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On 1-Sequence-Covering π-s-Images of Locally Separable Metric Spaces

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Abstract. In this paper, we give a characterization on 1-sequence-covering π-s-images of locally separable metric spaces by means of point-countable σ-strong sn-network consisting of cosmic spaces (sn-second countable spaces, $\aleph_0$-spaces). As an application, we get a new characterization on 1-sequence-covering, quotient π-s-images of locally separable metric spaces, which is helpful in solving Y. Tanaka and S. Xia's question in [21].

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1. Introduction

To determine what spaces the images of ŞniceŤ spaces under ŞniceŤ mappings are is one of the central questions of general topology [1]. In the past, many noteworthy results on images of metric spaces have been obtained. For a survey in this field,

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see [19], for example. Related to characterizations on images of metric spaces, Y. Tanaka and S. Xia posed the following question in [21].

Question 1.1 ([21]). *What is a nice characterization for a quotient s-image of a locally separable metric space?*

This question was partly answered by many authors [13], [14], [20]. It is known that 1-sequence-covering s-images of metric spaces have been characterized by point-countable sn-networks, and 1-sequence-covering π-images of metric spaces have been characterized by σ-strong sn-networks [16]. Also, as in the proofs of [10], 1-sequence-covering π-s-images of metric spaces can be characterized by point-countable σ-strong sn-networks. Recently, the characterizations on images of locally separable metric spaces cause attention once again, and 1-sequence-covering s-images of locally separable metric spaces have been characterized by point-countable sn-network consisting of cosmic spaces ($\aleph_0$-spaces) [5].

Taking these results into account, it is natural to be interested in the following question.

Question 1.2. *Are the following equivalent for a space X?*

1. *X is an 1-sequence-covering π-s-image of a locally separable metric space.*
2. *X has a point-countable σ-strong sn-network consisting of cosmic spaces ($\aleph_0$-spaces).*

In this paper, we give a characterization on 1-sequence-covering π-s-images of locally separable metric spaces by means of point-countable σ-strong sn-network consisting of cosmic spaces (sn-second countable spaces, $\aleph_0$-spaces). As an application, we get a new characterization on 1-sequence-covering, quotient π-s-images of locally separable metric spaces, which is helpful in solving the above Question 1.1 of Y. Tanaka and S. Xia.

Throughout this paper, all spaces are regular and T_1, $\mathbb{N}$ denotes the set of all natural numbers, $\omega = \mathbb{N} \cup \{0\}$, and a convergent sequence includes its limit point. Let $\mathscr{P}$ be a family of subsets of X and $x \in X$. Then $\bigcap \mathscr{P}$, and $st(x, \mathscr{P})$ denote the intersection $\bigcap\{P : P \in \mathscr{P}\}$, and the union $\bigcup\{P \in \mathscr{P} : x \in P\}$, respectively. A convergent sequence $\{x_n : n \in \omega\}$ converging to x_0 is *eventually* in a subset A of X, if $\{x_n : n \geq n_0\} \cup \{x_0\} \subset A$ for some $n_0 \in \mathbb{N}$.

For terms which are not defined here, please refer to [3].

2. Main Results

Definition 2.1. *Let P be a subset of a space X.*

(1) P is a sequential neighborhood *of x [4], if for every convergent sequence S converging to x in X, S is eventually in P.*

(2) P is a sequentially open *subset of X [4], if for every $x \in P$, P is a sequential neighborhood of x.*

Definition 2.2. *Let $\mathscr{P}$ be a family of subsets of a space X.*

(1) For each $x \in X$, $\mathscr{P}$ is a network at x *in X, if $x \in \bigcap \mathscr{P}$, and if $x \in U$ with U open in X, there exists $P \in \mathscr{P}$ such that $x \in P \subset U$.*

(2) $\mathscr{P}$ is a cs-network *of X [8], if for every convergent sequence S converging to $x \in U$ with U open in X, there exists $P \in \mathscr{P}$ such that S is eventually in $P \subset U$.*

(3) $\mathscr{P}$ is an sn-cover *of X [14], if each element of $\mathscr{P}$ is a sequential neighborhood of some point in X, and for each $x \in X$, some $P \in \mathscr{P}$ is a sequential neighborhood of x.*

Definition 2.3. *Let X be a space.*

(1) X is an $\aleph_0$-space *[17] (resp.,* cosmic *space [17],* sn-second countable *space [7]), if X has a countable cs-network (resp., countable network, countable sn-network).*

(2) X is a sequential space *[4], if every sequentially open subset of X is open.*

(3) X is sequentially separable *[2], if X has a countable subset D such that for each $x \in X$, there exists a sequence $L \subset D$ converging to x, where D is a* sequentially dense subset *of X.*

Definition 2.4. *Let $\mathscr{P} = \bigcup\{\mathscr{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that, for each $x \in X$, $\mathscr{P}_x$ is a network at x in X, and if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$.*

(1) $\mathscr{P}$ is a weak base *of X [18], if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathscr{P}_x$ satisfying $P \subset G$, then G is open in X*

(2) $\mathscr{P}$ is an sn-network *of X [12], if each member of $\mathscr{P}_x$ is a sequential neighborhood of x in X.*

(3) The above $\mathscr{P}_x$ is respectively a weak base*, and an sn*-network *at x in X [11].*

Remark 2.5 ([14]). *An sn-network of a sequential space is a weak base.*

Definition 2.6. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) f is an 1-sequence-covering *mapping [12], if for every $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$.*

(2) f is an 1-sequentially quotient *mapping [16], if for every $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_k : k \in \mathbb{N}\}$ converging to x_y in X with each $x_k \in f^{-1}(y_{n_k})$.*

(3) f is a π-mapping *[1], if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.*

(4) f is an s-mapping *[1], if $f^{-1}(y)$ is separable for every $y \in Y$.*

(5) f is a π-s-mapping *[10], if f is both π-mapping and s-mapping.*

Definition 2.7. *Let $\{\mathscr{P}_n : n \in \mathbb{N}\}$ be a refinement sequence of a space X, i.e., each $\mathscr{P}_n$ is a cover of X and $\mathscr{P}_{n+1}$ is a refinement of $\mathscr{P}_n$.*

(1) $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *is* σ-strong network *of* X *[9], if* $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ *is a network at* x *in* X *for every* $x \in X$.

(2) $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *is* weak development *of* X *[10], if* $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ *is a weak base at* x *in* X *for every* $x \in X$.

(3) $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *is a* σ-strong sn-network *of* X*, if* $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *is a* σ*-strong network and each* $\mathscr{P}_n$ *is an sn-cover of* X*. A* σ*-strong sn-network of* X *is a* point-star network of sn-covers *in the sense of [16].*

(4) $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *is an* sn-weak-development *of* X*, if* $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *is a weak-development and each* $\mathscr{P}_n$ *is an sn-cover of* X.

Definition 2.8. *Let* $\mathscr{P} = \bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ *be a* σ*-strong network of* X*. For every* $n \in \mathbb{N}$*, put* $\mathscr{P}_n = \{P_\alpha : \alpha \in A_n\}$*, and endowed* A_n *with discrete topology. Put*

$$M = \Big\{a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M*, which is a subspace of the product space* $\prod_{n\in\mathbb{N}} A_n$*, is a metric space,* x_a *is unique, and* $x_a = \bigcap_{n\in\mathbb{N}} P_{\alpha_n}$ *for every* $a \in M$*. Define* $f : M \to X$ *by choosing* $f(a) = x_a$*, then* f *is a mapping and* $(f, M, X, \{\mathscr{P}_n\})$ *is a* Ponomarev-system *[15].*

Theorem 2.9. *The following are equivalent for a space* X.

1. *X is an 1-sequence-covering π-s-image of a locally separable metric space.*

2. *X is an 1-sequentially-quotient π-s-image of a locally separable metric space.*

3. *X has a point-countable σ-strong sn-network consisting of sn-second countable spaces.*

4. *X has a point-countable σ-strong sn-network consisting of $\aleph_0$-spaces.*

5. *X has a point-countable σ-strong sn-network consisting of cosmic spaces.*

Proof.

(1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Let $f : M \longrightarrow X$ be an 1-sequentially-quotient π-s-mapping from a locally separable metric space M with a metric d onto X. For each $x \in X$, there exists $a_x \in f^{-1}(x)$ such that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X there exists a sequence $\{a_k : k \in \mathbb{N}\}$ converging to a_x in M with each $a_k \in f^{-1}(x_{n_k})$. Since M is locally separable metric, $M = \oplus_{\lambda\in\Lambda} M_\lambda$ by [3, 4.4.F], where each M_λ is a separable metric space with a metric d_λ. For each $\lambda \in \Lambda$, let D_λ be a countable dense subset of M_λ. For each $n \in \mathbb{N}$, put

$$\mathscr{B}_{\lambda,n,x} = \{B(a, 1/n) : a \in D_\lambda, a_x \in B(a, 1/n)\},$$

where $B(a, 1/n) = \{b \in M_\lambda : d_\lambda(a, b) < 1/n\}$, and put

$$\mathscr{B}_{n,x} = \bigcup\{\mathscr{B}_{\lambda,n,x} : \lambda \in \Lambda\}, \mathscr{B}_n = \bigcup\{\mathscr{B}_{n,x} : x \in X\}, \mathscr{B}_x = \bigcup\{\mathscr{B}_{n,x} : n \in \mathbb{N}\},$$

$$\mathscr{B} = \bigcup\{\mathscr{B}_n : n \in \mathbb{N}\} = \bigcup\{\mathscr{B}_x : x \in X\},$$

and

$$\mathscr{P}_{n,x} = f(\mathscr{B}_{n,x}), \mathscr{P}_n = \bigcup\{\mathscr{P}_{n,x} : x \in X\}, \mathscr{P}_x = \bigcup\{\mathscr{P}_{n,x} : n \in \mathbb{N}\},$$

$$\mathscr{P} = \bigcup\{\mathscr{P}_n : n \in \mathbb{N}\} = \bigcup\{\mathscr{P}_x : x \in X\}.$$

Then $\{\mathscr{P}_n : n \in \mathbb{N}\}$ is a refinement sequence of X. We shall prove that $\mathscr{P}$ is a point-countable σ-strong sn-network of X consisting of sn-second countable spaces by the following facts (a), (b), (c), and (d).

(a) $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ is a σ-strong network of X.

Let $x \in U$ with U open in X. Since f is a π-mapping, $d(f^{-1}(x), M - f^{-1}(U)) > 0$. It implies that $d(f^{-1}(x), M - f^{-1}(U)) > 2/n$ for some $n \in \mathbb{N}$. Let $x \in f(B(a, 1/n)) \in \mathscr{P}_n$ for some $B(a, 1/n) \in \mathscr{B}_{\lambda,n,x}$. We shall prove that $B(a, 1/n) \subset f^{-1}(U)$. In fact, if $B(a, 1/n) \not\subset f^{-1}(U)$, then there exists $b \in B(a, 1/n) - f^{-1}(U)$. Since $f^{-1}(x) \cap$

$B(a,1/n) \neq \emptyset$, there exists $c \in f^{-1}(x) \cap B(a,1/n)$. Then $d(f^{-1}(x), M - f^{-1}(U)) \leq d(c,b) \leq d(c,a) + d(a,b) < 2/n$. It is a contradiction. Then we get $f(B(a,1/n)) \subset U$. Therefore, $st(x, \mathscr{P}_n) = \bigcup\{f(B(a,1/n)) : x \in f(B(a,1/n)), a \in D_\lambda, \lambda \in \Lambda\} \subset U$. It implies that $\mathscr{P}$ is a σ-strong network of X.

(b) Each $\mathscr{P}_n$ is an sn-cover of X.

Let $x \in X$ and $P = f(B) \in \mathscr{P}_{n,x}$ for some $B \in \mathscr{B}_{n,x}$. We shall prove that P is a sequential neighborhood of x. Let S be a convergent sequence converging to x in X. Then there exists a convergent sequence L converging to a_x in M such that $f(L)$ is a subsequence of S. Since B is open, L is eventually in B. Hence $f(L_\lambda)$ is eventually in P. It implies that S is frequently in P. It follows from [6, Remark 1.4] that P is a sequential neighborhood of x. Therefore, $\mathscr{P}_n$ is an sn-cover of X.

(c) $\mathscr{P}$ is point-countable.

Let $x \in X$. Since f is an s-mapping, $f^{-1}(x)$ is separable. It implies that $f^{-1}(x)$ meets at most countably many M_λ's. Then $f^{-1}(x)$ meets at most countably many members of $\mathscr{B}_n$, i.e., x meets at most countable many members of $\mathscr{P}_n$. Therefore, $\mathscr{P}$ is point-countable.

(d) Each $P \in \mathscr{P}$ is an sn-second countable space.

Let $P = f(B)$ for some $B \in \mathscr{B}$. Since B is separable metric, P is sequentially separable by [14, Lemma 2.2]. Let D_P be a sequentially dense subset of P. For each $x \in P$, put $\mathscr{Q}_x = \{Q \cap P : Q \in \mathscr{P}_x, Q \cap D_P \neq \emptyset\}$, and put $\mathscr{Q} = \bigcup\{\mathscr{Q}_x : x \in P\}$. Since $\mathscr{P}$ is point-countable and D_P is countable, $\mathscr{Q}$ is countable. It suffices to prove the following facts (i), (ii), and (iv) for every $x \in P$.

(i) $\mathscr{Q}_x$ is a network at x in P.

Let $x \in U$ with U open in P. Then $x \in V$ with V open in X and $V \cap P = U$. Let S be a sequence in D_P converging to x. Since $\mathscr{P}$ is a σ-strong sn-network of X, $S \cup \{x\}$ is eventually in $Q \subset V$ with some $Q \in \mathscr{P}_x$. It implies that $Q \cap D_P \neq \emptyset$, and $x \in Q \cap P \subset V \cap P = U$. Therefore, $\mathscr{Q}_x$ is a network at x in P.

(ii) If $Q_1, Q_2 \in \mathcal{Q}_x$, then $Q \subset Q_1 \cap Q_2$ for some $Q \in \mathcal{Q}_x$.

Let $Q_1 = f(B_1) \cap P, Q_2 = f(B_2) \cap P$ for some $B_1, B_2 \in \mathcal{B}_x$. Let S be a sequence in D_P converging to x. Then there exists a sequence L converging to a_x in M such that $f(L)$ is a subsequence of S. Since $\mathcal{B}_x$ is a base at a_x in M, there exists $C \in \mathcal{B}_x$ such that $L \cup \{a_x\}$ is eventually in $C \subset B_1 \cap B_2$. Then $S \cup \{x\}$ is frequently in $f(C)$. It implies that $f(C) \cap D_P \neq \emptyset$. Put $Q = f(C) \cap P$. Then $Q \in \mathcal{Q}_x$, and $Q \subset Q_1 \cap Q_2$.

(iii) Each $Q \in \mathcal{Q}_x$ is a sequential neighborhood of x in P.

Let $Q = f(C) \cap P$ with some $C \in \mathcal{B}_x$, and $f(C) \cap D_P \neq \emptyset$, and let S be a convergent sequence converging to x in P. Then there exists a convergent sequence L converging to a_x in M such that $f(L)$ is a subsequence of S. Since L is eventually in C, S is frequently in Q. It follows from [6, Remark 1.4] that Q is a sequential neighborhood of x in P.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (1). Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-countable σ-strong sn-network of X consisting of cosmic spaces. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_{n,\lambda} : \lambda \in \Lambda_n\} = \bigcup\{\mathcal{P}_{n,x} : x \in X\}$, where each $\mathcal{P}_{n,x}$ is an sn-cover at x in X. Since each $P_{n,\lambda}$ is a cosmic space, $P_{n,\lambda}$ is a sequentially separable space by [14, Corollary 2.6]. Then $P_{n,\lambda}$ has a countable sequentially dense subset $D_{n,\lambda}$. For each $i \in \mathbb{N}$ and $x \in P_{n,\lambda}$, put $\mathcal{Q}_{n,\lambda,i,x} = \{P \cap P_{n,\lambda} : P \in \mathcal{P}_{i,x}, P \cap D_{n,\lambda} \neq \emptyset\}$, and put $\mathcal{Q}_{n,\lambda,i} = \bigcup\{\mathcal{Q}_{n,\lambda,i,x} : x \in P_{n,\lambda}\}$, and $\mathcal{Q}_{n,\lambda} = \bigcup\{\mathcal{Q}_{n,\lambda,i} : i \in \mathbb{N}\}$. Since $\mathcal{P}$ is a point-countable and $D_{n,\lambda}$ is countable, $\mathcal{Q}_{n,\lambda}$ is a countable. It is easy to see that $\{\mathcal{Q}_{n,\lambda,i} : i \in \mathbb{N}\}$ is a refinement sequence of $P_{n,\lambda}$. For each $x \in U$ with U open in $P_{n,\lambda}$, we get $x \in V$ with V open in X and $V \cap P_{n,\lambda} = U$. Since $\mathcal{P}$ is a σ-strong network of X, there exists $i \in \mathbb{N}$ such that $x \in st(x, \mathcal{P}_i) \subset V$. Let L be a sequence in $D_{n,\lambda}$ converging to x. Since $\mathcal{P}_i$ is an sn-cover of X, $L \cup \{x\}$ is eventually in $P \subset V$ for some $P \in \mathcal{P}_{i,x}$. Then $P \cap D_{n,\lambda} \neq \emptyset$, and $P \cap P_{n,\lambda} \in \mathcal{Q}_{n,\lambda,i}$. It implies that $x \in st(x, \mathcal{Q}_{n,\lambda,i}) = st(x, \mathcal{P}_i) \cap P_{n,\lambda} \subset V \cap P_{n,\lambda} = U$. Therefore, $\{\mathcal{Q}_{n,\lambda,i} : i \in \mathbb{N}\}$ is a σ-strong network of $P_{n,\lambda}$.

For each $x \in P_{n,\lambda}$ and $i \in \mathbb{N}$, let $Q \in \mathcal{Q}_{n,\lambda,i,x}$. Then $Q = P \cap P_{n,\lambda}$, where $P \in \mathcal{P}_{i,x}$ and $P \cap D_{n,\lambda} \neq \emptyset$. Let S be a convergent sequence converging to x in $P_{n,\lambda}$. Since $\mathcal{P}_i$ is an sn-cover of X, S is eventually in P. It implies that S is eventually in $P \cap P_{n,\lambda}$. Therefore, $\mathcal{Q}_{n,\lambda}$ is a σ-strong sn-network of $P_{n,\lambda}$.

By the above, the Ponomarev-system $(f_{n,\lambda}, M_{n,\lambda}, P_{n,\lambda}, \{\mathcal{Q}_{n,\lambda,i}\})$ exists. Since each $\mathcal{Q}_{n,\lambda,i}$ is countable, $M_{n,\lambda}$ is a separable metric space with the metric $d_{n,\lambda}$ described as follows. For $a = (\alpha_i), b = (\beta_i) \in M_{n,\lambda}$, if $a = b$, then $d_{n,\lambda}(a, b) = 0$, and otherwise, $d_{n,\lambda}(a, b) = 1/\min\{i \in \mathbb{N} : \alpha_i \neq \beta_i\}$. Put $M = \oplus\{M_{n,\lambda} : \lambda \in \Lambda_n, n \in \mathbb{N}\}$ and define $f : M \longrightarrow X$ by choosing $f(a) = f_{n,\lambda}(a)$ if $a \in M_{n,\lambda}$ with $\lambda \in \Lambda_n, n \in \mathbb{N}$. Then f is a mapping, and M is a locally separable metric space with the metric d described as follows. For each $a, b \in M$, if $a, b \in M_{n,\lambda}$ for some $\lambda \in \Lambda_n$ and $n \in \mathbb{N}$, then $d(a, b) = d_{n,\lambda}(a, b)$, and otherwise, $d(a, b) = 1$. We shall prove that f is a sequence-covering π-s-mapping by the following facts (a), (b), and (c).

(a) f is a π-mapping.

Let $x \in U$ with U open in X. Then $st(x, \mathcal{P}_m) \subset U$ for some $m \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$ with $x \in P_{n,\lambda}$, we get $st(x, \mathcal{Q}_{n,\lambda,m}) \subset U_{n,\lambda}$, where $U_{n,\lambda} = U \cap P_{n,\lambda}$. For each $a = (\alpha_i) \in M_{n,\lambda}$, if $d_{n,\lambda}(f^{-1}_{n,\lambda}(x), a) < 1/m$, then there exists $b = (\beta_i) \in f^{-1}_{n,\lambda}(x)$ such that $d_{n,\lambda}(a, b) < 1/m$. Hence $\alpha_i = \beta_i$ if $i \leq m$. Since $x \in Q_{\beta_m} \subset st(x, \mathcal{Q}_{n,\lambda,m}) \subset U_{n,\lambda}$, $f_{n,\lambda}(a) \in Q_{\alpha_m} = Q_{\beta_m} \subset U_{n,\lambda}$. It implies that $a \in f^{-1}_{n,\lambda}(U_{n,\lambda})$. Therefore, if $a \in M_{n,\lambda} - f^{-1}_{n,\lambda}(U_{n,\lambda})$, then $d_{n,\lambda}(f^{-1}_{n,\lambda}(x), a) \geq 1/m$. Hence $d_{n,\lambda}(f^{-1}_{n,\lambda}(x), M_{n,\lambda} - f^{-1}_{n,\lambda}(U_{n,\lambda})) \geq 1/m$. So we get

$$
\begin{aligned}
d(f^{-1}(x), M - f^{-1}(U)) &= \inf\{d(a, b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\
&= \min\Big\{1, \inf\{d_{n,\lambda}(f^{-1}_{n,\lambda}(x), M_{n,\lambda} - f^{-1}_{n,\lambda}(U_{n,\lambda})) : \lambda \in \Lambda_n, n \in \mathbb{N}\}\Big\} \\
&\geq 1/m > 0.
\end{aligned}
$$

It implies that f is a π-mapping.

(b) f is an s-mapping.

Let $x \in X$. For each $n \in \mathbb{N}$, since $\mathscr{P}_n$ is point-countable, $\Lambda_{n,x} = \{\lambda \in \Lambda_n : x \in P_{n,\lambda}\}$ is countable. Since each $M_{n,\lambda}$ is separable metric, $f_{n,\lambda}^{-1}(x)$ is separable. It implies that $f^{-1}(x) = \bigcup\{f_{n,\lambda}^{-1}(x) : \lambda \in \Lambda_{n,x}, n \in \mathbb{N}\}$ is separable. Then f is an s-mapping.

(c) f is 1-sequence-covering.

For each $x \in X$, since $\mathscr{P}$ is a σ-strong sn-network of X, there exists $P_{n,\lambda} \in \mathscr{P}$ such that for each convergent sequence S converging to x in X, S is eventually in $P_{n,\lambda}$. Since $\mathscr{Q}_{n,\lambda}$ is a countable σ-strong sn-network of $P_{n,\lambda}$, $f_{n,\lambda}$ is an 1-sequence-covering mapping as in the proof (3) ⇒ (1) of [16, Theorem 11]. Then there exists $a_x \in M_{n,\lambda}$ such that whenever $H_{n,\lambda}$ is a convergent sequence converging to x in $P_{n,\lambda}$ there exists a convergent sequence $K_{n,\lambda}$ converging to a_x in $M_{n,\lambda}$ with $f_{n,\lambda}(K_{n,\lambda}) = H_{n,\lambda}$. Put $H_{n,\lambda} = S \cap P_{n,\lambda}$, then $H_{n,\lambda}$ is a convergent sequence converging to x in $P_{n,\lambda}$. Since $S - P_{n,\lambda}$ is finite, $S - P_{n,\lambda} = f(F)$ for some finite subset F of M. Put $K = K_{n,\lambda} \cup F$, then K is a convergent sequence converging to a_x in M satisfying $f(K) = S$. It implies that f is 1-sequence-covering.

Corollary 2.10. *The following are equivalent for a space X.*

1. *X is an 1-sequence-covering, quotient π-s-image of a locally separable metric space.*
2. *X is an 1-sequentially-quotient, quotient π-s-image of a locally separable metric space.*
3. *X has a point-countable sn-weak-development consisting of sn-second countable spaces.*
4. *X has a point-countable sn-weak-development consisting of $\aleph_0$-spaces.*
5. *X has a point-countable sn-weak-development consisting of cosmic spaces.*

Proof. (1) ⇒ (2). It is obvious.

(2) ⇒ (3). By Theorem 2.9, X has a point-countable σ-strong sn-network $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ consisting of sn-second countable spaces. We shall prove that, for every $x \in X$, $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ is a weak base at x in X by the following facts (a), (b), and (c).

(a) $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ is a network at x in X.

It follows from the fact that $\bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ is a σ-strong network of X.

(b) If $st(x, \mathscr{P}_k), st(x, \mathscr{P}_l) \in \{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$, then there exists $m \in \mathbb{N}$ such that $st(x, \mathscr{P}_m) \subset st(x, \mathscr{P}_k) \cap st(x, \mathscr{P}_l)$.

It is clear by choosing $m = \max\{k, l\}$.

(c) Since X is a quotient image of a metric space, X is sequential. It is easy to see that each $st(x, \mathscr{P}_n)$ is a sequential neighborhood of x in X. Note that $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ is a network at x in X. Then $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ is an sn-network at x in X. It implies that $\{st(x, \mathscr{P}_n) : n \in \mathbb{N}\}$ is a weak base at x by Remark 2.5.

(3) ⇒ (4) ⇒ (5). It is obvious.

(5) ⇒ (1). Let $\mathscr{P} = \bigcup\{\mathscr{P}_n : n \in \mathbb{N}\}$ be a point-countable sn-weak-development of X consisting of cosmic spaces. By Theorem 2.9, X is an 1-sequence-covering π-s-image of a locally separable metric space under the mapping f. Since X has a weak-development, X is sequential. In fact, let A be a sequentially open subset of X and $x \in A$. For each $n \in \mathbb{N}$, if $st(x, \mathscr{P}_n) \not\subset A$, then there exists $x_n \in st(x, \mathscr{P}_n) - A$. We get that $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x. Then $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is eventually in A. It is a contradiction. Therefore, $st(x, \mathscr{P}_n) \subset A$ for some $n \in \mathbb{N}$. Hence A is open, i.e., X is sequential. Since X is sequential, f is quotient by [14, Lemma 3.5]. It implies that X is an 1-sequence-covering, quotient π-s-image of a locally separable metric space.

Remark 2.11. *Corollary 2.10 is a partly answer of Question 1.1.*

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