
0.50	0.46177	0.60	0.00008	0.75	1.222×10^{-0}	2.854×10^{-14}

expansion $H_{2,0}(ze^{-\pi i})$ in the remainder of the upper half-plane; a similar behaviour applies across the Stokes line arg $z = -\frac{1}{2}\pi$. The values of the absolute error in the second half of the table clearly indicate that a uniform accuracy over the sector $\frac{1}{2}\pi < \theta \le \frac{3}{4}\pi$ is achievable by discarding the exponential expansion in this sector in accordance with the second expansion in (56).

We remark that a detailed analysis of the Stokes multiplier has been carried out in [10] for the more general function ${}_p\Psi_0(z)$ with the parameters $\alpha_r = 1/n$ $(1 \le r \le p)$, for positive integer n, and general¶ a_r $(1 \le r \le p)$. It was shown that for large $|z|$ the leading behaviour of the Stokes multiplier $S(\theta)$ across the Stokes lines $\theta = \pm\pi\kappa$ is given by

$$S(\theta) \simeq \tfrac{1}{2} \pm \tfrac{1}{2}\mathrm{erf}\,[(\theta \mp \pi\kappa)(2\kappa/n)^{-1/2}(|z|/n)^{1/(2\kappa)}] \qquad (|z| \to \infty)$$

respectively, where erf denotes the error function and $\kappa = 1 - (p/n)$. Specialisation to the values $p = 2$, $n = 4$, to correspond to (53), shows the smooth transition of $S(\theta)$ across the Stokes lines $\theta = \pm\frac{1}{2}\pi$, thereby confirming the above viewpoint.

Appendix C. The Algebraic Expansion of $J_n(z)$ in the Case of Double Poles

When some, or all, of the poles in (12) are multiple, the analysis of the algebraic contributions to the integral $J_n(z)$ presented in Section 4.2 no longer applies. The treatment of the multiple-pole case in general would be very tedious. Accordingly, we demonstrate in the case $n = 2$ when double poles are present that the cancelation of the algebraic expansions associated with $J_n(z)$ continues to hold in the sector (34), where $m = m_1 + m_2$.

The algebraic expansion for the Wright function ${}_2\Psi_0(z)$ associated with the parameters α_r, a_r $(r = 1, 2)$ as $|z| \to \infty$ is given by (16) with $p = 2$, $q = 0$. The form of the expansion $H_{2,0}(ze^{-\pi i})$ in (13) and (14) has to be modified to take into account the presence of the double poles. From the Mellin-Barnes representation in (11), we have

$${}_2\Psi_0(z) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(s)\Gamma(a_1 - \alpha_1 s)\Gamma(a_2 - \alpha_2 s)(ze^{\pm\pi i})^{-s} ds \tag{57}$$

¶It is assumed the parameters are such that only simple poles arise in the corresponding integral (11).

valid in $|\arg(-z)| < \frac{1}{2}\pi(1+\alpha_1+\alpha_2)$, where the upper or lower sign is chosen according as $\arg z > 0$ or $\arg z < 0$, respectively. The two sequences of poles that contribute to the algebraic expansion are, from (12), $s_{m,r} = (a_r + m)/\alpha_r$ $(r = 1,2;\ m = 0,1,2,\ldots)$, where we suppose for some nonnegative integers k, ℓ that

$$s_{k,1} = \frac{a_1 + k}{\alpha_1} = \frac{a_2 + \ell}{\alpha_2} \tag{58}$$

for double poles to arise. The algebraic expansion $H_{2,0}(ze^{-\pi i})$, obtained by displacement of the integration path in (57) to the right over the above poles, then becomes

$$H_{2,0}(ze^{-\pi i}) = \sum_{j=1}^{2} \alpha_j^{-1}(ze^{-\pi i})^{-a_j/\alpha_j} S'_{2,0}(ze^{-\pi i}; j) + G(z) \tag{59}$$

where the prime denotes the deletion of the terms in the asymptotic sum (14) corresponding to the double poles. The contribution $G(z)$ resulting from the double poles may be shown to be

$$G(z) = \frac{1}{\alpha_1\alpha_2} \sum_{k,\ell} \frac{(-)^{k+\ell+1}}{k!\,\ell!}\, \Gamma(s_{k,1})(ze^{-\pi i})^{-s_{k,1}} \{\psi(s_{k,1}) - \alpha_1\psi(k+1) - \alpha_2\psi(\ell+1) - \log(ze^{-\pi i})\}, \tag{60}$$

where summation is over the integers k, ℓ satisfying (58) and $\psi(z)$ denotes the logarithmic derivative of the gamma function.

The integral $J_2(z)$ is associated with the function ${}_2\Psi_0(z)$ with the parameters $\alpha_r = m_r/\mu_r$, $a_r = \nu_r/\mu_r$ $(r = 1,2)$, where μ_r are positive even integers. The contribution to the algebraic expansion of $J_2(z)$ from the first series on the right-hand side of (59) (resulting from the simple poles) vanishes in the sector (34) by virtue of the discussion in Section 4.2. To deal with the contribution from the double poles, we note that $G(z)$ may be written in the form

$$G(z) = \sum_{k,\ell} (ze^{-\pi i})^{-\lambda_k} \{c_{k,\ell} + d_{k,\ell} \log(ze^{-\pi i})\},$$

where $c_{k,\ell}$ and $d_{k,\ell}$ are coefficients independent of z and

$$\lambda_k := \frac{\nu_1 + \mu_1 k}{m_1} = \frac{\nu_2 + \mu_2 \ell}{m_2}. \tag{61}$$

Recalling that $e(x) \equiv \exp(\pi i x)$, we then see that

$$e(\nu_1)G(ze(m_1)) = G(z) + \pi i m_1 \sum_{k,\ell} d_{k,\ell}(ze^{-\pi i})^{-\lambda_k}.$$

From (32) with $n = 2$, the contribution to the algebraic expansion of $J_2(z)$ resulting from the logarithmic series $G(z)$ in the common sector (34) is then

$$G(z) \quad - \quad e(\nu_1)\,G(ze(m_1)) - e(\nu_2)\,G(ze(m_2)) + e(\nu_1+\nu_2)\,G(ze(m_1+m_2))$$

$$
\begin{aligned}
&= -\pi i m_1 \sum_{k,\ell} d_{k,\ell}(ze^{-\pi i})^{-\lambda_k} - e(\nu_2)G(ze(m_2)) \\
&\quad + e(\nu_2)\left\{G(ze(m_2)) + \pi i m_1 \sum_{k,\ell} d_{k,\ell}(ze(m_2-1))^{-\lambda_k}\right\} \\
&= \pi i m_1 \sum_{k,\ell} d_{k,\ell}(ze^{-\pi i})^{-\lambda_k}\left\{e(\nu_2 - m_2\lambda_k) - 1\right\} \equiv 0
\end{aligned}
$$

since $e(\nu_2 - m_2\lambda_k) = 1$ by (61). Thus, the algebraic expansion associated with $J_2(z)$ when double poles are present similarly vanishes in the sector (34).