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Entropy solutions of nonlinear elliptic equations with measurable boundary conditions and without strict monotonocity conditions

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Abstract. We prove some existence results for nonlinear degenerate elliptic problems of the form

$$Au + g(x,u) = f - \text{div}F,$$

where $A(u) = -\text{div}a(x,u,\nabla u)$ is a Leray-Lions, operator defined form the weighted Sobolev space $W_0^{1,p}(\Omega,w)$ into its dual. The right hand side, $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*)$. Note that the Carathéodory function $a(x,s,\xi)$ satisfies only the large monotonicity instead of the monotonicity strict condition. We overcome this difficulty by using the L^1-version of Minty's lemma.

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1. Introduction

On a bounded open domain Ω of $\mathbb{R}^N$ $N \geq 2$ we consider the Dirichlet problem for the quasilinear degenerated elliptic equation,

$$\begin{cases} Au + g(x,u) = \mu \ \text{ in } \ \Omega \\ u = 0 \qquad \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where $Au = -\text{div}(a(x,u,\nabla u))$ is a Leray-Lions operators defined from the weighted Sobolev space $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ where $w = \{w_i,\ 0 \leq i \leq N\}$ is collection of weight functions on Ω, $1 < p < \infty$ and $w^* = \{w_i^{1-p'},\ 0 \leq i \leq N\}$.
Here $a(x,s,\xi)$ is a Carathéodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and $g(x,u)$ is a nonlinear term which satisfy some suitable conditions $(H_1)-(H_2)$ below. The second member μ is a

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measure which belongs in $L^1(\Omega) + W^{-1,p'}(\Omega, w^*)$.

The feature of this paper, is to treat a class of problems for which the classical monotone operator methods (developed by Visik [12], Minty [11], Browder [6], Brézis [5] and Lions [10] in non weighted case and by Akdim-Azroul [2] in weighted case and others) do not apply. The reason for this, is that $a(.)$ does not need to satisfy the strict monotonicity condition that is,

$$\langle a(x,s,\xi) - a(x,s,\eta), \xi - \eta\rangle > 0 \quad \text{for} \quad \text{all} \quad \xi \neq \eta \in \mathbb{R}^N, \tag{1.2}$$

of a typical Leray-Lions operator but only a large monotonicity that is

$$\langle a(x,s,\xi) - a(x,s,\eta), \xi - \eta\rangle \geq 0 \quad \text{for} \quad \text{all} \quad (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{1.3}$$

where $\langle , \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

The tool we use to overcome the difficulty of the not strict monotonicity (which can not guarantees the almost every where convergence of the gradient of approximation solution) is to investigate some techniques induced by Minty's lemma. The approach of pseudo-monotonicity can not be used due to the fact that $f \in L^1(\Omega)$. In order to prove the a.e. convergence of the gradient of the approximate solution u_n, the authors in [4] have show that u_n is bounded in the Marcinkiewicz space. While in our present work we prove the locally converge in measure of u_n (see step 2).

Thus our aim of this paper, is then to prove an existence of solution for the following problem,

$$(\mathscr{P})\begin{cases} -\mathrm{div} a(x,u,\nabla u) + g(x,u) = \mu \quad in \quad \Omega \\ \qquad\qquad u = 0 \qquad\qquad\qquad on \quad \partial\Omega \end{cases}$$

where $\mu = f - \mathrm{div} F$ with $f \in L^1(\Omega)$ and $F \in \Pi_{i=1}^N L^{p'}(\Omega, w_i^*)$.In the sense of entropy solution (see definition 2.1 below)

Note that, the existence of such entropy solution is proved by using only the large monotonicity (1.3).

This paper is organized as follows, section 2 contains some preliminaries and basic assumptions. In section 3 we give our main general result which is proved in section 4. Section 5 is devoted to an example which illustrated our abstract hypotheses.

2. Basic assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e. every component $w_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose in all our considerations that

$$w_i \in L^1_{\mathrm{loc}}(\Omega), \tag{2.1}$$

and

$$w_i^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega), \tag{2.2}$$

for any $0 \leq i \leq N$.

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfil

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \text{ for all } i=1,...,N,$$

which is a Banach space under the norm

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\right]^{\frac{1}{p}}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \ i=1,...,N\}$ and p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$ (for more details we refer to [8]).

Assumption$(A1)$

We assume that the norm :

$$|||u||| = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\right)^{\frac{1}{p}}, \tag{2.4}$$

is equivalent to the usual norm (2.3), and there exists a weight function $\sigma(x)$ on Ω and a parameter q, $1 < q < \infty$ such that the Hardy inequality,

$$\left(\int_\Omega |u(x)|^q \sigma(x)\,dx\right)^{\frac{1}{q}} \le c\left(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\right)^{\frac{1}{p}}$$

holds for every $u \in W_0^{1,p}(\Omega,w)$ with a constant $c > 0$ independent of u. Moreover, the imbedding,

$$W_0^{1,p}(\Omega,w) \hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{2.5}$$

is compact. Let A be a nonlinear operator from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ defined as

$$A(u) = -\text{div}(a(x,u,\nabla u))$$

where $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caradhéodory vector-valued function satisfies the following assumption.

Assumption$(A2)$
For $i = 1, ..., N$

$$|a_i(x,s,\xi)| \le \beta w_i^{\frac{1}{p}}(x) \ \ [k(x) + \sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{2.6}$$

for a.e., $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, some function $k(x) \in L^{p'}(\Omega)(\frac{1}{p} + \frac{1}{p'} = 1)$ and $\beta > 0$. Here σ and q are as in (A1).

$$\langle a(x,s,\xi) - a(x,s,\eta), \xi - \eta \rangle \ge 0 \ \text{ for } \ \text{all } \ (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{2.7}$$

$$\langle a(x,s,\xi), \xi \rangle \ge \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{2.8}$$

where α is strictly positive constant.
Moreover, the function $g(x,s)$ is a Carathéodory function satisfying

$$g(x,s)s \ge 0. \tag{2.9}$$

$$\sup_{|s| \le n} |g(x,s)| = h_n(x) \in L^1(\Omega) \tag{2.10}$$

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

Lemma 2.1. *(cf. [1]) Assume that (A1) holds. Let (u_n) be a sequence of $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$. Then $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega, w)$.*

3. Main Existence Theorem

Consider the following problem:

$$(\mathscr{P}) \begin{cases} -\text{div} a(x,u,\nabla u) + g(x,u) = f - \text{div}(F) & in \ \ \Omega \\ u = 0 & on \ \ \partial\Omega \end{cases}$$

where $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*)$.

Definition 3.1. .
An entropy solution of $(\mathscr{P})$ is a measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega, w)$ for every $k > 0$ and such that

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\varphi] \rangle \, dx + \int_\Omega g(x,u) T_k[u-\varphi] \, dx = \int_\Omega f \, T_k[u-\varphi] \, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi] \rangle \, dx$$

for every $\varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

Theorem 3.1. *Under the assumptions* $(A1)$ *and* $(A2)$ *there exist an entropy solution* u *of the problem* $(\mathscr{P})$*. i.e.* u *is a solution of* $(\mathscr{P})$ *in the following sense.*

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\varphi]\rangle\, dx + \int_\Omega g(x,u)T_k[u-\varphi]\, dx = \int_\Omega f\, T_k[u-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi]\rangle\, dx$$

for every $\varphi \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$*, for every* $k>0$*.*

Remark 3.1. *The statement of Theorem 3.1 generalizes in weighted case the analogous in [4] and [3](with* $g\equiv 0$*).*

4. Proof of Existence Theorem

4.1. Main Lemma

Lemma 4.1. *Let* u *be a measurable function such that* $T_k(u)$ *belongs to* $W_0^{1,p}(\Omega,w)$ *for every* $k>0$*. Then*

$$\int_\Omega \langle a(x,u,\nabla\varphi), \nabla T_k[u-\varphi]\rangle\, dx \le \int_\Omega f\, T_k[u-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi]\rangle\, dx. \tag{4.1}$$

is equivalent to

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\varphi]\rangle\, dx + \int_\Omega g(x,u)T_k[u-\varphi]\, dx = \int_\Omega f\, T_k[u-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi]\rangle\, dx. \tag{4.2}$$

for every φ *in* $W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$*, and for every* $k>0$*.*

Proof

In fact (4.2) implies (4.1) is easily proved adding and subtracting

$$\int_\Omega \langle a(x,u,\nabla\varphi), \nabla T_k[u-\varphi]\rangle\, dx$$

and then using assumption (2.7). Thus, it remains to prove that (4.1) implies (4.2). Let h and k be positive real numbers, let $\lambda \in\ ]-1,1[$ and $\psi \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$.
Choose, $\varphi = T_h(u-\lambda T_k(u-\psi)) \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$ as test function in (4.1), we have:

$$I_{hk} \le J_{hk} \tag{4.3}$$

with

$$\begin{aligned} I_{hk} = &\int_\Omega \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx \\ &+ \int_\Omega g(x,u)T_k(u-T_h(u-\lambda T_k(u-\psi)))\, dx = I'_{hk} + I''_{hk} \end{aligned}$$

and

$$J_{hk} = \int_{\Omega} f\, T_k(u - T_h(u - \lambda T_k(u-\psi)))\, dx + \int_{\Omega} \langle F, \nabla T_k(u - T_h(u - \lambda T_k(u-\psi)))\rangle\, dx.$$

Put
$A_{hk} = \{x \in \Omega,\ \ |u - T_h(u - \lambda T_k(u-\psi))| \le k\}$
and
$B_{hk} = \{x \in \Omega,\ \ |u - \lambda T_k(u-\psi)| \le h\}$.
Then, we obtain

$$I'_{hk} = \int_{A_{kh}\cap B_{hk}} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx$$

$$+ \int_{A_{kh}\cap B_{hk}^C} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx$$

$$+ \int_{A_{kh}^C} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx.$$

Since $\nabla T_k(u - T_h(u - \lambda T_k(u-\psi)))$ is different to zero only on A_{kh}, we have

$$\int_{A_{kh}^C} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx = 0. \tag{4.4}$$

Moreover, if $x \in B_{hk}^C$, we have $\nabla T_h(u - \lambda T_k(u-\psi)) = 0$ and using (2.8), we deduce that,

$$\int_{A_{kh}\cap B_{hk}^C} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx$$

$$= \int_{A_{kh}\cap B_{hk}^C} \langle a(x,u,0), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx = 0. \tag{4.5}$$

From (4.4) and (4.5), we obtain

$$I'_{hk} = \int_{A_{kh}\cap B_{hk}} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u - T_h(u-\lambda T_k(u-\psi)))\rangle\, dx.$$

Letting $h \to +\infty$, and $|\lambda| \le 1$, we have

$$A_{kh} \to \{x, |\lambda||T_k(u-\psi)| \le k\} = \Omega, \tag{4.6}$$

$$B_{hk} \to \Omega \quad \text{which implies} \;\; A_{kh}\cap B_{hk} \to \Omega. \tag{4.7}$$

Which and using Lebesgue theorem, we conclude that

$$\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} \langle a(x,u,\nabla T_h(u-\lambda T_k(u-\psi))), \nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\rangle \, dx \\ = \lambda\int_\Omega \langle a(x,u,\nabla(u-\lambda T_k(u-\psi)), \nabla T_k(u-\psi)\rangle \, dx. \tag{4.8}$$

i.e.,

$$\lim_{h\to+\infty} I'_{hk} = \lambda\int_\Omega \langle a(x,u,\nabla(u-\lambda T_k(u-\psi)), \nabla T_k(u-\psi)\rangle \, dx. \tag{4.9}$$

moreover it is easy to see that,

$$\lim_{h\to+\infty}\int_\Omega g(x,u)T_k(u-T_h(u-\lambda T_k(u-\psi)))\, dx = \lambda\int_\Omega g(x,u)T_k[u-\psi]\, dx$$

thus implies that,

$$\lim_{h\to+\infty} I_{hk} = \lambda\int_\Omega \langle a(x,u,\nabla(u-\lambda T_k(u-\psi)), \nabla T_k(u-\psi)\rangle \, dx + \lambda\int_\Omega g(x,u)T_k[u-\psi]\, dx \tag{4.10}$$

On the other hand, we have,

$$J_{hk} = \int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\psi)))\, dx + \int_\Omega \langle F, \nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\rangle \, dx.$$

Then

$$\lim_{h\to+\infty}\int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\psi)))\, dx + \int_\Omega \langle F, \nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\rangle \, dx$$

$$= \lambda\int_\Omega f\,T_k[u-\psi]\, dx + \lambda\int_\Omega \langle F, \nabla T_k[u-\psi]\rangle \, dx$$

i.e.,

$$\lim_{h\to+\infty} J_{hk} = \lambda\int_\Omega f\,T_k[u-\psi]\, dx + \lambda\int_\Omega \langle F, \nabla T_k[u-\psi]\rangle \, dx. \tag{4.11}$$

Together (4.10), (4.11) and passing to the limit in (4.3), we obtain,

$$\lambda\left(\int_\Omega \langle a(x,u,\nabla(u-\lambda T_k(u-\psi), \nabla T_k(u-\psi)\rangle \, dx + \int_\Omega g(x,u)T_k[u-\psi]\, dx\right) \\ \leq \lambda\left(\int_\Omega f\,T_k[u-\psi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\psi]\rangle \, dx\right)$$

for every $\psi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$, and for $k > 0$. Choosing $\lambda > 0$ dividing by λ, and then letting λ tend to zero , we obtain

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\varphi]\rangle\, dx + \int_\Omega g(x,u) T_k[u-\psi]\, dx \le \int_\Omega f\, T_k[u-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi]\rangle\, dx. \tag{4.12}$$

For $\lambda < 0$, dividing by λ, and then letting λ tend to zero , we obtain

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\varphi]\rangle\, dx + \int_\Omega g(x,u) T_k[u-\psi]\, dx \ge \int_\Omega f\, T_k[u-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi]\rangle\, dx. \tag{4.13}$$

Combining (4.12) and (4.13), we conclude the following equality :

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\varphi]\rangle\, dx + \int_\Omega g(x,u) T_k[u-\psi]\, dx = \int_\Omega f\, T_k[u-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u-\varphi]\rangle\, dx. \tag{4.14}$$

This completes the proof of Lemma 4.1.

4.2. Proof of Theorem 3.1

1. Approximate problem and a priori estimate

Let f_n be a sequence function of $L^\infty(\Omega)$ which is strongly convergent to f in $L^1(\Omega)$ such that $\|f_n\|_{L^1} \le \|f\|_{L^1}$, and let u_n be a solution in $W_0^{1,p}(\Omega, w)$ of the problem

$$\begin{cases} -\mathrm{div} a(x,u_n,\nabla u_n) + g_n(x,u_n) = f_n - \mathrm{div}(F) & in\ \ \Omega \\ \qquad u_n = 0 & on\ \ \partial\Omega \end{cases} \tag{4.15}$$

where

$$g_n(x,s) = \frac{g(x,s)}{1+\frac{1}{n}|g(x,s)|}\theta_n(x) \ \text{ and } \ \theta_n(x) = T_{\frac{1}{n}}(\sigma^{\frac{1}{q}}(x))$$

which exists thanks to [7].

Choosing $T_k(u_n)$ as test function in (4.15), we have

$$\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle\, dx + \int_\Omega g_n(x,u_n) T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n)\, dx + \int_\Omega \langle F, \nabla T_k(u_n)\rangle\, dx$$

using $\nabla T_k(u_n) = \nabla u_n \chi_{\{|u_n|\le k\}}$ and thanks to assumption (2.8), we obtain

$$\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle\, dx \ge \alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dx$$

then since $g_n(x,u_n)T_k(u_n) \ge 0$ we have,

$$\alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dx \le k\|f\|_{L^1} + \sum_{i=1}^N \int_\Omega F_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right| dx$$

$$\leq k\|f\|_{L^1} + \sum_{i=1}^{N}\int_{\Omega} F_i w_i^{\frac{-1}{p}} (\frac{\alpha}{2})^{\frac{-1}{p}} |\frac{\partial T_k(u_n)}{\partial x_i}| w_i^{\frac{1}{p}} (\frac{\alpha}{2})^{\frac{1}{p}}\, dx$$

by Young's inequality, we obtain

$$\alpha\sum_{i=1}^{N}\int_{\Omega} w_i |\frac{\partial T_k(u_n)}{\partial x_i}|^p\, dx \leq k\|f\|_{L^1} + \frac{c(\alpha)}{p'}\, |F\|_{\prod L^{p'}(\Omega, w_i^*)} + \frac{\alpha}{2}\sum_{i=1}^{N}\int_{\Omega} w_i |\frac{\partial T_k(u_n)}{\partial x_i}|^p\, dx.$$

Then,

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\Omega} w_i |\frac{\partial T_k(u_n)}{\partial x_i}|^p\, dx \leq k(\|f\|_{L^1} + \frac{c(\alpha)}{p'}\|F\|_{\prod L^{p'}(\Omega, w_i^*)}$$

for $k > 1$, which implies that

$$\left(\sum_{i=1}^{N}\int_{\Omega} |\frac{\partial T_k(u_n)}{\partial x_i}|^p w_i(x) dx\right)^{\frac{1}{p}} \leq c k^{\frac{1}{p}} \quad \forall k > 1. \tag{4.16}$$

2: Locally convergence of u_n in measure

We prove that u_n converges to some function u locally in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence), we shall show that u_n is a Cauchy sequence in measure in any ball B_R.
Let $k > 0$ large enough, by using (2.5), we have

$$\begin{aligned} k\ meas(\{|u_n| > k\} \cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\, dx \leq \int_{B_R} |T_k(u_n)|\, dx \\ &\leq \left(\int_{\Omega} |T_k(u_n)|^p w_0\, dx\right)^{\frac{1}{p}} . \left(\int_{B_R} w_0^{1-p'}\, dx\right)^{\frac{1}{q'}} \\ &\leq c_R \left(\int_{\Omega}\sum_{i=1}^{N} |\frac{\partial T_k(u_n)}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}} \\ &\leq c_1 k^{\frac{1}{p}}. \end{aligned}$$

which implies

$$meas(\{|u_n| > k\} \cap B_R) \leq \frac{c_1}{k^{1-\frac{1}{p}}} \quad \forall k > 1. \tag{4.17}$$

We have, for every $\delta > 0$,

$$\begin{aligned} meas(\{|u_n - u_m| > \delta\} \cap B_R) \leq\ & meas(\{|u_n| > k\} \cap B_R) + meas(\{|u_m| > k\} \cap B_R) \\ & + meas\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned} \tag{4.18}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, w)$, there exists some $v_k \in W_0^{1,p}(\Omega, w)$, such that

$$\begin{array}{ll} T_k(u_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,p}(\Omega, w) \\ T_k(u_n) \to v_k & \text{strongly in } L^q(\Omega, \sigma) \text{ and a.e. in } \Omega. \end{array}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then by (4.17) and (4.18), there exists some $k(\varepsilon) > 0$ such that $meas(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta, R)$. This proves that (u_n) is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{array}{lll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega, w), \\ T_k(u_n) \rightarrow T_k(u) & \text{strongly in } L^q(\Omega, \sigma) \text{ and a.e in } \Omega. \end{array} \tag{4.19}$$

3. Equi-integrability of nonlinearities

we need to prove that

$$g_n(x, u_n) \rightarrow g(x, u) \text{ strongly in } L^1(\Omega) \tag{4.20}$$

in particular it is enough to prove the equi-integrable of $g_n(x, u_n)$ to this purpose. We take $T_{l+1}(u_n) - T_l(u_n)$ as test function in (4.15), we obtain

$$\begin{aligned} \int_\Omega \langle a(x, u_n, \nabla u_n), \nabla(T_{l+1}(u_n) - T_l(u_n))\rangle \, dx + \int_\Omega g_n(x, u_n)(T_{l+1}(u_n) - T_l(u_n)) \, dx \\ = \int_\Omega f(T_{l+1}(u_n) - T_l(u_n)) \, dx \\ + \sum_{i=1}^N \int_\Omega F_i \nabla(T_{l+1}(u_n) - T_l(u_n)) \, dx \end{aligned}$$

which implies that,

$$\begin{aligned} \int_{\{l \leq |u_n| \leq l+1\}} \langle a(x, u_n, \nabla u_n), \nabla u_n\rangle \, dx + \int_{\{|u_n| \geq l+1\}} |g_n(x, u_n)| \, dx \\ \leq c \int_{\{|u_n| \geq l\}} |f| \, dx + \sum_{i=1}^N \int_{\{l \leq |u_n| \leq l+1\}} F_i w_i^{\frac{-1}{p}} (\frac{\alpha}{2})^{\frac{-1}{p}} |\nabla u_n| (\frac{\alpha}{2})^{\frac{1}{p}} \, dx \end{aligned}$$

by Young's inequality, we obtain

$$\begin{aligned} \int_{\{l \leq |u_n| \leq l+1\}} \langle a(x, u_n, \nabla u_n), \nabla u_n\rangle \, dx + \int_{\{|u_n| \geq l+1\}} |g_n(x, u_n)| \, dx \\ \leq c \int_{\{|u_n| \geq l\}} |f| \, dx + \frac{c(\alpha)}{p'} \sum_{i=1}^N \int_{\{|u_n| \geq l\}} |F_i|^{p'} w_i^{1-p'} \, dx \\ + \tfrac{\alpha}{2} \sum_{i=1}^N \int_{\{l \leq |u_n| \leq l+1\}} |\nabla u_n|^p w_i \, dx \end{aligned}$$

thus by (2.8), we have

$$\int_{\{|u_n| \geq l+1\}} |g_n(x, u_n)| \, dx \leq c \int_{\{|u_n| \geq l\}} |f_n| \, dx + \frac{c(\alpha)}{p'} \sum_{i=1}^N \int_{\{|u_n| \geq l\}} |F_i|^{p'} w_i^{1-p'} \, dx.$$

Let $\varepsilon > 0$, then there exist $l(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}} |g_n(x,u_n)|\, dx \leq \frac{\varepsilon}{2}. \tag{4.21}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |g_n(x,u_n)|\, dx &\leq \int_{E\cap\{|u_n|\leq l(\varepsilon)\}} |g_n(x,u_n)|\, dx + \int_{E\cap\{|u_n|>l(\varepsilon)\}} |g_n(x,u_n)|\, dx \\ &\leq \int_E |h_{l(\varepsilon)}(x)|\, dx + \int_{E\cap\{|u_n|>l(\varepsilon)\}} |g_n(x,u_n)|\, dx.\end{aligned}$$

In view to (2.10) there exist $\eta(\varepsilon) > 0$ such that

$$\int_E |h_{l(\varepsilon)}(x)|\, dx \leq \frac{\varepsilon}{2} \tag{4.22}$$

for all E such that meas$(E) < \eta(\varepsilon)$.

Finally, by combining (4.21) and (4.22) one easily has $\int_E |g_n(x,u_n)|\, dx \leq \varepsilon$, for all E such that meas$(E) < \eta(\varepsilon)$.

4. An intermediate Inequality

In this step, we shall prove that for $\varphi \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}&\int_\Omega \langle a(x,u_n,\nabla\varphi), \nabla T_k[u_n-\varphi]\rangle\, dx + \int_\Omega g_n(x,u_n) T_k[u_n-\varphi]\, dx \\ &\qquad \leq \int_\Omega f_n T_k[u_n-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u_n-\varphi]\rangle\, dx.\end{aligned} \tag{4.23}$$

We choose now $T_k(u_n-\varphi)$ as test function in (4.15), with φ in $W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$, we obtain

$$\begin{aligned}&\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k[u_n-\varphi]\rangle\, dx + \int_\Omega g_n(x,u_n) T_k[u_n-\varphi]\, dx \\ &\qquad = \int_\Omega f_n T_k[u_n-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u_n-\varphi]\rangle\, dx.\end{aligned}$$

Adding and subtracting the term $\int_\Omega \langle a(x,u_n,\nabla\varphi), \nabla T_k[u_n-\varphi]\rangle\, dx$ i.e.,

$$\begin{aligned}&\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k[u_n-\varphi]\rangle\, dx + \int_\Omega \langle a(x,u_n,\nabla\varphi), \nabla T_k[u_n-\varphi]\rangle\, dx \\ &\quad - \int_\Omega \langle a(x,u_n,\nabla\varphi), \nabla T_k[u_n-\varphi]\rangle\, dx + \int_\Omega g_n(x,u_n) T_k[u_n-\varphi]\, dx \\ &\quad = \int_\Omega f_n T_k[u_n-\varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u_n-\varphi]\rangle\, dx\end{aligned} \tag{4.24}$$

Thanks to assumption (2.7) and the definition of truncation function, we have

$$\int_\Omega \left\langle \left[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\varphi)\right], \nabla T_k[u_n-\varphi]\right\rangle dx \geq 0 \tag{4.25}$$

Combining (4.24) and (4.25), we obtain (4.23).

5. Passing to the limit

We shall prove that for $\varphi\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \langle a(x,u,\nabla\varphi),\nabla T_k[u-\varphi]\rangle\,dx+\int_\Omega g(x,u)T_k[u-\varphi]\,dx\leq\int_\Omega f\,T_k[u-\varphi]\,dx+\int_\Omega\langle F,\nabla T_k[u-\varphi]\rangle\,dx.$$

Firstly, we claim that

$$\int_\Omega \langle a(x,u_n,\nabla\varphi),\nabla T_k[u_n-\varphi]\rangle\,dx\to\int_\Omega\langle a(x,u,\nabla\varphi),\nabla T_k[u-\varphi]\rangle\,dx \text{ as } n\to+\infty.$$

Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^{1,p}(\Omega,w)$,with $M=k+\|\varphi\|_\infty$, then by Lemma 2.1, we have

$$T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi) \text{ in } W_0^{1.p}(\Omega,w), \tag{4.26}$$

which gives

$$\frac{\partial T_k}{\partial x_i}(u_n-\varphi)\rightharpoonup\frac{\partial T_k}{\partial x_i}(u-\varphi) \text{ weakly in } L^p(\Omega,w_i)\ \ \forall i=1,..,N. \tag{4.27}$$

Show that

$$a_i(x,T_M(u_n),\nabla\varphi)\to a_i(x,T_M(u),\nabla\varphi) \text{ strongly in } L^{p'}(\Omega,w_i^*)$$

Thanks to assumption (2.6), we obtain

$$\begin{aligned}|a_i(x,T_M(u_n),\nabla\varphi)|^{p'}w_i^{\frac{-p'}{p}} &\leq \beta[k(x)+|T_M(u_n)|^{\frac{q}{p'}}\sigma^{\frac{1}{p'}}+\sum_{j=1}^N|\frac{\partial\varphi}{\partial x_i}|^{p-1}w_i^{\frac{1}{p'}}]^{p'}\\ &\leq\gamma[k(x)^{p'}+|T_M(u_n)|^q\sigma+\sum_{j=1}^N|\frac{\partial\varphi}{\partial x_i}|^p w_i],\end{aligned} \tag{4.28}$$

with β and γ are positive constants. Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^{1,p}(\Omega,w)$ and $W_0^{1,p}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma)$, then $T_M(u_n)\to T_M(u)$ strongly in $L^q(\Omega,\sigma)$ and a.e. in Ω, hence

$$|a_i(x,T_M(u_n),\nabla\varphi)|^{p'}w_i^*\to|a_i(x,T_M(u),\nabla\varphi)|^{p'}w_i^* \ \ a.e.\text{in } \ \Omega.$$

and

$$\gamma\left[k(x)^{p'}+|T_M(u_n)|^q\sigma+\sum_{j=1}^N|\frac{\partial\varphi}{\partial x_i}|^p w_i\right]\to\gamma\left[k(x)^{p'}+|T_M(u)|^q\sigma+\sum_{j=1}^N|\frac{\partial\varphi}{\partial x_i}|^p w_i\right] \text{ a.e. in } \Omega.$$

Then, By Vitali's theorem, we deduce that

$$a_i(x, T_M(u_n), \nabla\varphi) \to a_i(x, T_M(u), \nabla\varphi) \text{ strongly in } L^{p'}(\Omega, w_i^*), \text{ as } n \to +\infty. \tag{4.29}$$

Combining (4.27) and (4.29), we obtain

$$\int_\Omega \langle a(x, u_n, \nabla\varphi), \nabla T_k[u_n - \varphi]\rangle\, dx \to \int_\Omega \langle a(x, u, \nabla\varphi), \nabla T_k[u - \varphi]\rangle\, dx, \text{ as } n \to +\infty. \tag{4.30}$$

Secondly, we show that

$$\int_\Omega f_n T_k[u_n - \varphi]\, dx \to \int_\Omega f\, T_k[u - \varphi]\, dx. \tag{4.31}$$

We have $f_n T_k[u_n - \varphi] \to f\, T_k[u - \varphi]$ *a.e.* in Ω and $|f_n T_k[u_n - \varphi]| \leq k|f_n|$ and $k|f_n| \to k|f|$ in $L^1(\Omega)$, then by using Vitali's theorem, we obtain (4.31).
Similarly thanks to (4.20) we can show that

$$\int_\Omega g_n(x, u_n) T_k[u_n - \varphi]\, dx \to \int_\Omega g(x, u) T_k[u - \varphi]\, dx \text{ as } n \to \infty. \tag{4.32}$$

Show that:

$$\int_\Omega \langle F, \nabla T_k[u_n - \varphi]\rangle\, dx \to \int_\Omega \langle F, \nabla T_k[u - \varphi]\rangle\, dx. \tag{4.33}$$

In view of (4.27) and since $F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*)$, we obtain (4.33).

Thanks to (4.30) , (4.31) and (4.33) allow to pass to the limit in the inequality (4.23), so that $\forall\varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$, we deduce

$$\int_\Omega \langle a(x, u, \nabla\varphi), \nabla T_k[u - \varphi]\rangle\, dx \leq \int_\Omega f\, T_k[u - \varphi]\, dx + \int_\Omega \langle F, \nabla T_k[u - \varphi]\rangle\, dx.$$

In view of Main Lemma, we can deduce that u is an entropy solution of the problem $(\mathscr{P})$. This completes the proof of Theorem 3.1.

Remark 4.1. *In the case where $F \equiv 0$, if we suppose that the second member are nonnegative, then we obtain a nonnegative solution.*

Indeed, If we take $v = T_h(u^+)$ in (P), we have

$$\begin{aligned} &\int_\Omega \langle a(x, u, \nabla u), \nabla T_k(u - T_h(u^+))\rangle\, dx \\ &\qquad + \int_\Omega g(x, u) T_k(u - T_h(u^+))\, dx \\ &\qquad \leq \int_\Omega f\, T_k(u - T_h(u^+))\, dx. \end{aligned}$$

Since $g(x,u)T_k(u - T_h(u^+)) \geq 0$, we deduce

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k(u - T_h(u^+))\rangle \, dx \leq \int_\Omega f \, T_k(u - T_h(u^+)) \, dx,$$

we remark also, by using $f \geq 0$

$$\int_\Omega f \, T_k(u - T_h(u^+)) \, dx \leq \int_{\{u \geq h\}} f \, T_k(u - T_h(u)) \, dx.$$

On the other hand, thanks to (2.8), we conclude

$$\alpha \int_\Omega \sum_{i=1}^N |\frac{\partial T_k(u^-)}{\partial x_i}|^p w_i \, dx \leq \int_{\{u \geq h\}} f \, T_k(u - T_h(u)) \, dx.$$

Letting h tend to infinity, we can easily deduce

$$T_k(u^-) = 0, \quad \forall k > 0,$$

which implies that

$$u \geq 0.$$

5. Example

Let us consider the following special case:

$$a_i(x,\eta,\xi) = w_i(x)|\xi_i|^{p-1} sgn(\xi_i) \quad i = 1, ..., N,$$

$$g(x,s) = \rho s|s|^r \qquad\qquad \rho > 0 \text{ and } r > 0$$

with $w_i(x)$ is a weight function ($i = 1, ..., N$).
For simplicity, we shall suppose that:

$$w_i(x) = w(x) \ \ for \ \ i = 1, ..., N-1, w_N(x) \equiv 0$$

it is easy to show that $a_i(x,s,\xi)$ are Caracthéodory function satisfying the growth condition (2.6) and the coercivity (2.8). On the other hand, the monotonicity condition is verified. In fact,

$$\sum_{i=1}^N (a_i(x,s,\xi) - a_i(x,s,\hat{\xi}))(\xi_i - \hat{\xi}_i) = w(x) \sum_{i=1}^{N-1} (|\xi_i|^{p-1} sgn(\xi_i) - |\hat{\xi}_i|^{p-1} sgn(\hat{\xi}_i))(\xi_i - \hat{\xi}_i) \geq 0$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in \mathbb{R}^N$. This last inequality can not be strict, since for $\xi \neq \hat{\xi}$ with $\xi_N \neq \hat{\xi}_N$ and $\xi_i = \hat{\xi}_i, \ \ i = 1, ..., N-1$. The corresponding expression is zero.

In particular, let us use special weight functions w and σ expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set

$$w(x) = d^{\lambda}(x), \quad \sigma(x) = d^{\mu}(x).$$

In this case, the Hardy inequality reads

$$\left(\int_{\Omega} |u(x)|^{q} d^{\mu}(x)\, dx \right)^{\frac{1}{q}} \leq c \left(\sum_{i=1}^{N-1} \int_{\Omega} |\frac{\partial u}{\partial x_i}|^{p} d^{\lambda}(x)\, dx \right)^{\frac{1}{p}}.$$

The corresponding imbedding is compact if:
(i) For, $1 < p \leq q < \infty$,

$$\lambda < p - 1, \quad \frac{N}{q} - \frac{N}{p} + 1 \geq 0, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0. \tag{5.1}$$

(ii) For $1 \leq q < p < \infty$,

$$\lambda < p - 1, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0. \tag{5.2}$$

Remark 5.1. *1.Condition* (5.1) *or* (5.2) *are sufficient for the compact imbedding* (2.5) *to hold; for example [[7], Example 1, [8] Example 1.5], and [9], Theorems 19.17, 19.22].*

Finally, the hypotheses of Theorem 3.1 are satisfied. Therefor the following problem

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega, w) \\ \displaystyle\int_{\Omega} \sum_{i=1}^{N} w_i(x) |\frac{\partial u}{\partial x_i}|^{p-1} \mathrm{sgn}(\frac{\partial u}{\partial x_i}) \frac{\partial T_k(u-\varphi)}{\partial x_i}\, dx \\ \displaystyle + \int_{\Omega} u\exp(u) T_k(u-\varphi)\, dx = \int_{\Omega} f\, T_k(u-\varphi)\, dx + \int_{\Omega} F \nabla T_k(u-\varphi)\, dx \\ \displaystyle f \in L^1(\Omega), \quad F \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*) \;\text{ and }\; \forall \varphi \in W_0^{1,p}(\Omega, w) \cap L^{\infty}(\Omega) \end{cases}$$

has at last one solution.

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