EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 6, No. 4, 2013, 435-450
ISSN 1307-5543 – www.ejpam.com

On the l_s-norm Generalization of the NLS Method for the Bass Model

Dragan Jukić

Department of Mathematics, J.J. Strossmayer University of Osijek, Trg Ljudevita Gaja 6, HR-31 000 Osijek, Croatia

Abstract. The best-known and widely used model in diffusion research is the Bass model. Estimation of its parameters has been approached in the literature by various methods, among which a very popular one is the nonlinear least squares (NLS) method proposed by Srinivasan and Mason. In this paper, we consider the l_s-norm ($1 \leq s < \infty$) generalization of the NLS method for the Bass model. Our focus is on the existence of the corresponding best l_s-norm estimate. We show that it is possible for the best l_s-norm estimate not to exist. As a main result, two theorems on the existence of the best l_s-norm estimate are obtained. One of them gives necessary and sufficient conditions which guarantee the existence of the best l_s-norm estimate.

2010 Mathematics Subject Classifications: 65D10, 65C20, 62J02, 91B26

Key Words and Phrases: Bass model, diffusion, l_s-norm estimate, least squares estimate, existence problem, data fitting

1. Introduction

The Bass model, introduced in 1969 (see Bass [4]), is the most popular first-purchase (adoption) diffusion model in marketing research. The main reason for this is that it finds its origin in a formal theory of product diffusion (see e.g. [30]), and that model parameters have an easy interpretation in terms of innovation and imitation effects. The model is in some respects similar to models of infectious diseases or contagion models which describe the spread of a disease through the population due to contact with infected persons (see [2, 3]). For general information on new product diffusion models we refer to Mahajan *et al.* [20].

In practice, the unknown parameters of the Bass model are not known in advance and must be estimated from the actual adoption data. There is no unique way to estimate the unknown parameters and many different methods have been proposed in the literature. Mahajan et al. [21] used real diffusion data for seven products to compare the performance of four estimation procedures: ordinary least squares estimation (OLS) proposed by Bass [4],

Email address: jukicd@mathos.hr

 © 2013 EJPAM All rights reserved.

maximum likelihood estimation (MLE) proposed by Schmittlein and Mahajan [32], nonlinear least squares estimation (NLS) suggested by Srinivasan and Mason [35], and algebraic estimation (AE) proposed by Mahajan and Sharma [22]. They concluded that, for the seven data sets considered in their study, the NLS procedure provides better predictions as well as more valid estimates of standard errors for the parameter estimates than the other three estimation procedures (see also [29, 36]). But, since each of these procedures has some advantages and disadvantages (see e.g. [21, 35, 36]), several other methods are proposed to estimate the unknown parameters in the Bass model. For example, Boswijk and Franses [7] have proposed an alternative to the Bass OLS regression.

The NLS estimation approach as proposed by Srinivasan and Mason has generally become the standard in diffusion research (see e.g. [20, 26, 27, 29]). In this paper, we consider the l_s-norm ($1 \le s < \infty$) generalization of the NLS approach for the Bass model. Our focus is on the existence of the corresponding best l_s-norm estimate. The structure of the paper is as follows. In Section 2, we briefly review the Bass model. In Section 3, both the NLS estimation approach and its generalization in the l_s-norm are described. We show that it is possible that the best l_s-norm estimate does not exist (Proposition 1). As our main results, two theorems on the existence of the best l_s-norm estimate are obtained in Section 4. One of them gives necessary and sufficient conditions which guarantee the existence of the best l_s-norm estimate. To the best of our knowledge, there is no previous paper that has focused on this existence problem.

2. Mathematical Formulation of the Bass Model

Bass [4] divided adopters (first-time buyers) of a new durable product into innovators and imitators. Imitators, unlike innovators, are those buyers who are influenced in their adoption by the number of previous buyers. The Bass diffusion model has three parameters: the coefficient of innovation or external influence ($p > 0$), the coefficient of imitation or internal influence ($q \ge 0$), and the total market potential ($m > 0$), i.e. the maximum cumulative number of adopters that diffusion is expected to reach. According to the model, if $N(t)$ is the cumulative number of adopters at time t, then the adoption rate $\frac{dN(t)}{dt}$ is described by the following differential equation:

$$\frac{dN(t)}{dt} = p[m - N(t)] + \frac{q}{m}N(t)[m - N(t)], \quad N(0) = 0, \quad t \ge 0. \tag{1}$$

In Equation (1), the first term $p[m - N(t)]$ represents adoptions due to innovators, whereas the second term, $\frac{q}{m}N(t)[m - N(t)]$, represents adoptions due to imitators.

The closed form solution of (1) is given by

$$N(t; m, p, q) = m\frac{1 - \mathrm{e}^{-(p+q)t}}{1 + \frac{q}{p}\,\mathrm{e}^{-(p+q)t}}, \quad t \ge 0.$$

The graph of the function N, known as the Bass cumulative adoption curve, is an “S-shaped” curve. If $q > p$, for this curve the point of inflection occurs at $t_I := \frac{1}{p+q}\ln(q/p)$ with

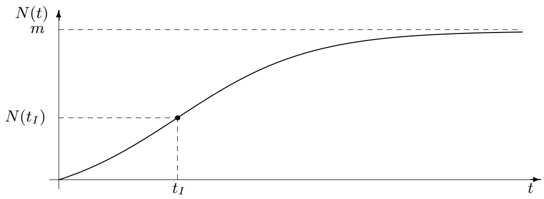

Figure 1: A Typical S-shaped Bass Cumulative Adoption Curve.

$N(t_I;m,p,q) = m\frac{(q-p)}{2q}$ (see Fig. 1). For $q \leq p$, the graph is still S-shaped, but the point of inflection occurs at a negative value of t.

The Bass model has been extensively used by marketing researchers primarily for the purpose of modelling diffusion processes and forecasting the sales of products, but it has also been used for various types of diffusion analysis in applied research, such as industrial technology, biology, medicine, engineering, computing, agriculture, social sciences, etc. For a review of the Bass model and its applications, see e.g. [20, 28].

The problem of nonlinear weighted least squares and total least squares fitting of the Bass cumulative adoption curve is considered by Jukić in [15] and [14], respectively. The nonlinear weighted least squares fitting of the Bass adoption curve is considered in [23].

3. l_s-norm Generalization of the NLS Method for the Bass Model

In practice, the unknown parameters of the Bass model are not known in advance and they must be estimated from the actual adoption data. Suppose we are given the data (t_i, X_i), $i = 1,\ldots,n$, $n > 3$, where

$$0 < t_1 < t_2 < \ldots < t_n \tag{2}$$

denotes the times at which incremental sales of the product are observed, and

$$X_i > 0, \quad i = 1,\ldots n, \tag{3}$$

is the observed number of new adopters in the time interval $(t_{i-1}, t_i]$. Here, by definition, $t_0 = 0$. Note that conditions (2) and (3) are natural.

The formulation of the NLS approach for the Bass model is as follows: The observed number of new adopters X_i in the time interval $(t_{i-1}, t_i]$ is modeled as

$$X_i = N(t_i;m,p,q) - N(t_{i-1};m,p,q) + \varepsilon_i, \quad i = 1,\ldots,n,$$

where ε_i is an additive error term. Here, by definition, $t_0 = 0$. Based on these equations, Srinivasan and Mason [35] proposed to estimate the unknown parameters p, q and m in the sense of least squares (LS) by minimizing functional

$$S(m,p,q) = \sum_{i=1}^{n}[N(t_i;m,p,q) - N(t_{i-1};m,p,q) - X_i]^2$$

on the set

$$\mathscr{P} := \{(m,p,q) : m,p > 0,\ q \geq 0\}.$$

This problem is a nonlinear l_2-norm problem. During the last few decades, an increased interest in the alternative l_s-norm has become apparent (see e.g. [1, 12, 33]). For example, l_1-norm criteria are more suitable if there are wild points (outliers) in the data. Therefore, instead of minimizing functional S, sometimes a more adequate criterion for estimation of unknown parameters m, p and q of the Bass model is to use some weighted l_s-norm, i.e. to minimize on the set $\mathscr{P}$ the following functional:

$$F_s(m,p,q) = \sum_{i=1}^{n} w_i |N(t_i;m,p,q) - N(t_{i-1};m,p,q) - X_i|^s. \tag{4}$$

where $w_i > 0$ are some weights, and where s ($1 \leq s < \infty$) is an arbitrary fixed number. A point $(m^\star, p^\star, q^\star) \in \mathscr{P}$ such that

$$F_s(m^\star, p^\star, q^\star) = \inf_{(m,p,q)\in\mathscr{P}} F_s(m,p,q)$$

is called the best l_s*-norm estimate*, if it exists. For $s = 2$, the best l_2-norm estimate is the familiar weighted LS estimate.

The above weighted l_s-norm minimization problem is a nonlinear problem which can only be solved in an iterative way. Before starting an iterative procedure, it is still necessary to question whether the best l_s-norm estimate exists. Even in the case of nonlinear LS problems ($s = 2$), it is still extremely difficult to answer this question (see [5, 6, 8–11, 13, 15–19, 24, 25, 31, 34]).

The following proposition shows that there exist data such that the best l_s-norm estimate does not exist.

Proposition 1. *Let (w_i, i, X_i), $i = 1, \ldots, n$, $n > 3$, be the data. If the data are such that*

i) the points (i, X_i), $i = 1, \ldots, n$ all lie on some exponential curve $y(t) = be^{ct}$, $b, c > 0$, or

ii) $0 < X_1 = X_2 = \ldots = X_n =: k$,

then the best l_s-norm estimate does not exist.

Proof. (i) Since $F_s(m,p,q) \geq 0$ for all $(m,p,q) \in \mathscr{P}$, and

$$\begin{aligned}
&\lim_{x\to\infty} F_s\Big(\frac{xb}{1-\mathrm{e}^{-c}}, \frac{c}{x+1}, \frac{cx}{x+1}\Big) \\
&\quad = \lim_{x\to\infty} \sum_{i=1}^{n} w_i \left| \frac{xb}{1-\mathrm{e}^{-c}} \frac{1-\mathrm{e}^{-ci}}{1+x\,\mathrm{e}^{-ci}} - \frac{xb}{1-\mathrm{e}^{-c}} \frac{1-\mathrm{e}^{-c(i-1)}}{1+x\,\mathrm{e}^{-c(i-1)}} - X_i \right|^s \\
&\quad = \sum_{i=1}^{n} w_i |be^{ci} - X_i|^s = 0,
\end{aligned}$$

this means that $\inf_{(m,p,q)\in\mathscr{P}} F_s(m,p,q) = 0$. Furthermore, since the graph of any function of the form

$$t \mapsto N(t;m,p,q) - N(t-1;m,p,q) = \frac{m(1+\frac{q}{p})(\mathrm{e}^{p+q}-1)\,\mathrm{e}^{-(p+q)t}}{(1+\frac{q}{p}\,\mathrm{e}^{-(p+q)t})(1+\frac{q}{p}\,\mathrm{e}^{p+q}\,\mathrm{e}^{-(p+q)t})}, \quad t \geq 1, \tag{5}$$

where $(m,p,q)\in\mathscr{P}$, intersects the graph of function $y(t) = be^{ct}$ in three points at most, and $n > 3$, it follows that $F_s(m,p,q) > 0$ for all $(m,p,q)\in\mathscr{P}$, and hence the best l_s-norm estimate does not exist.

(ii) Consider the following class of Bass functions

$$t \mapsto N\left(t;\frac{2}{x},\frac{kx}{2},\frac{kx}{2}\right) = 2\frac{\frac{1-\mathrm{e}^{-ktx}}{x}}{1+\mathrm{e}^{-ktx}}, \quad x > 0.$$

Using L'Hospital rule, it is easy to show that

$$\begin{aligned}\lim_{x\to 0+} F_s\left(\frac{2}{x},\frac{kx}{2},\frac{kx}{2}\right) &= \lim_{x\to 0+}\sum_{i=1}^{n} w_i \left|2\frac{\frac{1-\mathrm{e}^{-kxi}}{x}}{1+\mathrm{e}^{-kxi}} - 2\frac{\frac{1-\mathrm{e}^{-kx(i-1)}}{x}}{1+\mathrm{e}^{-kx(i-1)}} - X_i\right|^s \\ &= \sum_{i=1}^{n} w_i|k-N_i|^s = 0.\end{aligned}$$

This means that $\inf_{(m,p,q)\in\mathscr{P}} F_s(m,p,q) = 0$. Furthermore, since the graph of any function of type (5) intersects the line $y = k$ in two points at most, and $n > 3$, it follows that $F_s(m,p,q) > 0$ for all $(m,p,q)\in\mathscr{P}$, and hence the best l_s-norm estimate does not exist. □

4. The Existence Theorems

The following theorem, which is our main result, gives a necessary and sufficient condition which guarantees the existence of the best l_s-norm estimate. First, let us introduce the following notation:

$$E_s^\star := \inf_{b,c>0}\sum_{i=1}^{n} w_i|b\,\mathrm{e}^{ct_i} - b\,\mathrm{e}^{c_{t_{i-1}}} - X_i|^s. \tag{6}$$

By carefully examining the proof of Theorem 1 one can see that $E_s^\star$ is a so-called existence level for functional F_s (see e.g. [8]).

Theorem 1 (Necessary and sufficient condition). *Suppose that the data (w_i, t_i, X_i), $i = 1,\dots,n$, $n > 3$, satisfy conditions* (2) *and* (3). *Then functional F_s defined by* (4) *attains its infimum on $\mathscr{P}$ (i.e. the best l_s-norm estimate exists) if and only if there is a point $(m_0,p_0,q_0)\in\mathscr{P}$ such that $F_s(m_0,p_0,q_0) \leq E_s^\star$.*

In practice, we usually have observations of a diffusion process at certain equispaced time intervals (say yearly, quarterly, or monthly), so that $t_i = i\delta$, $i = 1,\dots,n$, where δ denotes the calendar time between two successive observations. In this case, by using substitutions $\alpha := b(1-\mathrm{e}^{-c\delta})$ and $\beta := c\delta$ in (6), it is easy to show that $E_s^\star = \inf_{\alpha,\beta>0}\sum_{i=1}^n w_i|\alpha\,\mathrm{e}^{\beta t_i} - X_i|^s$. Therefore, under the assumptions of the theorem, the best l_s-norm estimate exists if and only if there is at least one function of type (5) which is in an l_s-norm fitting sense as good as or better than the best exponential curve of type $t \mapsto \alpha\,\mathrm{e}^{\beta t}$, where $\alpha,\beta > 0$.

It is clear that, regardless of how much effort is put into marketing, there is a certain upper bound, say M, for the market potential m (i.e., the maximum number of adopters). In most cases management has a judgment, a strong intuitive feel, about the upper bound M, but if not, the upper bound M can be the size of the relevant population. The following theorem tells us that if parameter m is bounded above, then the l_s-norm estimate will exist. First, let us introduce the following notation: Given any real number $M > 0$, let

$$\mathscr{P}_M := \{(m,p,q) : 0 < m \le M, p > 0,\, q \ge 0\}.$$

Theorem 2. *Suppose that the data* (w_i,t_i,X_i), $i = 1,\dots,n$, $n > 3$, *satisfy conditions* (2) *and* (3). *Then functional* F_s *defined by* (4) *attains its infimum on* $\mathscr{P}_M$, *i.e. there exists a point* $(m^\star,p^\star,q^\star) \in \mathscr{P}_M$ *such that* $F_s(m^\star,p^\star,q^\star) = \inf_{(m,p,q)\in\mathscr{P}_M} F_s(m,p,q)$.

The proof of this theorem is omitted; it is the same for respective parts of the proof of Theorem 1, with the exception that we do not have to prove that $m^\star < \infty$.

The following lemma will be used in the proof of Theorem 1.

Lemma 1. *Suppose that the data* (w_i,t_i,X_i), $i = 1,\dots,n$, $n > 3$, *satisfy conditions* (2) *and* (3). *Then given any* $i_0 \in \{2,\dots,n\}$ *there exists a point in* $\mathscr{P}$ *at which functional* F_s *defined by* (4) *attains a value less than* $\sum_{\substack{i=1\\ i\neq i_0-1,i_0}}^n w_i|X_i|^s$.

Proof. In order to simplify the notation in the proof, we denote

$$(\tau_i,\xi_i) := (t_{i_0-2+i}, X_{i_0-2+i}), \quad i = 0,1,2.$$

Let $x_0 \in (0,\infty)$ be any point such that

$$\frac{\xi_2(\mathrm{e}^{-x\tau_0} - \mathrm{e}^{-x\tau_1}) - \xi_1(\mathrm{e}^{-x\tau_1} - \mathrm{e}^{-x\tau_2})}{\xi_1(\mathrm{e}^{-x\tau_1} - \mathrm{e}^{-x\tau_2})\,\mathrm{e}^{-x\tau_0} - \xi_2(\mathrm{e}^{-x\tau_0} - \mathrm{e}^{-x\tau_1})\,\mathrm{e}^{-x\tau_2}} > 0$$

for all $x \in (x_0,\infty)$. Since both the numerator and the denominator of the above expression are positive for all sufficiently large values of x, such x_0 exists. Now define functions $\alpha, m, p, q : (x_0,\infty) \to (0,\infty)$ by

$$\begin{aligned}
\alpha(x) &:= \frac{\xi_2(\mathrm{e}^{-x\tau_0} - \mathrm{e}^{-x\tau_1}) - \xi_1(\mathrm{e}^{-x\tau_1} - \mathrm{e}^{-x\tau_2})}{\xi_1(\mathrm{e}^{-x\tau_1} - \mathrm{e}^{-x\tau_2})\,\mathrm{e}^{-x\tau_0} - \xi_2(\mathrm{e}^{-x\tau_0} - \mathrm{e}^{-x\tau_1})\,\mathrm{e}^{-x\tau_2}},\\
m(x) &:= \frac{\xi_2(1+\alpha(x)\,\mathrm{e}^{-x\tau_1})(1+\alpha(x)\,\mathrm{e}^{-x\tau_2})}{(1+\alpha(x))(\mathrm{e}^{-x\tau_1} - \mathrm{e}^{-x\tau_2})},
\end{aligned}$$

$$p(x) := \frac{x}{1+\alpha(x)},$$
$$q(x) := \frac{x\alpha(x)}{1+\alpha(x)}.$$

By a straightforward but tedious calculation, one can verify that for all $x \in (x_0, \infty)$,

$$\frac{\xi_2(1+\alpha(x)\,\mathrm{e}^{-x\tau_2})}{\mathrm{e}^{-x\tau_1}-\mathrm{e}^{-x\tau_2}} = \frac{\xi_1(1+\alpha(x)\,\mathrm{e}^{-x\tau_0})}{\mathrm{e}^{-x\tau_0}-\mathrm{e}^{-x\tau_1}} \tag{7}$$

and

$$\lim_{x\to\infty} \alpha(x)\,\mathrm{e}^{-xt} = \begin{cases} 0 & \text{, if } t > \tau_1 \\ \frac{\xi_2}{\xi_1} & \text{, if } t = \tau_1 \\ \infty & \text{, if } t < \tau_1. \end{cases} \tag{8}$$

Note that $(m(x), p(x), q(x)) \in \mathscr{P}$ for all $x \in (x_0, \infty)$. It is easy to verify that

$$N(t; m(x), p(x), q(x)) = \frac{\xi_2(1+\alpha(x)\,\mathrm{e}^{-x\tau_1})(1+\alpha(x)\,\mathrm{e}^{-x\tau_2})}{(1+\alpha(x))(\mathrm{e}^{-x\tau_1}-\mathrm{e}^{-x\tau_2})} \frac{1-\mathrm{e}^{-xt}}{1+\alpha(x)\,\mathrm{e}^{-xt}}$$

and

$$\begin{aligned}\Delta N(t;x) :=& N(t; m(x), p(x), q(x)) - N(\tau_1; m(x), p(x), q(x)) \\ =& \frac{\xi_2(1+\alpha(x)\,\mathrm{e}^{-x\tau_2})}{\mathrm{e}^{-x\tau_1}-\mathrm{e}^{-x\tau_2}} \frac{\mathrm{e}^{-x\tau_1}-\mathrm{e}^{-xt}}{1+\alpha(x)\,\mathrm{e}^{-xt}}.\end{aligned} \tag{9}$$

Note that

$$N(t_i; m(x), p(x), q(x)) - N(t_{i-1}; m(x), p(x), q(x)) = \Delta(t_i; x) - \Delta(t_{i-1}; x), \quad i = 1, \dots, n.$$

Also note that due to (7) equation (9) can be rewritten in the form

$$\Delta N(t;x) = \frac{\xi_1(1+\alpha(x)\,\mathrm{e}^{-x\tau_0})}{\mathrm{e}^{-x\tau_0}-\mathrm{e}^{-x\tau_1}} \frac{\mathrm{e}^{-x\tau_1}-\mathrm{e}^{-xt}}{1+\alpha(x)\,\mathrm{e}^{-xt}}. \tag{10}$$

It follows immediately from (9) and (10) that

$$\Delta N(\tau_2; x) = \xi_2, \quad \Delta N(\tau_1; x) = 0 \text{ and } \Delta N(\tau_0; x) = -\xi_1. \tag{11}$$

Now we are going to show that

$$\lim_{x\to\infty} \Delta N(t;x) = \begin{cases} -\xi_1 & \text{, if } t < \tau_1 \\ \xi_2 & \text{, if } t > \tau_1. \end{cases} \tag{12}$$

To do this, first note that $\Delta N(t;x)$ can be rewritten in the following two equivalent forms:

$$\Delta N(t;x) = \frac{\xi_2(1+\alpha(x)\,\mathrm{e}^{-x\tau_2})}{1-\mathrm{e}^{-x(\tau_2-\tau_1)}} \frac{\mathrm{e}^{-x(\tau_1-t)}-1}{\mathrm{e}^{-x(\tau_1-t)}+\alpha(x)\,\mathrm{e}^{-x\tau_1}} \tag{13}$$

or

$$\Delta N(t;x) = \frac{\xi_2(1+\alpha(x)\,\mathrm{e}^{-x\tau_2})}{1-\mathrm{e}^{-x(\tau_2-\tau_1)}}\,\frac{1-\mathrm{e}^{-x(t-\tau_1)}}{1+\alpha(x)\,\mathrm{e}^{-xt}}. \tag{14}$$

If $t < \tau_1$, then taking the limit as $x \to \infty$ in (13) and using (8) it is easy to show that $\lim_{x\to\infty} \Delta N(t;x) = -\xi_1$, whereas, if $t > \tau_1$, it follows immediately from (14) and (8) that $\lim_{x\to\infty} \Delta N(t;x) = \xi_2$.

Let $x > x_0$ be sufficiently large, so that

$$0 < \Delta N(t_i;x) - \Delta N(t_{i-1};x) \le X_i, \quad i = 1,\dots,n$$

whereby the equality holds only if $i = i_0$ or $i = i_0 - 1$. Due to (11) and (12), such x exists. Then

$$\begin{aligned} F_s(m(x),p(x),q(x)) &= \sum_{i=1}^{n} w_i |\Delta N(t_i;x) - \Delta N(t_{i-1};x) - X_i|^s \\ &= \sum_{\substack{i=1\\ i\neq i_0-1,i_0}}^{n} w_i |\Delta N(t_i;x) - \Delta N(t_{i-1};x) - X_i|^s \\ &< \sum_{\substack{i=1\\ i\neq i_0-1,i_0}}^{n} w_i |X_i|^s. \end{aligned}$$

□

Proof of Theorem 1. Assume first that $(m^\star, p^\star, q^\star) \in \mathscr{P}$ is the best l_s-norm estimate, and then show that $F_s(m^\star, p^\star, q^\star) \le E_s^\star$. In order to do this, first note that for all $b, c, x > 0$,

$$\begin{aligned} F_s(m^\star, p^\star, q^\star) \le & F_s\big(xb, \frac{c}{x+1}, \frac{cx}{x+1}\big) \\ =& \sum_{i=1}^{n} w_i \Big| xb\frac{1-\mathrm{e}^{-ct_i}}{1+x\,\mathrm{e}^{-ct_i}} - xb\frac{1-\mathrm{e}^{-ct_{i-1}}}{1+x\,\mathrm{e}^{-ct_{i-1}}} - X_i \Big|^s, \end{aligned}$$

from where taking the limit as $x \to \infty$ it follows that

$$F_s(m^\star, p^\star, q^\star) \le \sum_{i=1}^{n} w_i |b\,\mathrm{e}^{ct_i} - b\,\mathrm{e}^{ct_{i-1}} - X_i|^s.$$

From the last inequality and the definition of $E_s^\star$ we obtain that $F_s(m^\star, p^\star, q^\star) \le E_s^\star$.

Let us show the converse of the theorem. Suppose that there is a point $(m_0, p_0, q_0) \in \mathscr{P}$ such that $F_s(m_0, p_0, q_0) \le E_s^\star$. Since functional F_s is nonnegative, there exists $F_s^\star := \inf_{(m,p,q)\in\mathscr{P}} F_s(m,p,q)$. It should be shown that the best l_s-norm estimate exists, i.e. that there exists a point $(m^\star, p^\star, q^\star) \in \mathscr{P}$ such that $F_s(m^\star, p^\star, q^\star) = F_s^\star$. To do this, first note that

$$F_s^\star \le F_s(m_0, p_0, q_0) \le E_s^\star.$$

If $F_s^\star = F_s(m_0, p_0, q_0)$, to complete the proof it is enough to set $(m^\star, p^\star, q^\star) = (m_0, p_0, q_0)$. Hence, we can further assume that

$$F_s^\star < F_s(m_0, p_0, q_0) \le E_s^\star. \tag{15}$$

Let (m_k, p_k, q_k) be a sequence in $\mathscr{P}$, such that

$$\begin{aligned} F_s^\star &= \lim_{k\to\infty} F_s(m_k, p_k, q_k) \\ &= \lim_{k\to\infty} \sum_{i=1}^{n} w_i |N(t_i; m_k, p_k, q_k) - N(t_{i-1}; m_k, p_k, q_k) - X_i|^s. \end{aligned} \tag{16}$$

Without loss of generality, in further consideration we may assume that sequences (m_k), (p_k) and (q_k) are monotone. This is possible because the sequence (m_k, p_k, q_k) has a subsequence $(m_{l_k}, p_{l_k}, q_{l_k})$, such that all its component sequences (m_{l_k}), (p_{l_k}) and (q_{l_k}) are monotone; and since $\lim_{k\to\infty} F_s(m_{l_k}, p_{l_k}, q_{l_k}) = \lim_{k\to\infty} F_s(m_k, p_k, q_k) = F_s^\star$.

Since each monotone sequence of real numbers converges in the extended real number system $\overline{\mathbb{R}}$, define

$$m^\star := \lim_{k\to\infty} m_k, \quad p^\star := \lim_{k\to\infty} p_k, \quad q^\star := \lim_{k\to\infty} q_k.$$

Note that $0 \le m^\star, p^\star, q^\star \le \infty$, because $(m_k, p_k, q_k) \in \mathscr{P}$.

To complete the proof it is enough to show that $(m^\star, p^\star, q^\star) \in \mathscr{P}$, i.e. that $0 < m^\star < \infty$, $0 < p^\star < \infty$ and $0 \le q^\star < \infty$. The continuity of functional F_s will then imply that $F_s^\star = \lim_{k\to\infty} F_s(m_k, p_k, q_k) = F_s(m^\star, p^\star, q^\star)$.

It remains to show that $(m^\star, p^\star, q^\star) \in \mathscr{P}$. The proof will be done in three steps. In step 1 we will show that $0 < m^\star < \infty$. It is also the most difficult part of the proof. In step 2 we will show that $0 < p^\star + q^\star < \infty$, which will imply that $0 \le p^\star, q^\star < \infty$. The proof that $p^\star > 0$ will be done in step 3.

Before continuing the proof, note that all sequences

$$(N(t_i; m_k, p_k, q_k) - N(t_{i-1}; m_k, p_k, q_k)), \quad i = 1, \dots, n,$$

must converge to a finite number because otherwise, if

$$\lim_{k\to\infty} [N(t_{i_0}; m_k, p_k, q_k) - N(t_{i_0-1}; m_k, p_k, q_k)] = \infty$$

for some i_0, then it would follow from (16) that

$$F_s^\star \ge \lim_{k\to\infty} w_i |N(t_{i_0}; m_k, p_k, q_k) - N(t_{i_0-1}; m_k, p_k, q_k)|^s = \infty,$$

which is impossible. Now, since $N(t_0; m_k, p_k, q_k) = 0$, it follows readily that all limits

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k), \quad i = 1, \dots, n$$

are finite.

Step 1. Let us first show that $m^\star < \infty$. We prove this by contradiction. Suppose on the contrary that $m^\star = \infty$. Without loss of generality, by taking appropriate subsequences if necessary, we may assume that the sequence $(\frac{q_k}{p_k m_k})$ is monotone. Let $l^\star := \lim_{k\to\infty} \frac{q_k}{p_k m_k}$. Then only one of the following three cases can occur:

(i) $l^\star = \infty$,

(ii) $0 < l^\star < \infty$, or

(iii) $l^\star = 0$.

Now, we are going to show that functional F_s cannot attain its infimum in either of these three cases, which will prove that $m^\star < \infty$. Before continuing the proof, let us note that

$$N(t_i; m_k, p_k, q_k) = \frac{1 - \mathrm{e}^{-(p_k+q_k)t_i}}{\frac{1}{m_k} + \frac{q_k}{p_k m_k}\,\mathrm{e}^{-(p_k+q_k)t_i}}, \quad i = 1, \ldots, n. \tag{17}$$

Case (i): $l^\star = \infty$. First note that $0 \le p^\star + q^\star \le \infty$. If $0 \le p^\star + q^\star < \infty$, then from (17) it easily follows that

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = 0, \quad i = 1, \ldots, n$$

and hence from (16) it would follow that $F_s^\star = \sum_{i=1}^n w_i |X_i|^s$. Since according to Lemma 1 there exists a point in $\mathscr{P}$ at which functional F_s attains a value smaller than $\sum_{i=1}^n w_i |X_i|^s$, this means that in this way functional F_s cannot attain its infimum.

It remains to consider the case when $p^\star + q^\star = \infty$. If $\lim_{k\to\infty} \frac{q_k}{p_k m_k}\,\mathrm{e}^{-(p_k+q_k)t_n} = 0$, then it would follow from (17) that $\lim_{k\to\infty} N(t_n; m_k, p_k, q_k) = \infty$, which is impossible because, as we know, all these limits must be finite. If $\lim_{k\to\infty} \frac{q_k}{p_k m_k}\,\mathrm{e}^{-(p_k+q_k)t_n} > 0$, regardless of whether this limit is finite or infinite, then from the equalities

$$\frac{q_k}{p_k m_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \frac{q_k}{p_k m_k}\,\mathrm{e}^{-(p_k+q_k)t_n} \cdot \mathrm{e}^{-(p_k+q_k)(t_i-t_n)}, \quad i = 1, \ldots, n$$

it follows readily that

$$\lim_{k\to\infty} \frac{q_k}{p_k m_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \infty, \quad i = 1, \ldots, n-1.$$

Due to this, now it is easy to show that from (17) it follows that

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = 0, \quad i = 1, \ldots, n-1,$$

and therefore from (16) it would follow that $F_s^\star \ge \sum_{i=1}^{n-1} w_i |X_i|^s$. Again, according to Lemma 1, there exists a point in $\mathscr{P}$ at which functional F_s attains a value smaller than $\sum_{i=1}^{n-1} w_i |X_i|^s$. This means that in this way functional F_s cannot attain its infimum.

Case (ii): $0 < l^\star < \infty$.If $p^\star + q^\star = 0$, then from (17) it easily follows that

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = 0, \quad i = 1, \dots, n$$

and therefore we would obtain that $F_s^\star = \sum_{i=1}^n w_i |X_i|^s$. As already shown in case (i), there exists a point in $\mathscr{P}$ at which functional F_s attains a value smaller than $\sum_{i=1}^n w_i |X_i|^s$. Therefore, in this way functional F_s cannot attain its infimum.

If $p^\star + q^\star = \infty$, then from (17) it follows that $\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = \infty$, $i = 1, \dots, n$, which is impossible because, as we know, all these limits must be finite.

Finally, if $0 < p^\star + q^\star < \infty$, then

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = \frac{1}{l^\star}(\mathrm{e}^{(p^\star+q^\star)t_i} - 1), \quad i = 1, \dots, n.$$

In this case we would have

$$F_s^\star = \lim_{k\to\infty} F_s(m_k, p_k, q_k) = \sum_{i=1}^n w_i \Big| \frac{1}{l^\star} \mathrm{e}^{(p^\star+q^\star)t_i} - \frac{1}{l^\star} \mathrm{e}^{(p^\star+q^\star)t_{i-1}} - X_i \Big|^s \geq E_s^\star,$$

which contradicts assumption (15). This means that in this way functional F_s cannot attain its infimum.

Case (iii): $l^\star = 0$. If $0 < p^\star + q^\star \leq \infty$, then

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = \infty, \quad i = 1, \dots, n.$$

As concluded in case (ii), in this way functional F_s cannot attain its infimum.

Let us now suppose that $p^\star + q^\star = 0$. By the Lagrange mean value theorem, for every $k \in \mathbb{N}$ there exist real numbers $\vartheta_{i,k} \in (0, 1)$, $i = 1, \dots, n$, such that

$$\begin{aligned} N(t_i; m_k, p_k, q_k) &= \frac{m_k(p_k + q_k)t_i \, \mathrm{e}^{-\vartheta_{i,k}(p_k+q_k)t_i}}{1 + \frac{q_k}{p_k} \mathrm{e}^{-(p_k+q_k)t_i}} \\ &= m_k p_k t_i \, \mathrm{e}^{-\vartheta_{i,k}(p_k+q_k)t_i} \left(\frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k} \mathrm{e}^{-(p_k+q_k)t_i}} \right). \end{aligned} \tag{18}$$

Since $\mathrm{e}^{-(p_k+q_k)t_i} < 1$ for every $k \in \mathbb{N}$, it is easy to check that

$$1 < \frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k} \mathrm{e}^{-(p_k+q_k)t_i}} < \mathrm{e}^{(p_k+q_k)t_i}, \quad i = 1, \dots, n$$

from where passing to the limit as $k \to \infty$ we obtain

$$\lim_{k\to\infty} \frac{1 + \frac{q_k}{p_k}}{1 + \frac{q_k}{p_k} \mathrm{e}^{-(p_k+q_k)t_i}} = 1, \quad i = 1, \dots, n.$$

Now, by using (18) we obtain

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = k_0 t_i, \quad i = 1, \ldots, n, \tag{19}$$

where $k_0 := \lim_{k\to\infty}(m_k p_k)$ is finite or infinite.

If $k_0 = 0$, from (16) and (19) it follows that $F_s^\star = \sum_{i=1}^n w_i |X_i|^s$. If $k_0 = \infty$, then we would have $\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = \infty$, $i = 1, \ldots, n$. As already shown in case (i), in these two ways ($k_0 = 0$ and $k_0 = \infty$) functional F_s cannot attain its infimum. Now suppose that $0 < k_0 < \infty$. Then by using (16) and (19) we would obtain

$$F_s^\star = \sum_{i=1}^n w_i |k_0(t_i - t_{i-1}) - X_i|^s. \tag{20}$$

Furthermore, since by the definition of $E_s^\star$,

$$\sum_{i=1}^n w_i \Big| \frac{1}{c}\,\mathrm{e}^{ck_0 t_i} - \frac{1}{c}\,\mathrm{e}^{ck_0 t_{i-1}} - X_i \Big|^s \geq E_s^\star \quad \text{for every } c > 0,$$

taking the limit as $c \to 0+$ it follows that $\sum_{i=1}^n w_i |k_0(t_i - t_{i-1}) - X_i|^s \geq E_s^\star$. Due to this and (20) we would have that $F_s^\star \geq E_s^\star$, which contradicts assumption (15). This means that in this way functional F_s cannot attain its infimum.

Thus, we have proved that $m^\star < \infty$. It is easy to show that $m^\star > 0$. We prove this by contradiction. If $m_k \to 0$, then from the inequalities

$$0 \leq m_k \frac{1 - \mathrm{e}^{-(p_k+q_k)t_i}}{1 + \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i}} < m_k, \quad i = 1, \ldots, n$$

we would have

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = 0, \quad i = 1, \ldots, n.$$

As shown in case (i), in this way functional F_s cannot attain its infimum.

We completed the proof that $0 < m^\star < \infty$.

Step 2. Let us first show that $p^\star + q^\star < \infty$. Suppose on the contrary that $p^\star + q^\star = \infty$.

If $\lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \infty$ for all $i = 1, \ldots, n$, then

$$N(t_i; m_k, p_k, q_k) \to 0, \quad i = 1, \ldots, n$$

As already shown in case (i) from step 1, in this way functional F_s cannot attain its infimum.

It remains to consider the case when $0 \leq \lim_{k\to\infty} \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} < \infty$ for at least one index $i \geq 1$. Let i_0 be the minimal index with this property. By using equalities

$$\frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_i} = \frac{q_k}{p_k}\,\mathrm{e}^{-(p_k+q_k)t_{i_0}} \cdot \mathrm{e}^{-(p_k+q_k)(t_i - t_{i_0})}, \quad i = 1, \ldots, n$$

it is easy to show that

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = \begin{cases} 0 & , \text{if } i < i_0 \\ m^\star & , \text{if } i > i_0. \end{cases}$$

Due to this and (16), now it is easy to show that if $i_0 < n$, we would have $F_s^\star \geq \sum_{\substack{i=1 \\ i\neq i_0, i_0+1}}^{n} w_i |X_i|^s$, whereas, if $i_0 = n$, we would have $F_s^\star \geq \sum_{i=1}^{n-1} w_i |X_i|^s$. Since according to Lemma 1 in both subcases ($i_0 < n$ and $i_0 = n$) there exists a point in $\mathscr{P}$ at which functional F_s attains a smaller value, this means that in this way functional F_s cannot attain its infimum. In this way we completed the proof that $p^\star + q^\star < \infty$.

Now, we are going to show that $0 < p^\star + q^\star$. We prove this by contradiction. Suppose on the contrary that $p^\star + q^\star = 0$. Then from the inequalities

$$0 \leq m_k \frac{1 - \mathrm{e}^{-(p_k+q_k)t_i}}{1 + \frac{q_k}{p_k}\mathrm{e}^{-(p_k+q_k)t_i}} < m_k(1 - \mathrm{e}^{-(p_k+q_k)t_i}), \quad i = 1, \ldots, n$$

we would have

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = 0, \quad i = 1, \ldots, n.$$

As shown in case (i) from step 1, in this way functional F_s cannot attain its infimum.

So far,we have shown that $0 < m^\star < \infty$ and $0 < p^\star + q^\star < \infty$. By using this, in the next step we will show that $p^\star > 0$.

Step 3. Let us show that $p^\star > 0$. We prove this by contradiction. Suppose on the contrary that $p^\star = 0$. Then from the inequalities $0 < p^\star + q^\star < \infty$ it follows that $q^\star > 0$. Now it is easy to conclude that

$$\lim_{k\to\infty} \frac{q_k}{p_k} \mathrm{e}^{-(p_k+q_k)t_i} = \infty, \quad i = 1, \ldots, n$$

and therefore

$$\lim_{k\to\infty} N(t_i; m_k, p_k, q_k) = 0, \quad i = 1, \ldots, n.$$

As shown in case (i) from step 1, in this way functional F_s cannot attain its infimum. Thus, we proved that $p^\star > 0$ and herewith we completed the proof.

□

References

[1] A. Atieg and G.A. Watson. Use of l_p norms in fitting curves and surfaces to data. *Australian and New Zealand Industrial and Applied Mathematics Journal*, 45(E):C187-C200, 2004.

[2] N.T.J. Bailey. *The Mathematical Theory of Infectious Diseases and its Applications*. Griffin, London, 1975.

[3] N.T.J. Bailey. *The Mathematical Theory of Epidemics*. Griffin, London, 1957.

[4] F.M. Bass. A new product growth model for consumer durables. *Management Science*, 15:215-227, 1969.

[5] D.M. Bates and D.G. Watts. *Nonlinear Regression Analysis and its Applications*. Wiley, New York, 1988.

[6] Å. Björck. *Numerical Methods for Least Squares Problems*. SIAM, Philadelphia, 1996.

[7] H.P. Boswijk and P.H. Franses. On the econometrics of the Bass diffusion model. *Journal of Business and Economic Statistics*, 23:255-268, 2005.

[8] E. Demidenko. Criteria for unconstrained global optimization. *Journal of Optimization Theory and Applications*, 136:375-395, 2008.

[9] E. Demidenko. Criteria for global minimum of sum of squares in nonlinear regression. *Computational Statistics & Data Analysis*, 51:1739-1753, 2006.

[10] E. Demidenko. On the existence of the least squares estimate in nonlinear growth curve models of exponential type. *Communications in Statistics - Theory and Methods*, 25:159-182, 1996.

[11] P.E. Gill, W. Murray, and M.H Wright. *Practical Optimization*. Academic Press, London, 1981.

[12] R. Gonin and A.H. Money. *Nonlinear L_p-Norm Estimation*. Marcel Dekker, New York, 1989.

[13] K.P. Hadeler, D. Jukić, and K. Sabo. Least squares problems for Michaelis Menten kinetics. *Mathematical Methods in the Applied Sciences*, 30:1231-1241, 2007.

[14] D. Jukić. Total least squares fitting Bass diffusion model. *Mathematical and Computer Modelling*, 53:1756-1770, 2011.

[15] D. Jukić. On nonlinear weighted least squares estimation of Bass diffusion model. *Applied Mathematics and Computation*, 219:7891–7900. 2013.

[16] D. Jukić and D. Marković. On nonlinear weighted errors-in-variables parameter estimation problem in the three-parameter Weibull model. *Applied Mathematics and Computation*, 215:3599-3609, 2010.

[17] D. Jukić. On the existence of the best discrete approximation in l_p norm by reciprocals of real polynomials. *Journal of Approximation Theory*, 156:212-222, 2009.

[18] D. Jukić, M. Benšić, and R. Scitovski. On the existence of the nonlinear weighted least squares estimate for a three-parameter Weibull distribution. *Computational Statistics & Data Analysis*, 52:4502-4511, 2008.

[19] D. Jukić, G. Kralik, and R. Scitovski. Least squares fitting Gompertz curve. *Journal of Computational and Applied Mathematics*, 169:359-375, 2004.

[20] V. Mahajan, E. Muller, and Y. Wind (Eds.). *New-Product Diffusion Models*. Kluwer Academic Publishers, London, 2000.

[21] V. Mahajan, C.H. Mason, and V. Srinivasan. An evaluation of estimation procedures for new product diffusion models. In V. Mahajan and Y. Wind, editors, *Innovation Diffusion Models of New Product Acceptance.*, pages 203-232, Cambridge, 1986, Ballinger Publishing Company.

[22] V. Mahajan and S. Sharma. Simple algebraic estimation procedure for innovation diffusion models of new product acceptance. *Technological Forecasting and Social Change*, 30:33l-346, 1986.

[23] D. Marković and D. and Jukić. On parameter estimation in the Bass model by nonlinear least squares fitting the adoption curve, *International Journal of Applied Mathematics and Computer Science*, 23:145-155, 2013.

[24] D. Marković and D. Jukić. On nonlinear weighted total least squares parameter estimation problem for the three-parameter Weibull density, *Applied Mathematical Modelling*, 34:1839-1848, 2010.

[25] D. Marković, D. Jukić and M. Benšić. Nonlinear weighted least squares estimation of a three-parameter Weibull density with a nonparametric start. *Journal of Computational and Applied Mathematics*, 228:304-312, 2009.

[26] P.M. Parker. Aggregate diffusion forecasting modcls in marketing: A critical review. *International Journal of Forecasting*, 10:353-380, 1994.

[27] P.M. Parker. Choosing among diffusion models: Some empirical evidence. *Marketing Letters*, 4:81-94, 1993.

[28] R. Peres, E. Muller and V. Mahajan. Innovation diffusion and new product growth models: A critical review and research directions. *International Journal of Research in Marketing*, 27:91-106, 2000.

[29] W.P. Putsis., Jr. and V. Srinivasan. Estimation techniques for macro diffusion models. In V. Mahajan, E. Muller and Y. Wind, editors, *New-Product Diffusion Models*, pages 263-293, Kluwer Academic Publishers, Boston, 2000.

[30] E.M. Rogers. *Diffusion of Innovations*. The Free Press, New York, 1962.

[31] G.J.S. Ross. *Nonlinear Estimation*. Springer, New York, 1990.

[32] D. Schmittlein and V. Mahajan. Maximum likelihood estimation for an innovation diffusion model of new product acceptance. *Marketing Science*, 1:57-78, 1982.

[33] R. Scitovski and M. Meler. Solving parameter estimation problem in new product diffusion models. *Applied Mathematics and Computation*, 127:45-63, 2002.

[34] G.A.F Seber and C.J. Wild. *Nonlinear Regression*. Wiley, New York, 1989.

[35] V. Srinivasan and C.H. Mason. Nonlinear least squares estimation of new product diffusion models. *Marketing Science*, 5:169-178, 1986.

[36] R. Venkatesan, T.V. Krishnan, and V. Kumar. Evolutionary estimation of macro-level diffusion models using genetic algorithms: An alternative to nonlinear least squares. *Marketing Science*, 24:451-464, 2004.