EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 4, 2011, 424-434
ISSN 1307-5543 – www.ejpam.com

Lattice Structures on $\mathscr{Z}^+$ Induced by Convolutions

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Abstract. A Convolution C is a mapping of the set $\mathscr{Z}^+$ of positive integers into the power set $P(\mathscr{Z}^+)$ such that every member of $C(n)$ is a divisor of n. If for any n, $D(n)$ is the set of all positive divisors of n , then D is called the Dirichlet's convolution. It is well known that $\mathscr{Z}^+$ has the structure of a distributive lattice with respect to the division order. Corresponding to any general convolution C, one can define a binary relation $\leq_C$ on Z^+ by ' $m \leq_C n$ if and only if $m \in C(n)$ ' . In this paper we characterize Convolutions C which induce partial orders with respect to which Z^+ has the structure of a semi lattice or lattice and various lattice theoretic properties are discussed in terms of convolution.

2000 Mathematics Subject Classifications: 06B99,11A99.

Key Words and Phrases: Poset,Lattice,Support,Convolution,Multiplicative,Relatively Prime.

1. Introduction

A Convolution is a mapping $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ such that $\mathscr{C}(n)$ is a set of positive divisors on n, $n \in \mathscr{C}(n)$ and $\mathscr{C}(n) = \bigcup_{m \in \mathscr{C}(n)} \mathscr{C}(m)$, for any $n \in \mathscr{Z}^+$. Popular examples are the Dirichlet's convolution D and the Unitary convolution U defined respectively by

$$D(n) = \text{ The set of all positive divisors of } n$$
$$\text{and } U(n) = \text{ The set of Unitary divisors of } n$$

for any $n \in \mathscr{Z}^+$. If $\mathscr{C}$ is a convolution, then the binary relation $\leq_{\mathscr{C}}$ on $\mathscr{Z}^+$, defined by,

$$m \leq_{\mathscr{C}} n \text{ if and only if } m \in \mathscr{C}(n),$$

is a partial order on $\mathscr{Z}^+$ and is called the partial order induced by $\mathscr{C}$ [3]. It is well known that the Dirichlet's convolution induces the division order on $\mathscr{Z}^+$ with respect to which $\mathscr{Z}^+$ becomes a distributive lattice, where, for any $a, b \in \mathscr{Z}+$, the greatest common divisor(GCD) and the least common multiple(LCM) of a and b are respectively the greatest lower bound(glb)

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and the least upper bound(lub) of a and b . In fact, with respect to the division order, the lattice $\mathscr{Z}^+$ satisfies the infinite join distributive law given by

$$(a \vee (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \vee b_i))$$

for any $a \in \mathscr{Z}^+$ and $\{b_i\}_{i\in I} \subseteq \mathscr{Z}^+$. In this paper, we discuss various aspects of the lattice structures on $\mathscr{Z}^+$ induced by general convolutions.

2. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leq$ on X which is reflexive ($a \leq a$), transitive ($a \leq b, b \leq c \Longrightarrow a \leq c$) and antisymmetric ($a \leq b, b \leq a \Longrightarrow a = b$) and that a pair $(X, \leq)$ is called a partially ordered set(poset) if X is a non-empty set and $\leq$ is a partial order on X. For any $A \subseteq X$ and $x \in X$, x is called a lower(upper) bound of A if $x \leq a$(respectively $a \leq x$) for all $a \in A$. We have the usual notations of the greatest lower bound(glb) and least upper bound(lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A(lub of A) is denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$). A partially ordered set $(X, \leq)$ is called a meet semi lattice if $a \wedge b$ (=glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leq)$ is called a join semi lattice if $a \vee b$ (=lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leq)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system $(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leq$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leq$ are related by

$$(a = a \wedge b \quad \Longleftrightarrow \quad a \leq b \quad \Longleftrightarrow \quad a \vee b = b).$$

Throughout the paper, $\mathscr{Z}^+$ and $\mathscr{N}$ denote the set of positive integers and the set of non-negative integers respectively.

Definition 1. *A mapping $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ is called a convolution if the following are satisfied for any $n \in \mathscr{Z}^+$.*

(1). $\mathscr{C}(n)$ is a set of positive divisors of n

(2). $n \in \mathscr{C}(n)$

(3). $\mathscr{C}(n) = \bigcup_{m\in\mathscr{C}(n)} \mathscr{C}(m)$.

Definition 2. *For any convolution $\mathscr{C}$ and m and $n \in \mathscr{Z}^+$, we define*

$$\big(m \leq n \textit{ if and only if } m \in \mathscr{C}(n)\big)$$

Then $\leq_{\mathscr{C}}$ is a partial order on $\mathscr{Z}^+$ and is called the partial order induced by $\mathscr{C}$ on $\mathscr{Z}^+$.

In fact, for any mapping $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ such that each member of $\mathscr{C}(n)$ is a divisor of n, $\leq_{\mathscr{C}}$ is a partial order on $\mathscr{Z}^+$ if and only if $\mathscr{C}$ is a convolution, as defined above[4]. It is known that, for any convolution $\mathscr{C}$, the poset $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ satisfies the Descending Chain Condition(DCC) in the sense that any non-empty subset of $\mathscr{Z}^+$ has minimal member.

3. Semilattice Structures on $\mathscr{Z}^+$

In this section, we discuss possible semi lattice structures on $\mathscr{Z}^+$ induced by convolutions. Recall that the Dirichlet's convolution induces a lattice structure on $\mathscr{Z}^+$, while the Unitary convolution induces only a meet semi lattice structure on $\mathscr{Z}^+$.

Definition 3. *Let $\mathscr{C}$ be a convolution.*

*(1). $\mathscr{C}$ is said to satisfy the **finite intersection property**(FIP) if*

$$\mathscr{C}(n_1) \cap \mathscr{C}(n_2) \cap \cdots \cap \mathscr{C}(n_r) \neq \emptyset \textit{ for any } n_1, n_2, \cdots, n_r \in \mathscr{Z}^+.$$

*(2). $\mathscr{C}$ is said to be **closed under finite intersections(unions)** if, for any $n_1, n_2, \cdots, n_r \in \mathscr{Z}^+$, there exists $n \in \mathscr{Z}^+$ such that*

$$\mathscr{C}(n_1) \cap \mathscr{C}(n_2) \cap \cdots \cap \mathscr{C}(n_r) = \mathscr{C}(n).$$
$$\scriptstyle (\textit{respectively } \mathscr{C}(n_1) \cup \mathscr{C}(n_2) \cup \cdots \cup \mathscr{C}(n_r) = \mathscr{C}(n))$$

*(3). $\mathscr{C}$ is said to be **closed under non-empty intersections** if, for any non-empty subset A of $\mathscr{Z}^+$, there exists $n \in \mathscr{Z}^+$ such that $\bigcap\limits_{a \in A} \mathscr{C}(a) = \mathscr{C}(n)$.*

Theorem 1. *Let $\mathscr{C}$ be a convolution and $\leq_{\mathscr{C}}$ the partial order induced by $\mathscr{C}$ on $\mathscr{Z}^+$. Then the following are equivalent to each other.*

(1). $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a semilattice

(2). $\mathscr{C}$ is closed under finite intersections

(3). $\mathscr{C}$ is closed under non-empty intersections

(4). Every non-empty subset of $\mathscr{Z}^+$ has glb in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$.

Proof. $(1) \Longrightarrow (2)$: Suppose that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet semilattice.
Then every non-empty finite subset of $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a semilattice.
Let $n_1, n_2, \cdots, n_r \in \mathscr{Z}^+$ and $\big(n = glb\ \{n_1, n_2, \cdots, n_r\}\big)$. Then, for any $a \in \mathscr{Z}^+$,

$$\begin{aligned} a \in \bigcap_{i=1}^{r} \mathscr{C}(n_i) &\iff a \leq_{\mathscr{C}} n_i \text{ for all } 1 \leq i \leq r \\ &\iff a \leq_{\mathscr{C}} n, \text{ since } n = glb\{n_1, n_2, \cdots, n_r\} \\ &\iff a \in \mathscr{C}(n) \end{aligned}$$

and hence $\left(\bigcap_{i=1}^{r} \mathscr{C}(n_i) = \mathscr{C}(n)\right)$. Thus $\mathscr{C}$ is closed under finite intersections.
$(2) \Longrightarrow (3)$: Suppose that $\mathscr{C}$ is closed under finite intersections. Let A be a non-empty subset of $\mathscr{Z}^+$. We have to prove that $\bigcap_{a\in A} \mathscr{C}(a) = \mathscr{C}(n)$ for some $n \in \mathscr{Z}^+$. If A is finite, we are through. Suppose that A is an infinite subset of $\mathscr{Z}^+$. Then A is countably infinite and hence we can write

$$A = \{a_1, a_2, a_3, \cdots\}$$

For each $r \in \mathscr{Z}^+$, there exists $b_r \in \mathscr{Z}^+$ such that $\bigcap_{i=1}^{r} \mathscr{C}(a_i) = \mathscr{C}(b_r)$.
Then, we have, for any $r \in \mathscr{Z}^+$,

$$b_{r+1} \in \mathscr{C}(b_{r+1}) = \bigcap_{i=1}^{r+1} \mathscr{C}(a_i) \subseteq \bigcap_{i=1}^{r} \mathscr{C}(a_i) = \mathscr{C}(b_r)$$

so that $b_{r+1} \leq_{\mathscr{C}} b_r$ for all $r \in \mathscr{Z}^+$.
Let $B = \{b_1, b_2, b_3, \cdots\}$. Since $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ satisfies the descending chain condition, B has a minimal element, say n. Then $n = b_r$ for some r and, since $b_{r+k} \leq b_r = n$ and since n is minimal in B, it follows that $b_{r+k} = n = b_r$ for all $k \in \mathscr{Z}^+$.
Now, $\bigcap_{i=1}^{\infty} \mathscr{C}(a_i) = \bigcap_{i=1}^{\infty} \mathscr{C}(b_i) = \mathscr{C}(b_1) \cap \cdots \cap \mathscr{C}(b_r) = \mathscr{C}(b_r) = \mathscr{C}(n)$. Thus $\mathscr{C}$ is closed under non-empty intersections.
$(3) \Longrightarrow (4)$ is similar to that of $(1) \Longrightarrow (2)$. $(4) \Longrightarrow (1)$ is trivial.

The dual of the above theorem is not true; that is, even if $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a join semilattice, $\mathscr{C}$ may not be closed under finite unions. However, we have the following other extreme.

Theorem 2. *Let $\mathscr{C}$ be a convolution and m and $n \in \mathscr{Z}^+$. Then $\mathscr{C}(m) \cup \mathscr{C}(n) = \mathscr{C}(a)$ for some $a \in \mathscr{Z}^+$ if and only if $\mathscr{C}(m) \subseteq \mathscr{C}(n)$ or $\mathscr{C}(n) \subseteq \mathscr{C}(m)$.*

Proof. If $\mathscr{C}(m) \cup \mathscr{C}(n) = \mathscr{C}(a)$, then $a \in \mathscr{C}(a) = \mathscr{C}(m) \cup \mathscr{C}(n)$ and hence $a \in \mathscr{C}(m)$ or $a \in \mathscr{C}(n)$ so that

$$\mathscr{C}(n) \subseteq \mathscr{C}(a) \subseteq \mathscr{C}(m) \text{ or } \mathscr{C}(m) \subseteq \mathscr{C}(a) \subseteq \mathscr{C}(n).$$

The converse is trivial.

In fact, any convolution $\mathscr{C}$ is never closed under finite (or infinite) unions. For, consider two distinct primes p and q. Then neither $\mathscr{C}(p) \subseteq \mathscr{C}(q)$ nor $\mathscr{C}(q) \subseteq \mathscr{C}(p)$ and hence, by the above theorem $\mathscr{C}(p) \cup \mathscr{C}(q) \neq \mathscr{C}(a)$ for any $a \in \mathscr{Z}^+$. Though $\mathscr{C}$ is never closed under finite unions, it is quite possible that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a join semi lattice. For, consider the Dirichlet's convolution D. Then $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ is a lattice.

Recall that a partially ordered set $(X, \leq)$ is called directed below(above) if, for any a and $b \in X$, there exists $x \in X$ such that $x \leq a$ and $x \leq b$ (respectively $a \leq x$ and $b \leq x$).

Theorem 3. *Let $\mathscr{C}$ be a convolution which is closed under finite intersections and $\leq_{\mathscr{C}}$ be the partial order on $\mathscr{Z}^+$ induced by $\mathscr{C}$. Then $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice if and only if it is directed above.*

Proof. From the hypothesis and Theorem 1, it follows that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet semilattice. Also, every non-empty subset of $\mathscr{Z}^+$ has glb in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$.
Now, suppose that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is directed above. Let a and $b \in \mathscr{Z}^+$ and

$$\left(A = \{n \in \mathscr{Z}^+ | a \leq_{\mathscr{C}} n \text{ and } b \leq_{\mathscr{C}} n\}\right)$$

Then, since the poset is directed above, A is a non-empty subset of $\mathscr{Z}^+$ and hence A has glb. Then it can be easily verified that the glb of A is the lub of $\{a, b\}$. Thus $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a join semilattice also and hence a lattice. The converse is trivial.

Unlike in Theorem 1, $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ may be a join semi lattice and not every non-empty subset has lub in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. In this context, note that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ can never possess the largest element; for, $\mathscr{C}(n)$ is a finite set for all $n \in \mathscr{Z}^+$. Theorem 3 can be dualised as given in the following theorem, whose proof is a consequence of the fact that the set of lower bounds of any non-empty subset of $\mathscr{Z}^+$ is finite.

Theorem 4. *Let $\mathscr{C}$ be a convolution such that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a join semilattice. Then $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice if and if it is directed below.*

Definition 4. *Let $\mathscr{C}$ be a convolution and p a prime number. Define a relation $\leq^p_{\mathscr{C}}$ on the set $\mathscr{N}$ of non-negative integers by*

$$\left(a \leq^p_{\mathscr{C}} b \textit{ if and only if } p^a \in \mathscr{C}(p^b)\right)$$

for any a and $b \in \mathscr{N}$.

It can be easily verified that $\leq^p_{\mathscr{C}}$ is a partial order on $\mathscr{N}$, for each prime p. The following is a direct verification.

Theorem 5. *Let $\mathscr{C}$ be a convolution.*

(1). If $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet(join) semilattice, then so is $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for each prime p.

(2). If $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice, then so is $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for each prime p.

The converse of the assertions made in the above theorem are not true in general. For, consider the following.

Example 1. *Define $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ by*

$$\mathscr{C}(n) = \begin{cases} \{1, 2, 5, 10\} & \textit{if } n = 10 \\ \{1, 2, 5, 20\} & \textit{if } n = 20 \\ \{1, n\} & \textit{otherwise} \end{cases}$$

Then $\mathscr{C}$ is a convolution. In this case, for any prime p and $a \in \mathscr{N}$, we have $\mathscr{C}(p^a) = \{1, p^a\}$ and hence 0 is the only lower bound for any two distinct a and b in $(\mathscr{N}, \leq^p_{\mathscr{C}})$. This implies that, for each prime p, $(\mathscr{N}, \leq^p_{\mathscr{C}})$ is a meet semilattice. However, $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is not a meet semilattice, since the set $\{10, 20\}$ has no glb in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$.

However, the converse of Theorem 5 are true if the convolution satisfies certain additional conditions.

4. Multiplicative Convolutions

Multiplicative convolutions are of special importance, for the single reason that the partial orders induced by them on $\mathscr{Z}^+$ can be characterized by those on $\mathscr{N}$. In this section, we discuss the order structures on $\mathscr{Z}^+$ induced by multiplicative convolutions.

Definition 5. *A convolution $\mathscr{C}$ is said to be* ***multiplicative*** *if, for any relatively prime integers m and n,*

$$\mathscr{C}(mn) = \mathscr{C}(m)\mathscr{C}(n) := \{ab | a \in \mathscr{C}(m) \text{ and } b \in \mathscr{C}(n)\}.$$

It can be verified that a convolution $\mathscr{C}$ is multiplicative if and only if, for any distinct primes $p_1, p_2, \cdots, p_r$ and non-negative integers $a_1, a_2, \cdots, a_r$,

$$\mathscr{C}(\prod_{i=1}^{r} p_i^{a_i}) = \prod_{i=1}^{r} \mathscr{C}(p_i^{a_i}) := \{m_1 m_2 \cdots m_r | m_i \in \mathscr{C}(p_i^{a_i})\}.$$

Multiplicative convolutions can be characterized in terms of the orders induced by them on $\mathscr{Z}^+$ and $\mathscr{N}$, as given in the following whose proof is a straight forward verification.

Theorem 6. *Let $\mathscr{C}$ be a convolution and $\leq_{\mathscr{C}}$ and $\leq_{\mathscr{C}}^p$ be the partial orders induced by $\mathscr{C}$ on $\mathscr{Z}^+$ and $\mathscr{N}$ respectively, for each prime p. Let*

$$\sum_P \mathscr{N} = \{f : P \longrightarrow \mathscr{N} | f(p) = 0 \text{ for all but finite number of } p's\}$$

where P is the set of all primes. Define

$$\theta : \mathscr{Z}^+ \longrightarrow \sum_P \mathscr{N}$$

by $\theta(n)(p) = a$, where a is the largest in $\mathscr{N}$ such that p^a divides n. Then θ is a bijection. Further the convolution $\mathscr{C}$ is multiplicative if and only if α is an order isomorphism of $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ onto $(\sum_P \mathscr{N}, \leq_{\mathscr{C}})$, where $\leq_{\mathscr{C}}$ also denotes the point-wise order on $\sum_P \mathscr{N}$ defined by

$$f \leq_{\mathscr{C}} g \Longleftrightarrow f(p) \leq_{\mathscr{C}}^p g(p) \text{ for all } p \in P.$$

Corollary 1. *Let $\mathscr{C}$ be a multiplicative convolution. Then $(\mathscr{N}, \leq_{\mathscr{C}}^p)$ is a meet(join) semilattice for each prime p if and only if $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet (respectively join) semilattice.*

Corollary 2. *For any multiplicative convolution $\mathscr{C}$, $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice if and only if $(\mathscr{N}, \leq_{\mathscr{C}}^p)$ is a lattice for each prime p.*

Example 2. *(1). Let D be the Dirichlet's convolution defined by*

$$D(n) = \text{ The set of all positive divisors of } n.$$

Then D is multiplicative. $\leq_D$ is precisely the division order on $\mathscr{Z}^+$ and, for each prime p, $\leq_D^p$ is the usual order on $\mathscr{N}$. $(\mathscr{Z}^+, \leq_D)$ is known to be distributive lattice.

(2). Let $U(n)$ be the Unitary convolution defined by

$$U(n) = \{d \in D(n) | d \text{ and } \frac{n}{d} \text{ are relatively prime}\}.$$

Then U is multiplicative and $(\mathscr{Z}^+, \leq_U)$ is a meet semilattice, but not a join semilattice.

(3). Let F_2 be the square-free convolution defined by

$$F_2(n) = \{n\} \cup \{d \in D(n) | p^2 \text{ does not divide } n \text{ for any prime } p\}.$$

Then F_2 is a multiplicative convolution and $(\mathscr{Z}^+, \leq_{F_2})$ is a meet semilattice but not a join semilattice.

(4). For any $k \in \mathscr{Z}^+$, a positive integer d is said to be k-free if p^k does not divide d for any prime p. Let $F_k(n)$ be the set of all k-free divisors of n together with n. Then F_k is a multiplicative convolution and $(\mathscr{Z}^+, \leq_{F_k})$ is a meet semilattice but not a join semi lattice.

In our further discussions, we assume that a convolution $\mathscr{C}$ satisfies the additional property that $1 \in \mathscr{C}(n)$ for all $n \in \mathscr{Z}^+$. Note that this is equivalent to saying that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ has least element and that this is further equivalent to saying $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is directed below. By assuming that $1 \in \mathscr{C}(n)$, we are not losing any generality, since we are interested in convolutions $\mathscr{C}$ with respect to which $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet semilattice.

Theorem 7. *A convolution $\mathscr{C}$ is multiplicative if and only if the following conditions are satisfied for any relatively prime integers m and n.*

(1). $m \vee n$ exists in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ and is equal to mn.

(2). $x \wedge (m \vee n) = (x \wedge m) \vee (x \wedge n)$ for all $x \in \mathscr{Z}^+$, in the sense that, if one side is defined then the other side is also defined and they are equal.

Proof. Let $\mathscr{C}$ be a convolution. Suppose that $\mathscr{C}$ is multiplicative. Then the mapping $\theta : \mathscr{Z}^+ \longrightarrow \sum_P \mathscr{N}$, defined in Theorem 6, is an order isomorphism of $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ onto $(\sum_P \mathscr{N}, \leq_{\mathscr{C}})$. Let m and $n \in \mathscr{Z}^+$ such that $(m,n) = 1$. We shall prove that $mn = lub\{m,n\}$ in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. Since $m \in \mathscr{C}(m)$ and $1 \in \mathscr{C}(n)$, we have, $m = m.1 \in \mathscr{C}(m).\mathscr{C}(n) = \mathscr{C}(mn)$. And hence $m \leq_{\mathscr{C}} mn$ and similarly $n \leq_{\mathscr{C}} mn$. If r is any upper bound of $\{m,n\}$ in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$, then

$$\theta(m) \leq_{\mathscr{C}} \theta(r) \text{ and } \theta(n) \leq_{\mathscr{C}} \theta(r) \text{ in } (\sum_P \mathscr{N}, \leq_{\mathscr{C}})$$

and hence, for any prime p,

$$\theta(m)(p) \leq^p_{\mathscr{C}} \theta(r)(p) \text{ and } \theta(n)(p) \leq^p_{\mathscr{C}} \theta(r)(p).$$

Also, since m and n are relatively prime, we have for any prime p, $\theta(m)(p) = 0$ or $\theta(n)(p) = 0$. Now, $\theta(mn)(p) = \theta(m)(p) + \theta(n)(p) = \theta(m)(p)$ or $\theta(n)(p)$. And hence $\theta((mn))(p) \leq^p_{\mathscr{C}} \theta(r)(p)$. Therefore $\theta(mn) \leq_{\mathscr{C}} \theta(r)$ and $mn \leq_{\mathscr{C}} r$. Thus mn is the least upper bound of m and n. This proves (1).

To prove (2), let $x \in \mathscr{Z}^+$. Suppose that $x \wedge (m \vee n)$ exists in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. Suppose $m \vee n$ exists and is equal to mn , we are given that $x \wedge (mn)$ exists. We shall prove that $x \wedge m$ and $x \wedge n$ exists in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. To prove this, it is enough if we prove that $\theta(x)(p) \wedge \theta(m)(p)$ and $\theta(x)(p) \wedge \theta(n)(p)$ exist in $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for any prime p. Since $\theta(mn)(p) = \theta(m)(p) + \theta(n)(p) = \theta(m)(p)$ or $\theta(n)(p)$ and since $\theta(x)(p) \wedge \theta(m)(p)$ exists in $(\mathscr{N}, \leq^p_{\mathscr{C}})$ it follows that $\theta(x)(p) \wedge \theta(m)(p)$ and $\theta(x)(p) \wedge \theta(n)(p)$ exist in $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for any prime p. Therefore $x \wedge m$ and $x \wedge n$ exist in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. Now, since $x \wedge m \leq_{\mathscr{C}} m$, $x \wedge m \in \mathscr{C}(m) \subseteq D(m)$ and hence $x \wedge m$ is a divisor of m. Similarly $x \wedge n$ is a divisor of n. Since $(m, n) = 1$, we get that $(x \wedge m, x \wedge n) = 1$ and hence by (1), $(x \wedge m) \vee (x \wedge n)$ exists in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ and is equal to the product $(x \wedge m)(x \wedge n)$. Now, for any $p \in P$, we have

$$\begin{aligned}
\theta(x \wedge (m \vee n))(p) &= \theta(x \wedge (mn))(p) \\
&= \theta(x)(p) \wedge \theta(mn)(p) \\
&= \theta(x)(p) \wedge \theta(m)(p) \text{ or } \theta(x)(p) \wedge \theta(n)(p) \\
&= \theta(x \wedge m)(p) \text{ or } \theta(x \wedge n)(p) \\
&= \theta((x \wedge m)(x \wedge n))(p) \\
&= \theta((x \wedge m) \vee (x \wedge n))(p)
\end{aligned}$$

and hence $\theta(x \wedge (m \vee n)) = \theta((x \wedge m) \vee (x \wedge n))$, so that $x \wedge (m \vee n) = (x \wedge m) \vee (x \wedge n)$. Similarly, we can prove that the left hand side of the equation exists if the right hand side exists and that they are equal. This proves (2). Conversely suppose that the conditions (1) and (2) are satisfied for any relatively prime positive integers m and n. To prove that $\mathscr{C}$ is multiplicative, let us consider, m and $n \in \mathscr{Z}^+$ such that $(m, n) = 1$. Then $(y, z) = 1$ for all $y \in \mathscr{C}(m)$ and $z \in (C)(n)$ and hence, by (1), $y \vee z$ exists and is equal to yz in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ whenever $y \leq_{\mathscr{C}} m$ and $z \leq_{\mathscr{C}} n$ and, by (2),

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ for all } x \in \mathscr{Z}^+.$$

Now consider

$$\begin{aligned}
x \in \mathscr{C}(mn) &\implies x \leq_{\mathscr{C}} mn = m \vee n \\
&\implies x = x \wedge (m \vee n) = (x \wedge m) \vee (x \wedge n) \\
&\implies x = (x \wedge m)(x \wedge n), x \wedge m \in \mathscr{C}(m),\ x \wedge n \in \mathscr{C}(n) \\
&\implies x \in \mathscr{C}(m)\mathscr{C}(n)
\end{aligned}$$

Therefore $\mathscr{C}(mn) \subseteq \mathscr{C}(m)\mathscr{C}(n)$.

$$\begin{aligned}
\text{On the other hand } x \in \mathscr{C}(m)\mathscr{C}(n) &\implies x = yz, y \in \mathscr{C}(m) \text{ and } z \in \mathscr{C}(n)\\
&\implies x = y \vee z, y \leq_{\mathscr{C}} m, \text{ and } z \leq_{\mathscr{C}} n\\
&\implies x = y \vee z \leq_{\mathscr{C}} m \vee n = mn\\
&\implies x \in \mathscr{C}(mn)
\end{aligned}$$

Therefore $\mathscr{C}(m)\mathscr{C}(n) \subseteq \mathscr{C}(mn)$. Thus $\mathscr{C}(mn) = \mathscr{C}(m)\mathscr{C}(n)$ and hence $\mathscr{C}$ is multiplicative.

Actually, the existence of $m \vee n$ in condition (1) above is a consequence of (2). For, choose a prime p which divides neither m nor n. Then $p \wedge m = 1 = p \wedge n$ and hence $(p \wedge m) \vee (p \wedge n)$ exists which, by(2), implies that $p \wedge (m \vee n)$ exists and, in particular $m \vee n$ exists.

Theorem 8. *A convolution $\mathscr{C}$ is multiplicative if and only if the following are satisfied in the poset $(\mathscr{Z}^+, \leq_{\mathscr{C}})$.*

(1). For any $m, n \in \mathscr{Z}^+$ with $(m,n) = 1, m \vee n$ exists and is equal to the product mn

(2). For any x, m and $n \in \mathscr{Z}^+$ with $(x,m) = 1 = (x,n)$, $x \vee (m \wedge n) = (x \vee m) \wedge (x \vee n)$.

Proof. Let $\mathscr{C}$ be a convolution and $\leq_{\mathscr{C}}$ be the corresponding partial order on $\mathscr{Z}^+$. Suppose that $\mathscr{C}$ is multiplicative. Then by Theorem 7, (1) holds good. To prove (2), let x, m and $n \in \mathscr{Z}^+$ such that $(x,m) = 1 = (x,n)$. By (1), $x \vee m$ and $x \vee n$ exist and are equal to xm and xn respectively in $(\mathscr{Z}^+, \leq \mathscr{C})$. Suppose that $(x \vee m) \wedge (x \vee n)$ exists. Then $\theta(xm)(p) \wedge \theta(xn)(p)$ exists in $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for any prime p. We have $\theta(xm)(p) = \theta(x)(p) + \theta(m)(p)$ and $\theta(xn)(p) = \theta(x)(p) + \theta(n)(p)$. If $\theta(x)(p) = 0$, then $\theta(m)(p) = \theta(xm)(p)$ and $\theta(n)(p) = \theta(xn)(p)$ which implies that $\theta(m)(p) \wedge \theta(n)(p)$ exists in $(\mathscr{N}, \leq^p_{\mathscr{C}})$. On the other hand, if $\theta(x)(p) \neq 0$, then $\theta(m)(p) = 0 = \theta(n)(p)$ and trivially $\theta(m)(p) \wedge \theta(n)(p)$ exists. Thus $\theta(m)(p) \wedge \theta(n)(p)$ exists in $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for all primes p. Again, $m \wedge n$ exists in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. Also, since $m \wedge n \leq_{\mathscr{C}} n$, $m \wedge n$ is a divisor of m and hence $(x, m \wedge n) = 1$. Therefore $x \vee (m \wedge n)$ exists. By evaluating $\theta(x \vee (m \wedge n))(p)$ and $\theta((x \vee m) \wedge (x \vee n))(p)$, we get that they are equal for all primes p. Thus $x \vee (m \wedge n)$ exists and is equal to $(x \vee m) \wedge (x \vee n)$. The other implication can also be proved similarly. Thus the condition (2) holds good.

Conversely suppose that the conditions (1) and (2) are satisfied in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. To prove the multiplicativity of $\mathscr{C}$, let m and $n \in \mathscr{Z}^+$ such that $(m,n) = 1$. Then by (1), $m \vee n$ exists and is equal to mn. In particular $m \leq_{\mathscr{C}} mn$ and $n \leq_{\mathscr{C}} mn$.

$$\begin{aligned}
x \in \mathscr{C}(m)\mathscr{C}(n) &\implies x = ab, a \in \mathscr{C}(m) \text{ and } b \in \mathscr{C}(n)\\
&\implies x = ab = a \vee b, (\text{since } (a,b) = 1)\\
&\implies x \leq_{\mathscr{C}} m \vee n = mn(\text{ since } a \leq_{\mathscr{C}} m, b \leq_{\mathscr{C}} n)\\
&\implies x \in \mathscr{C}(mn)
\end{aligned}$$

Therefore $\mathscr{C}(m)\mathscr{C}(n) \subseteq \mathscr{C}(n)$. On the other hand, let $x \in \mathscr{C}(mn)$. Put $y = (x,m)$ and $z = (x,n)$. Since $(m,n) = 1$, it follows that $x = yz$ and $(y,z) = 1 = (y,n) = (z,m) = (m,n)$

and hence $m \vee n = mn$ and $m \vee z = mz$. Since $m \leq_{\mathscr{C}} m \vee n = mn$ and $z \leq_{\mathscr{C}} y \vee z = yz = x \leq_{\mathscr{C}} mn$, we get that $m \vee z \leq_{\mathscr{C}} mn = m \vee n$. Therefore $(m \vee z) \wedge (m \vee n)$ exists and is equal to $m \vee z$. By (2), $m \vee (z \wedge n)$ exists and $m \vee z = (m \vee z) \wedge (m \vee n) = m \vee (z \wedge n) = m(z \wedge n)$. Since $(z, m) = 1$, z should divide $z \wedge n$. But $z \wedge n \leq_{\mathscr{C}} z$ and hence $z \wedge n \in \mathscr{C}(z)$. Therefore $z = z \wedge n \leq_{\mathscr{C}} n$. Similarly $y \leq_{\mathscr{C}} m$ and hence $y \in \mathscr{C}(m)$ and $z \in \mathscr{C}(n)$ and $x = yz \in \mathscr{C}(m)\mathscr{C}(n)$. Therefore $\mathscr{C}(mn) \subseteq \mathscr{C}(m)\mathscr{C}(n)$.Thus $\mathscr{C}(mn) = \mathscr{C}(m)\mathscr{C}(n)$. Thus $\mathscr{C}$ is multiplicative.

Let us recall a subset A of $\mathscr{Z}^+$ is **multiplicatively closed** if the product of any two relatively prime elements of A is again an element of A. In the following, we obtain a condition equivalent to (1) above.

Theorem 9. *The following are equivalent for any convolution $\mathscr{C}$.*

(1). $m \vee n$ exists in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ and is equal to mn, for all relatively prime m, n in $\mathscr{Z}^+$.

(2). (i) $\mathscr{C}$ is multiplicatively closed for all $n \in \mathscr{Z}^+$ and

(ii) $\mathscr{C}(m)\mathscr{C}(n) \subseteq \mathscr{C}(mn)$ whenever $(m, n) = 1$.

Proof. (1)$\Longrightarrow$(2): Let $n \in \mathscr{Z}^+$ and $x, y \in \mathscr{C}(n)$ such that $(x, y) = 1$. Then $x \vee y$ exists and is equal to xy . Since $x \leq_{\mathscr{C}} n$ and $y \leq_{\mathscr{C}} n$, we get that $xy = x \vee y \leq_{\mathscr{C}} n$ and hence $xy \in \mathscr{C}(n)$. Therefore $\mathscr{C}(n)$ is multiplicatively closed. Next, let $m, n \in \mathscr{Z}^+$ such that $(m, n) = 1$. Then, $m \vee n$ exists and is equal to mn and

$$\begin{aligned} x \in \mathscr{C}(m) \text{ and } y \in \mathscr{C}(n) &\Longrightarrow x \leq_{\mathscr{C}} m \text{ and } y \leq_{\mathscr{C}} n \\ &\Longrightarrow (x, y) = 1 \text{ and } x \vee y \leq_{\mathscr{C}} m \vee n \\ &\Longrightarrow xy = x \vee y \in \mathscr{C}(m \vee n) = \mathscr{C}(mn) \end{aligned}$$

Thus $\mathscr{C}(m).\mathscr{C}(n) \subseteq \mathscr{C}(mn)$.
(2)$\Longrightarrow$(1) : Let $m, n \in \mathscr{Z}^+$ such that $(m, n) = 1$. Then by (2)(ii), $\mathscr{C}(m).\mathscr{C}(n) \subseteq \mathscr{C}(mn)$ and therefore

$$m = m.1 \in \mathscr{C}(m).\mathscr{C}(n) \subseteq \mathscr{C}(mn) \text{ and } n = n.1 \in \mathscr{C}(m).\mathscr{C}(n) \subseteq \mathscr{C}(mn)$$

and hence $m \leq_{\mathscr{C}} mn$ and $n \leq_{\mathscr{C}} mn$. If x is any upper bound of m and n in $(\mathscr{Z}^+, \leq_{\mathscr{C}})$, then m and $n \in \mathscr{C}(x)$ and hence $mn = m \vee n \leq_{\mathscr{C}} x$.
Thus, mn is the least upper bound of m and n.

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