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Some Spectral Properties of the Generalized Difference Operator Δ_v

Ali M. Akhmedov[1,*], Saad R. El-Shabrawy [2]

[1,2] *Baku State University, Faculty of Mech. & Math., Z. Khalilov Str., 23, AZ 1148, Baku, Azerbaijan*
[2] *Permanent Address: Mansoura University, Damietta Branch, Faculty of Science, Mathematics Department, New Damietta, Egypt.*

Abstract. In this paper we consider the generalized difference operator Δ_v on the sequence spaces l_1 and c_0. The operator Δ_v is represented by a lower triangular double band matrix whose nonzero entries are the elements of a sequence (v_k) with certain conditions. We mainly review several recent results concerning the fine spectrum of the operator Δ_v over the sequence spaces l_1 and c_0. Also, we provide some new results. Following that we give some illustrative examples which motivate the main results. Finally, we give notes on the fine spectrum of the operator Δ_v. These notes attempt to present some ideas about changing the conditions on the sequence (v_k) in the fine spectrum of the operator Δ_v. The new results of this paper generalize and improve some recent results that appeared recently in the literature.

Key Words and Phrases: Spectrum of an operator; Generalized difference operator; The sequence spaces l_1 and c_0.

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1. Introduction

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We summarize the knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c has been studied by Altay and Başar [9]. Akhmedov and Başar [1, 2] have studied the fine spectrum of the difference operator Δ over the sequence spaces l_p and bv_p, where $1 \le p < \infty$. Note that the sequence space bv_p was studied by Başar and Altay [12] and Akhmedov and Başar [2]. Malafosse [22]

*Corresponding author.

Email addresses: akhmedovali@rambler.ru (A. Akhmedov), srshabrawy@yahoo.com (S. El-Shabrawy)

has studied the spectrum and the fine spectrum of the difference operator Δ over the space s_r, where s_r denotes the Banach space of all sequences $x=(x_k)$ normed by

$$\|x\|_{s_r}=\sup_{k\in\mathbb{N}}\frac{|x_k|}{r^k}\quad (r>0).$$

The fine spectrum of the Zweier matrix operator Z^s over the sequence spaces l_1 and bv has been examined by Altay and Karakuş [11]. The fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c has been studied by Altay and Başar [10]. Also, the fine spectrum of the operator $B(r,s)$ over the sequence spaces l_1 and bv has been examined by Furkan et al. [18]. The fine spectrum of the operator $B(r,s)$ over the sequence spaces l_p and bv_p, where $1<p<\infty$ has been determined by Bilgiç and Furkan [13]. The fine spectrum of the operator $B(r,s,t)$ over the sequence spaces c_0 and c has been studied by Furkan et al. [16]. Also, the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_p and bv_p, where $1<p<\infty$ has been determined by Furkan et al. [17]. Panigrahi and Srivastava [23] have studied the fine spectrum of the generalized second order difference operator Δ^2_{uv} over the sequence space c_0. The fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence spaces c_0 and c has been studied by Akhmedov and El-Shabrawy [3,5]. The fine spectrum of the upper triangular double-band matrices over the sequence spaces c_0 and c has been determined by Karakaya and Altun [20].

The operator Δ_v has been introduced firstly by Srivastava and Kumar [24]. The operator $\Delta_v : (l_1\to l_1, c_0\longrightarrow c_0)$ is defined as follows:

$$\Delta_v x=\Delta_v(x_k)=(v_kx_k-v_{k-1}x_{k-1})_{k=0}^{\infty}\text{ with } x_{-1}=v_{-1}=0, \tag{1}$$

where the sequence (v_k) is assumed to be either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty}v_k=L>0\text{ and }\sup_k v_k\le 2L. \tag{2}$$

It is easy to verify that the operator Δ_v is represented by a lower triangular double band matrix of the form

$$\Delta_v=\begin{pmatrix} v_0 & 0 & 0 & \cdots\\ -v_0 & v_1 & 0 & \cdots\\ 0 & -v_1 & v_2 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}. \tag{3}$$

The fine spectrum of the generalized difference operator Δ_v over the sequence spaces c_0 and l_1 was investigated by Srivastava and Kumar [24, 25]. In [7, 8], Akhmedov and El-Shabrawy have proved by counterexamples that some of the main results in [24, 25] are incorrect and the corresponding corrected results are provided. The fine spectrum of the operator Δ_v over the sequence space c has been examined by Akhmedov and El-Shabrawy [6]. Recently, El-Shabrawy [15] has studied the fine spectrum of the operator Δ_v over the sequence space l_p, where $1<p<\infty$. Akhmedov and El-Shabrawy [4] have modified the definition of the operator Δ_v and have determined the fine spectrum of the modified operator Δ_v over the sequence spaces c and l_p, where $1 < p < \infty$.

Note that, if (v_k) is a constant sequence, say $v_k = L \neq 0$ for all $k \in \mathbb{N}$, then the operator Δ_v is reduced to the operator $B(r,s)$ with $r = L$, $s = -L$ and the results for the spectrum and the fine spectrum of the operator Δ_v on the sequence spaces l_1 and c_0 follow immediately from the corresponding results in [10, 18]. Then, throughout this paper, the case when (v_k) is a constant sequence is not considered.

The rest of the paper is organized as follows. Section 2 presents some basic concepts of spectral theory concerning the spectrum and the fine spectrum of linear operators. Next, in Section 3, we mainly review several recent results concerning the fine spectrum of operator Δ_v over the sequence spaces l_1 and c_0. Also, some new results are obtained. In Section 4 we give some illustrative examples to support the main results. In Section 5 we show some ideas about changing the conditions on the sequence (v_k) in the fine spectrum of the operator Δ_v.

2. Preliminaries

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a *sequence space*. We shall write l_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by l_1, l_p and bv_p we denote the spaces of all absolutely summable sequences, p-absolutely summable sequences and p-bounded variation sequences, respectively.

A triangle is a lower triangular matrix with all of the principal diagonal elements nonzero. Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{4}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By (λ, μ), we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda, \mu)$ if and only if the series on the right side of (4) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. We use the convention that any term with negative subscript is equal to naught.

We recall some basic concepts of spectral theory which are needed for our investigation [see 21, pp. 370-372].

Let X be a Banach space and $T : X \to X$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \left\{ y \in X : y = Tx, x \in X \right\}.$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by
$(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

If $T : l_1 \to l_1$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : l_1^* \to l_1^*$ is defined by the transpose of the matrix A. It is well-known that the dual space l_1^* of l_1 is isomorphic to l_∞. Also, if $T : c_0 \to c_0$ is a bounded linear operator with matrix A then the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose of the matrix A. The dual space c_0^* of c_0 is isomorphic to the Banach space l_1.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I, \tag{5}$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{6}$$

and call it the *resolvent operator* of T. Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. The boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects.

Definition 1. *Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value λ of T is a complex number such that*

(R1) T_λ^{-1} *exists,*

(R2) T_λ^{-1} *is bounded,*

(R3) T_λ^{-1} *is defined on a set which is dense in X.*

The *resolvent set* of T, denoted by $\rho(T,X)$, is the set of all regular values λ of T. Its complement $\sigma(T,X) = \mathbb{C} \backslash \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

Hence if $(T - \lambda I)x = \theta$ for some $x \neq \theta$, then $\lambda \in \sigma_p(T,X)$, by definition, that is, λ is an eigenvalue of T. The vector x is then called an *eigenvector* of T corresponding to the eigenvalue λ.

Now, we may give:

Lemma 1 ([19, p. 59]). *T has a dense range if and only if T^* is one to one.*

3. Recent and New Results on the Fine Spectrum of the Operator Δ_v on l_1 and c_0

In this section we mainly review several recent results concerning the fine spectrum of the operator Δ_v on the sequence spaces l_1 and c_0. Also, we provide some new results. As we mentioned before, the case when (v_k) is a constant sequence is not considered here. So, throughout this section, the sequence (v_k) is assumed to be a strictly decreasing sequence of positive real numbers satisfying the conditions (2).

3.1. The Fine Spectrum of the Operator Δ_v on l_1

Srivastava and Kumar [25] investigated the fine spectrum of the operator Δ_v on the sequence space l_1. But, incorrect results are obtained. Akhmedov and El-Shabrawy [8] have proved by a counterexample that the results concerning the point spectrum and the residual spectrum are incorrect. In this subsection we summarize the main results.

Theorem 1. *The operator $\Delta_v : l_1 \longrightarrow l_1$ is a bounded linear operator and*

(i) $\left\|\Delta_v\right\|_{l_1} = 2v_0$.

(ii) $\sigma(\Delta_v, l_1) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$.

(iii) $\sigma_p(\Delta_v, l_1) = \phi$.

(iv) $\sigma_p(\Delta_v^*, l_1^*) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$.

(v) $\sigma_r(\Delta_v, l_1) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$.

(vi) $\sigma_c(\Delta_v, l_1) = \phi$.

The results (i), (ii), (iv) and (vi) of Theorem 1 have been given by Srivastava and Kumar [25] and the results (iii) and (v) of Theorem 1 have been proved by Akhmedov and El-Shabrawy [8].

To help understanding we give the following example which disproves the statements of Srivastava and Kumar [25] concerning the point spectrum and the residual spectrum of the operator Δ_v on l_1.

Example 1. *Consider the sequence (v_k), where $v_k = \frac{k+2}{k+1}, k \in \mathbb{N}$. Clearly, (v_k) is a strictly decreasing sequence of positive real numbers satisfying the conditions* (2)*; where $\lim_{k\to\infty} v_k = L = 1$, $\sup_k v_k = v_0 = 2 \leq 2L$. We can prove that $v_0 = 2 \notin \sigma_p(\Delta_v, l_1)$. Indeed, suppose for contrary that there exists $x = (x_k) \neq \theta$ in l_1 such that $\Delta_v x = v_0 x$. Then*

$$(v_0 - v_0)x_0 = 0 \text{ and } -v_k x_k + (v_{k+1} - v_0)x_{k+1} = 0,$$

for all $k \in \mathbb{N}$. If $x_0 = 0$, then $x_k = 0$, for all $k \geq 1$, and so we have a contradiction since $x \neq \theta$. Also, if $x_0 \neq 0$ then we can easily see that

$$\left|x_k\right| \geq \left|x_0\right|, \text{ for all } k \geq 1,$$

and so we have a contradiction since $x \in l_1$. Then $v_0 \notin \sigma_p(\Delta_v, l_1)$. Similarly, we can prove that $v_k \notin \sigma_p(\Delta_v, l_1)$ for all $k \geq 1$. Thus $\sigma_p(\Delta_v, l_1) = \phi$.

Now, the operator $\Delta_v - v_0 I$ on l_1 is defined by

$$(\Delta_v - v_0 I)x = (0, -v_0 x_0 + (v_1 - v_0)x_1, -v_1 x_1 + (v_2 - v_0)x_2, \ldots), \tag{7}$$

where $x = (x_k) \in l_1$. The operator $(\Delta_v - v_0 I)^{-1}$ exists since $v_0 \notin \sigma_p(\Delta_v, l_1)$. But $(\Delta_v - v_0 I)^{-1}$ does not satisfy (R3). Indeed, consider the sequence $y = (1, 0, 0, \ldots)$ in l_1 and let y be the center of a small ball, say, of radius $1/3$. Clearly, by (7), this ball does not intersect the range of the operator $\Delta_v - v_0 I$. Then, the operator $\Delta_v - v_0 I$ does not have a dense range in l_1. Hence, by definition, $v_0 \in \sigma_r(\Delta_v, l_1)$.

3.2. The Fine Spectrum of the Operator Δ_v on c_0

Srivastava and Kumar [24] investigated the fine spectrum of the operator Δ_v on the sequence space c_0. But, incorrect results are also obtained. Akhmedov and El-Shabrawy [7] have proved by a counterexample that the results concerning the point spectrum, the residual spectrum and the continuous spectrum are incorrect. In this subsection we summarize the main results.

Theorem 2. *The operator $\Delta_v : c_0 \longrightarrow c_0$ is a bounded linear operator and*

(i) $\left\|\Delta_v\right\|_{c_0} = v_0 + v_1$.

(ii) $\sigma(\Delta_v, c_0) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$.

(iii) $\sigma_p(\Delta_v, c_0) = \varnothing$.

The bounded linearity of the operator Δ_v on c_0 has been given by the Srivastava and Kumar [24] and the norm of the operator Δ_v on c_0 has been revised by Akhmedov and El-Shabrawy [7]. Also, the result (ii) of Theorem 2 has been proved by Srivastava and Kumar [24] and the result (iii) of Theorem 2 has been proved by Akhmedov and El-Shabrawy [7].

The results concerning the point spectrum of the adjoint operator Δ_v^* of Δ_v are given by the following theorem.

Theorem 3 ([7]).

(i) $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{v_0\} \subseteq \sigma_p(\Delta_v^*, c_0^*)$,

(ii) $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \subseteq \sigma_p(\Delta_v^*, c_0^*)$,

(iii) $\sigma_p(\Delta_v^*, c_0^*) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$.

We give the following example to support the results in Theorem 3.

Example 2. *Consider the sequence* (v_k), *where* $v_k = \frac{(k+3)^2}{(k+2)^2+(k+3)^2}$, $k \in \mathbb{N}$. *We can show that* $1 \in \sigma_p(\Delta_v^*, c_0^*)$. *But* $1 \notin \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{v_0\}$ *and* $1 \notin \left\{\lambda \in \mathbb{C} : \sup_n \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$. *On the other hand if* $v_k = \frac{k+3}{2k+5}$, $k \in \mathbb{N}$, *then* $1 \in \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$. *But* $1 \notin \sigma_p(\Delta_v^*, c_0^*)$.

The following theorem gives some results on the residual spectrum of the operator Δ_v on c_0.

Theorem 4 ([7]).

(i) $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup \{v_0\} \subseteq \sigma_r(\Delta_v, c_0)$,

(ii) $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_v, c_0)$,

(iii) $\sigma_r(\Delta_v, c_0) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| < 1\right\}$.

For the continuous spectrum of the operator Δ_v on c_0, we have the following theorem.

Theorem 5 ([7]).

(i) $\sigma_c(\Delta_v, c_0) \subseteq \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \setminus \{v_0\}$,

(ii) $\{\lambda \in \mathbb{C} : |\lambda - L| \leq L\} \cap \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - v_k}{v_k}\right| \geq 1\right\} \subseteq \sigma_c(\Delta_v, c_0)$.

Also, we have the following theorem.

Theorem 6 ([7]).

(i) $\sigma_r(\Delta_v, c_0) = \sigma_p(\Delta_v^*, c_0^*)$.

(ii) $\sigma_c(\Delta_v, c_0) = \sigma(\Delta_v, c_0) \setminus \sigma_p(\Delta_v^*, c_0^*)$.

Now, we give the following new results:

Theorem 7. $\sigma_p(\Delta_v^*, c_0^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H$, *where*

$$H = \left\{\lambda \in \mathbb{C} : |\lambda - L| = L, \sum_{k=0}^{\infty} \left|\prod_{i=0}^{k} \frac{\lambda - v_i}{v_i}\right| < \infty\right\}.$$

Proof. Suppose that $\Delta_v^* f = \lambda f$ for $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong l_1$. Then, by solving the system of equations

$$\begin{gathered} v_0 f_0 - v_0 f_1 = \lambda f_0, \\ v_1 f_1 - v_1 f_2 = \lambda f_1, \\ \vdots \\ v_k f_k - v_k f_{k+1} = \lambda f_k, \\ \vdots \end{gathered}$$

we obtain

$$f_{k+1} = \frac{v_k - \lambda}{v_k} f_k, \text{ for all } k \in \mathbb{N}.$$

Then we must take $f_0 \neq 0$, since otherwise we would have $f = \theta$. It is clear that, for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, \ldots, f_k, 0, 0, \ldots)$ is an eigenvector of the operator Δ_v^* corresponding to the eigenvalue $\lambda = v_k$, where $f_0 \neq 0$ and $f_n = \frac{v_{n-1}-\lambda}{v_{n-1}} f_{n-1}$ for all $n = 1, 2, 3, \ldots, k$. Thus $\{v_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_v^*, c_0^*)$. On the other hand if $\lambda \neq v_k$ for all $k \in \mathbb{N}$, then we can see that $\sum_k |f_k| < \infty$ if $\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right| = \left|\frac{\lambda - L}{L}\right| < 1$. Also, it can be proved that $H \subseteq \sigma_p(\Delta_v^*, c_0^*)$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H \subseteq \sigma_p(\Delta_v^*, c_0^*).$$

The second inclusion can be proved analogously.

Theorem 8. $\sigma_r(\Delta_v,\, c_0) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup H.$

Proof. The proof follows immediately from Theorems 6(i) and 7.

Theorem 9. $\sigma_c(\Delta_v,\, c_0) = \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \setminus H.$

Proof. The proof follows immediately from Theorems 2(ii), 2(iii) and 8.

4. Illustrative Examples

In this section we give some illustrative examples on the fine spectrum of the operator Δ_v on the sequence spaces l_1 and c_0.

In the following example, we consider a strictly decreasing sequence (v_k) of positive real numbers satisfying the conditions (2). It will be shown that the following equalities are not hold;

$$\sigma_p(\Delta_v, l_1) = \{v_0, v_1, v_2, \ldots\}$$

and

$$\sigma_r(\Delta_v, l_1) = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\} \setminus \{v_0, v_1, v_2, \ldots\}.$$

Example 3 ([8]). *Consider the sequence (v_k), where $v_k = \frac{k+3}{2k+5}$, $k \in \mathbb{N}$. Clearly, (v_k) is a strictly decreasing sequence of positive real numbers satisfying the conditions* (2)*; where $\lim_{k\to\infty} v_k = L = 1/2$, $\sup_k v_k = 3/5 \leq 1 = 2L$. We can prove that $v_0 = 3/5 \notin \sigma_p(\Delta_v, l_1)$. Indeed, suppose for contrary that there exists $x = (x_k) \neq \theta$ in l_1 such that $\Delta_v x = v_0 x$. Then*

$$(v_0 - v_0)x_0 = 0 \textit{ and } -v_k x_k + (v_{k+1} - v_0)x_{k+1} = 0,$$

for all $k \in \mathbb{N}$. If $x_0 = 0$, then $x_k = 0$, for all $k \geq 1$, and so we have a contradiction since $x \neq \theta$. Also, if $x_0 \neq 0$ then

$$\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right| = \left|\frac{L}{L - v_o}\right| = 5 > 1,$$

and so we have a contradiction since $x \in l_1$. Then $v_0 \notin \sigma_p(\Delta_v, l_1)$. Similarly, we can prove that $v_k \notin \sigma_p(\Delta_v, l_1)$ for all $k \geq 1$, and so $\sigma_p(\Delta_v, l_1) = \phi$.

Now, the operator $\Delta_v - v_0 I$ on l_1 is defined by

$$(\Delta_v - v_0 I)x = (0, -v_0 x_0 + (v_1 - v_0)x_1,\ -v_1 x_1 + (v_2 - v_0)x_2, \ldots), \tag{8}$$

where $x = (x_k) \in l_1$. The operator $(\Delta_v - v_0 I)^{-1}$ exists since $v_0 \notin \sigma_p(\Delta_v, l_1)$. But $(\Delta_v - v_0 I)^{-1}$ does not satisfy (R3). Indeed, consider the sequence $y = (1, 0, 0, \ldots)$ in l_1 and let y be the center of a small ball, say, of radius $1/3$. Clearly, by (8), this ball does not intersect the range of the operator $\Delta_v - v_0 I$. Then, the operator $\Delta_v - v_0 I$ does not have a dense range in l_1. Hence, by definition, $v_0 \in \sigma_r(\Delta_v, l_1)$.

The following example disproves the statements of Srivastava and Kumar [24] concerning the point spectrum, the residual spectrum and the continuous spectrum of the operator Δ_v on c_0. More precisely, we consider a strictly decreasing sequence (v_k) of positive real numbers satisfying the conditions (2) and it will be shown that the following equalities are not hold;

$$\begin{aligned}
\sigma_p(\Delta_v, c_0) &= \{v_0, v_1, v_2, \ldots\}, \\
\sigma_r(\Delta_v, c_0) &= \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \setminus \{v_0, v_1, v_2, \ldots\}, \\
\sigma_p(\Delta_v^*, c_0^*) &= \{\lambda \in \mathbb{C} : |\lambda - L| < L\}, \\
\sigma_c(\Delta_v, c_0) &= \{\lambda \in \mathbb{C} : |\lambda - L| = L\} \setminus \{v_0\}.
\end{aligned}$$

Example 4 ([7])**.** *Consider the sequence (v_k), where $v_k = \frac{(k+3)^2}{(k+2)^2+(k+3)^2}$, $k \in \mathbb{N}$. The sequence (v_k) is a strictly decreasing sequence of positive real numbers satisfying the conditions (2); where $\lim\limits_{k\to\infty} v_k = L = 1/2$, $\sup\limits_k v_k = 9/13 \leq 1 = 2L$. We can prove, as in Example 3, that $v_k \notin \sigma_p(\Delta_v, c_0)$ for all $k \in \mathbb{N}$. Also, we can prove that $v_0 \in \sigma_r(\Delta_v, c_0)$.*

On the other hand, for $\lambda = 1$, we have

$$\frac{\lambda - v_k}{v_k} = \frac{1 - v_k}{v_k} = \left(\frac{k+2}{k+3}\right)^2.$$

If we suppose that $\Delta_v^* f = (1) f$ for some $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong l_1$, then we obtain that $f_k = \frac{v_{k-1}-1}{v_{k-1}} f_{k-1}$, $k \geq 1$. If we take $f_0 = 0$, then $f = \theta$ and we have a contradiction since $f \neq \theta$. If $f_0 \neq 0$, then

$$\sum_{k=0}^{\infty} \left|f_k\right| = \left|f_0\right| + \left|f_0\right| \sum_{k=1}^{\infty} \left|\frac{1 - v_0}{v_0}\right| \left|\frac{1 - v_1}{v_1}\right| \cdots \left|\frac{1 - v_{k-1}}{v_{k-1}}\right| = \left|f_0\right| + 4\left|f_0\right| \sum_{k=1}^{\infty} \left(\frac{1}{k+2}\right)^2 < \infty.$$

Then $1 \in \sigma_p(\Delta_v^*, c_0^*)$, and consequently $1 \notin \sigma_c(\Delta_v, c_0)$.

In the following example, we consider a sequence of positive real numbers (not necessarily strictly decreasing) satisfying the conditions (2) and we calculate the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the operator Δ_v on c_0.

Example 5. *Consider the sequence* (v_k), *where* $v_k = \frac{(k+2)^2}{(k+2)^2+(k+3)^2}$, $k \in \mathbb{N}$. *We can prove that the operator* $\Delta_v : c_0 \longrightarrow c_0$ *is a bounded linear operator with the norm* $\left\|\Delta_v\right\|_{c_0} = 1$ *and*

$$\sigma(\Delta_v, c_0) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\},$$
$$\sigma_p(\Delta_v, c_0) = \varnothing.$$
$$\sigma_p(\Delta_v^*, c_0^*) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\},$$
$$\sigma_r(\Delta_v, c_0) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\},$$
$$\sigma_c(\Delta_v, c_0) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}.$$

In Example 5, we see that although the sequence (v_k) is not strictly decreasing, the residual spectrum and the continuous spectrum in addition to the spectrum and the point spectrum of the operator Δ_v are completely determined. In fact, If (v_k) is assumed to be a sequence of positive real numbers (not necessarily strictly decreasing) satisfying the conditions (2), then we can have results similar to those in Section 3.2. This means that the condition that (v_k) is a strictly decreasing is not an effective condition. In the next section we modify the definition of the operator Δ_v in two ways by dropping the condition that (v_k) is strictly decreasing sequence of positive real numbers and replacing the conditions (2) by another conditions.

5. Notes on the Fine Spectrum of the Operator Δ_v on c_0 and l_1

In this section we are going to show some ideas about changing the conditions on the sequence (v_k) in the fine spectrum of the operator Δ_v. We consider two modifications of the operator Δ_v. More precisely, we modify the definition of the operator Δ_v by changing the conditions on the sequence (v_k) in two ways. First, we consider the sequence (v_k) of nonzero real numbers such that

$$\lim_{k\to\infty} v_k = L > 0 \text{ and } \sup_k v_k \leq L, \tag{9}$$

and we study the fine spectrum of the modified operator Δ_v on c_0. Second, we consider the sequence (v_k) of nonzero real numbers such that

$$\lim_{k\to\infty} v_k = L > 0,\ v_k \geq L \text{ and } v_k \neq 2L, \text{ for all } k \in \mathbb{N}, \tag{10}$$

and we study the fine spectrum of the modified operator Δ_v on l_1.

We should indicate the reader that we use the same symbol for the operator Δ_v and its modifications here, since they have the same matrix representation and the difference between them lies in the conditions on the sequence (v_k).

5.1. The fine Spectrum of the Modified Operator Δ_ν on c_0

In this subsection we calculate the fine spectrum of the modified operator Δ_ν, which is represented by the matrix in (3) such that the conditions (9) are satisfied, on the sequence space c_0. The modified operator Δ_ν of this form has been introduced and studied by Akhmedov and El-Shabrawy [4] over the sequence spaces c and l_p, where $1 < p < \infty$. The results of this section improve the corresponding results in Section 3.2.

We begin by determining when a matrix A induces a bounded linear operator from c_0 to itself.

Lemma 2 ([26, p. 129]). *The matrix $A = \left(a_{nk}\right)$ gives rise to a bounded linear operator $T \in B\left(c_0\right)$ from c_0 to itself if and only if*

(i) The rows of A are in l_1 and their l_1 norms are bounded,

(ii) The columns of A are in c_0.

The operator norm of T is the supremum of the l_1 norms of the rows.

Corollary 1. *The modified operator $\Delta_\nu : c_0 \to c_0$ is a bounded linear operator with the norm $\left\|\Delta_\nu\right\|_{c_0} = \sup\limits_k(\left|\nu_k\right| + \left|\nu_{k-1}\right|)$.*

Theorem 10. *Let $D = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\}$ and $E = \left\{\nu_k : k \in \mathbb{N}, \left|\nu_k - L\right| > L\right\}$. Then $\sigma(\Delta_\nu, c_0) = D \cup E$.*

Proof. First, we prove that $(\Delta_\nu - \lambda I)^{-1}$ exists and is in $B(c_0)$ for $\lambda \notin D \cup E$ and then the operator $\Delta_\nu - \lambda I$ is not invertible for $\lambda \in D \cup E$.
Let $\lambda \notin D \cup E$. Then, $|\lambda - L| > L$ and $\lambda \neq \nu_k$ for all $k \in \mathbb{N}$. So, $\Delta_\nu - \lambda I$ is triangle, and hence $(\Delta_\nu - \lambda I)^{-1}$ exists. We can calculate that

$$(\Delta_\nu - \lambda I)^{-1} = \begin{pmatrix} \frac{1}{(\nu_0-\lambda)} & 0 & 0 & \cdots \\ \frac{\nu_0}{(\nu_0-\lambda)(\nu_1-\lambda)} & \frac{1}{(\nu_1-\lambda)} & 0 & \cdots \\ \frac{\nu_0\nu_1}{(\nu_0-\lambda)(\nu_1-\lambda)(\nu_2-\lambda)} & \frac{\nu_1}{(\nu_1-\lambda)(\nu_2-\lambda)} & \frac{1}{(\nu_2-\lambda)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, the rows of $(\Delta_\nu - \lambda I)^{-1}$ are in l_1 and the supremum of the l_1 norms of the rows of $(\Delta_\nu - \lambda I)^{-1}$ is $\sup\limits_k S_k$, where

$$S_k = \left[\frac{1}{\left|\nu_k - \lambda\right|} + \frac{\left|\nu_{k-1}\right|}{\left|\nu_k - \lambda\right|\left|\nu_{k-1} - \lambda\right|} + \ldots. + \frac{\left|\nu_{k-1}\right|\left|\nu_{k-2}\right|\ldots\left|\nu_0\right|}{\left|\nu_k - \lambda\right|\left|\nu_{k-1} - \lambda\right|\ldots\left|\nu_0 - \lambda\right|}\right], \quad k \in \mathbb{N}.$$

Then, we can easily prove that $\sup\limits_k S_k < \infty$. Also, it is clear that the columns of $(\Delta_\nu - \lambda I)^{-1}$ are in c_0. From Lemma 2, $(\Delta_\nu - \lambda I)^{-1} \in (c_0, c_0)$. Thus $\sigma(\Delta_\nu, c_0) \subseteq D \cup E$.

Conversely, suppose that $\lambda \notin \sigma(\Delta_v, c_0)$. Then $(\Delta_v - \lambda I)^{-1} \in B(c_0)$. Since $(\Delta_v - \lambda I)^{-1}$-transform of the unit sequence $e = (1,0,0,\ldots)$ is in c_0, we have $\lim_{k\to\infty}\left|\frac{v_k}{v_{k+1}-\lambda}\right| = \left|\frac{L}{L-\lambda}\right| \leq 1$ and $\lambda \neq v_k$ for all $k \in \mathbb{N}$. Then $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \subseteq \sigma(\Delta_v, c_0)$ and $\{v_k : k \in \mathbb{N}\} \subseteq \sigma(\Delta_v, c_0)$. But $\sigma(\Delta_v, c_0)$ is compact set, and so it is closed. Then $D = \{\lambda \in \mathbb{C} : |\lambda - L| \leq L\} \subseteq \sigma(\Delta_v, c_0)$ and $E = \left\{v_k : k \in \mathbb{N}, \left|v_k - L\right| > L\right\} \subseteq \sigma(\Delta_v, c_0)$. This completes the proof.

The point spectrum of the operator Δ_v on c_0 is given by the following theorem.

Theorem 11. $\sigma_p(\Delta_v, c_0) = E$.

Proof. Suppose $\Delta_v x = \lambda x$ for $x \neq \theta = (0,0,0,\ldots)$ in c_0. Then by solving the system of equations

$$\left.\begin{aligned} v_0 x_0 &= \lambda x_0 \\ -v_0 x_0 + v_1 x_1 &= \lambda x_1 \\ -v_1 x_1 + v_2 x_2 &= \lambda x_2 \\ &\vdots \end{aligned}\right\}$$

we obtain

$$(v_0 - \lambda)x_0 = 0 \text{ and } -v_k x_k + (v_{k+1} - \lambda)x_{k+1} = 0, \text{ for all } k \in \mathbb{N}.$$

Hence, for all $\lambda \notin \{v_k : k \in \mathbb{N}\}$, we have $x_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. This shows that $\sigma_p(\Delta_v, c_0) \subseteq \{v_k : k \in \mathbb{N}\}$. Also, if $\lambda = L$, then we can easily prove that $\lambda \notin \sigma_p(\Delta_v, c_0)$. Thus $\sigma_p(\Delta_v, c_0) \subseteq \{v_k : k \in \mathbb{N}\} \setminus \{L\}$. Now, we will prove that

$$\lambda \in \sigma_p(\Delta_v, c_0) \text{ if and only if } \lambda \in E.$$

If $\lambda \in \sigma_p(\Delta_v, c_0)$, then $\lambda = v_j \neq L$ for some $j \in \mathbb{N}$ and there exists $x \in c_0$, $x \neq \theta$ such that $\Delta_v x = v_j x$. Then

$$\lim_{k\to\infty}\left|\frac{x_{k+1}}{x_k}\right| = \left|\frac{L}{L - v_j}\right| \leq 1.$$

But $\left|\frac{L}{L-v_j}\right| \neq 1$. Then $\lambda = v_j \in \left\{v_k : k \in \mathbb{N}, \left|v_k - L\right| > L\right\} = E$. Thus $\sigma_p(\Delta_v, c_0) \subseteq E$.

Conversely, let $\lambda \in E$. Then there exists $i \in \mathbb{N}$ such that $\lambda = v_i \neq L$ and so we can take $x \neq \theta$ such that $\Delta_v x = v_i x$ and

$$\lim_{k\to\infty}\left|\frac{x_{k+1}}{x_k}\right| = \left|\frac{L}{L - v_i}\right| < 1,$$

that is, $x \in c_0$. Thus $E \subseteq \sigma_p(\Delta_v, c_0)$. This completes the proof.

We give the following lemma which is required in the proof of the next theorem.

Lemma 3. *Let $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - L| = L\}$. Then the series*

$$\sum_k \left|\frac{(v_0 - \lambda)(v_1 - \lambda)\ldots(v_k - \lambda)}{v_0 v_1 \ldots v_k}\right|,$$

is not a convergent series.

Proof. Let $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ such that $|\lambda - L| = L$. Then

$$|\lambda|^2 = \lambda_1^2 + \lambda_2^2 = 2\lambda_1 L.$$

Also,

$$\begin{aligned} \left|v_k - \lambda\right|^2 &= v_k^2 + (\lambda_1^2 + \lambda_2^2) - 2\lambda_1 v_k \\ &= v_k^2 - 2\lambda_1(v_k - L) \\ &\geq v_k^2. \end{aligned}$$

Therefore

$$\left|\frac{v_k - \lambda}{v_k}\right| \geq 1, \text{ for all } k \in \mathbb{N}.$$

This completes the proof.

Theorem 12. $\sigma_p(\Delta_v^*, c_0^*) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup E$.

Proof. Suppose that $\Delta_v^* f = \lambda f$ for $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong l_1$. Then, by solving the system of equations

$$\begin{gathered} v_0 f_0 - v_0 f_1 = \lambda f_0 \\ v_1 f_1 - v_1 f_2 = \lambda f_1 \\ \vdots \\ v_k f_k - v_k f_{k+1} = \lambda f_k, \\ \vdots \end{gathered}$$

we obtain

$$f_{k+1} = \frac{v_k - \lambda}{v_k} f_k, k \in \mathbb{N}.$$

Therefore, we must take $f_0 \neq 0$, since otherwise we would have $f = \theta$.

It is clear that, for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, \ldots, f_k, 0, 0, \ldots)$ is an eigenvector of the operator Δ_v^* corresponding to the eigenvalue $\lambda = v_k$, where $f_0 \neq 0$ and $f_n = \frac{v_{n-1} - \lambda}{v_{n-1}} f_{n-1}$, for all $n = 1, 2, \ldots, k$. Thus $\{v_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_v^*, c_0^*)$. Also, if $\lambda \neq v_k$ for all $k \in \mathbb{N}$, then $f_k \neq 0$ for all $k \in \mathbb{N}$, and so, $\sum_k \left|f_k\right| < \infty$ if $\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right| = \left|\frac{\lambda - L}{L}\right| < 1$. Thus $\{\lambda \in \mathbb{C} : |\lambda - L| < L\} \cup E \subseteq \sigma_p(\Delta_v^*, c_0^*)$.

Conversely, if $\lambda \in \sigma_p(\Delta_v^*, c_0^*)$, then there exists $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong l_1$, $\Delta_v^* f = \lambda f$. Then, $f_{k+1} = \frac{v_k - \lambda}{v_k} f_k$, $k \in \mathbb{N}$ and $\sum_k \left|f_k\right| < \infty$. Therefore $\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right| = \left|\frac{\lambda - L}{L}\right| < 1$ or $\lambda \in \{v_k : k \in \mathbb{N}\}$ (note that $|\lambda - L| = L$ contradicts with $\sum_k \left|f_k\right| < \infty$, by using Lemma 3. This completes the proof.

Theorem 13. $\sigma_r(\Delta_v, c_0) = \sigma_p(\Delta_v^*, c_0^*) \backslash \sigma_p(\Delta_v, c_0)$.

Proof. The proof follows immediately from the definition of the residual spectrum and Lemma 1.

Theorem 14. $\sigma_r(\Delta_v, c_0) = \{\lambda \in \mathbb{C} : |\lambda - L| < L\}$.

Proof. The proof follows immediately from Theorems 11, 12 and 13.

Theorem 15. $\sigma_c(\Delta_v, c_0) = \sigma(\Delta_v, c_0) \backslash \sigma_p(\Delta_v^*, c_0^*)$.

Proof. The proof follows immediately from Theorems 11, 12 and 13.

Theorem 16. $\sigma_c(\Delta_v, c_0) = \{\lambda \in \mathbb{C} : |\lambda - L| = L\}$.

Proof. The proof follows immediately from Theorems 10, 12 and 15.

5.2. The Fine Spectrum of the Modified Operator Δ_v on l_1

In this subsection we calculate the fine spectrum of the operator Δ_v, which is represented by the matrix in (3) such that the conditions (10) are satisfied, on the sequence space l_1. The results of this section generalize the corresponding results in Section 3.1.

We begin by determining when a matrix A induces a bounded linear operator from l_1 to itself.

Lemma 4 ([14, p. 253, Theorem 34.16]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Corollary 2. *The operator $\Delta_v : l_1 \to l_1$ is a bounded linear operator with the norm*

$$\left\|\Delta_v\right\|_{l_1} = 2 \sup_k v_k.$$

By using arguments similar to those used in Section 5.1, we can prove the following main theorem.

Theorem 17. *(i)* $\sigma(\Delta_v, l_1) = D \cup E$.

(ii) $\sigma_p(\Delta_v, l_1) = E$.

(iii) $\sigma_p(\Delta_v^*, l_1^*) = D \cup E$.

(iv) $\sigma_r(\Delta_v, l_1) = D$.

(v) $\sigma_c(\Delta_v, l_1) = \phi$.

6. Conclusion

In this paper, the fine spectrum of the generalized difference operator Δ_v on the sequence spaces c_0 and l_1 is commented on, and some new results are obtained. Illustrative examples are given as well. These examples are used not only to apply new results but also to disprove some recent results. Finally, two modifications of the operator Δ_v are introduced. The new results of this paper generalize and improve some recent results that appeared in the literature.

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