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Probabilistic Proofs of Some Relationships Between the Bernoulli and Euler Polynomials

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Abstract. The main purpose of this article is to provide probabilistic proofs of the relationships between the generalized Bernoulli (or Nörlund) polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of (real or complex) order α and degree n in x, which were proved recently by Srivastava and Pintér [11]. Some other approaches to these relationships and their seemingly interesting generalizations are also investigated.

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1. Introduction, Definitions and Preliminaries

Throughout this paper, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \cdots\},\ \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [2, Vol.

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III, p. 253 *et seq.*], [4, Section 2.8] and [9, p. 61 *et seq.*]; see also [1], [2, Vol. I, p. 35 *et seq.*], [5], [10, p. 81 *et seq.*] and [8], and the references cited therein):

$$\left(\frac{t}{e^t-1}\right)^{\alpha}\cdot e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}\ \left(|t|<2\pi;\ 1^{\alpha}:=1\right) \tag{1}$$

and

$$\left(\frac{2}{e^t+1}\right)^{\alpha}\cdot e^{xt}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{t^n}{n!}\ \left(|t|<\pi;\ 1^{\alpha}:=1\right), \tag{2}$$

so that, obviously, the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are given, respectively, by

$$B_n(x):=B_n^{(1)}(x)\ \text{and}\ E_n(x):=E_n^{(1)}(x)\qquad \left(n\in\mathbb{N}_0\right). \tag{3}$$

For the classical Bernoulli numbers B_n and the classical Euler numbers E_n, we have

$$B_n:=B_n(0)=B_n^{(1)}(0)\ \text{and}\ E_n:=E_n(0)=E_n^{(1)}(0)\qquad \left(n\in\mathbb{N}_0\right), \tag{4}$$

respectively.

Recently, for the generalized Bernoulli (or Nörlund) polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials of order α and degree n in x, Srivastava and Pintér [11] proved the following two theorems.

Theorem 1 (see Srivastava and Pintér [11, p. 379, Theorem 1]). *The following identity holds true:*

$$B_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(\alpha)}(y)+\frac{k}{2}B_{k-1}^{(\alpha-1)}(y)\right]E_{n-k}(x) \tag{5}$$
$$(\alpha\in\mathbb{C};\ n\in\mathbb{N}_0).$$

Theorem 2 (see Srivastava and Pintér [11, p. 380, Theorem 2]). *The following identity holds true:*

$$E_n^{(\alpha)}(x+y)=\sum_{k=0}^{n}\frac{2}{k+1}\left[E_{k+1}^{(\alpha-1)}(y)-E_{k+1}^{(\alpha)}(y)\right]B_{n-k}(x) \tag{6}$$
$$(\alpha\in\mathbb{C};\ n\in\mathbb{N}_0).$$

The main objective of this sequel to the aforementioned work by Srivastava and Pintér [11] is to provide probabilistic proofs of the relationships between the Bernoulli and Euler polynomials, which are asserted by Theorems 1 and 2. Some other approaches to the Srivastava-Pintér identities and their seemingly interesting generalizations are also investigated.

2. A Set of Useful Probabilistic Tools

In this section, we recall several probabilistic tools which will be needed in Section 3 for the probabilistic proofs of Theorems 1 and 2. First of all, Sun [12] gave the following probabilistic representation of the Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli (or Norlünd) polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$. Throughout this paper, we follow the usual convention and *tacitly* assume that an empty sum and an empty product are interpreted to be 0 and 1, respectively.

Lemma 1 (see Sun [12]). *Given a sequence $\{L_n\}_{n\in\mathbb{N}}$ of independent random variables, each with the Laplace distribution $\frac{1}{2}\exp(-|x|)$ $(x \in \mathbb{R})$, define the random variable $\mathscr{L}_B$ by*

$$\mathscr{L}_B = \sum_{k=1}^{\infty} \frac{L_k}{2\pi k}. \tag{7}$$

Then each of the following probabilistic representations holds true:

$$B_n(x) = \mathbb{E}\left[\left(\imath\mathscr{L}_B + x - \frac{1}{2}\right)^n\right] \qquad (n \in \mathbb{N}_0;\ x \in \mathbb{R};\ \imath^2 = -1) \tag{8}$$

and

$$B_n^{(\alpha)}(x) = \mathbb{E}\left[\left(x + \sum_{i=1}^{\alpha}\left(\imath\mathscr{L}_B^{(i)} - \frac{1}{2}\right)\right)^n\right] \qquad (n \in \mathbb{N}_0;\ \alpha \in \mathbb{N}_0;\ x \in \mathbb{R}), \tag{9}$$

where the random variables $\left\{\mathscr{L}_B^{(i)}\right\}_{1\le i\le \alpha}$ are independent and distributed as $\mathscr{L}_B$ in (7).

Remark 1. *The symbol $\mathbb{E}_X$ denotes the expectation operator given by*

$$\mathbb{E}_X\left[g(X)\right] = \int f_X(x)\, g(x)\, dx,$$

where f_X is the probability density of the relevant random variable X. Moreover, in the absence of ambiguity, we will use the simple notation $\mathbb{E}$.

Remark 2. *The random variable $\mathscr{L}_B$, defined by* (7) *as an infinite sum of independent random variables, may seem to be difficult to use. We, therefore, propose the following characterization, which can be easily proved by looking at the characteristic function of the random variable $\mathscr{L}_B$ as defined by* (7).

Lemma 2. *The random variable $\mathscr{L}_B$ in* (7) *follows a logistic distribution with the density given by*

$$f_{\mathscr{L}_B}(x) = \frac{\pi}{2}\,\mathrm{sech}^2(\pi x) \qquad (x \in \mathbb{R}). \tag{10}$$

Sun [12] also derived the following formulas for the Euler polynomials $E_n(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$.

Lemma 3 (see Sun [12]). *If the random variable $\mathscr{L}_E$ is defined by*

$$\mathscr{L}_E = \sum_{k=1}^{\infty} \frac{L_k}{(2k-1)\pi} \tag{11}$$

where $\{L_k\}_{k\in\mathbb{N}}$ are independent Laplace random variables, then each of the following probabilistic representations holds true:

$$E_n(x) = \mathbb{E}\left[\left(\imath\mathscr{L}_E + x - \frac{1}{2}\right)^n\right] \qquad (n \in \mathbb{N}_0;\ x \in \mathbb{R}). \tag{12}$$

More generally, for $\alpha \in \mathbb{C}$,

$$E_n^{(\alpha)}(x) = \mathbb{E}\left[\left(x + \sum_{i=1}^{\alpha}\left(\imath\mathscr{L}_E^{(i)} - \frac{1}{2}\right)\right)^n\right] \qquad (n \in \mathbb{N}_0;\ \alpha \in \mathbb{C};\ x \in \mathbb{R}) \tag{13}$$

where the random variables $\left\{\mathscr{L}_E^{(i)}\right\}_{1\le i\le \alpha}$ are independent and distributed as $\mathscr{L}_E$ in (11).

Remark 3. *As in the case of the Bernoulli polynomials, a more convenient characterization of the random variable $\mathscr{L}_E$ is provided by the following lemma.*

Lemma 4. *The random variable $\mathscr{L}_E$ follows the hyperbolic secant distribution*

$$f_{\mathscr{L}_E}(x) = \operatorname{sech}(\pi x). \tag{14}$$

Lemma 5 below provides a fundamental property of each of the random variables $\mathscr{L}_B$ and $\mathscr{L}_E$.

Lemma 5. *If U_B is uniformly distributed over $[0,1]$ and independent of $\mathscr{L}_B$, then, for any entire function $\varphi(x)$ and for all $x \in \mathbb{C}$,*

$$\mathbb{E}\left[\varphi\left(x + U_B + \imath\mathscr{L}_B - \frac{1}{2}\right)\right] = \varphi(x). \tag{15}$$

Furthermore, if U_E is a Bernoulli random variable:

$$\Pr\{U_E = 0\} = \Pr\{U_E = 1\} = \frac{1}{2}$$

independent of $\mathscr{L}_E$, then

$$\mathbb{E}\left[\varphi\left(x + U_E + \imath\mathscr{L}_E - \frac{1}{2}\right)\right] = \varphi(x) \tag{16}$$

for any entire function $\varphi(x)$ and for all $x \in \mathbb{C}$.

Proof. It suffices to check, with

$$\mathscr{L} = \mathscr{L}_B \text{ or } \mathscr{L}_E \qquad \text{and} \qquad U = U_B \text{ or } U_E,$$

that

$$\mathbb{E}\left[\left(x + U + \imath\mathscr{L} - \frac{1}{2}\right)^n\right] = x^n \qquad (n \in \mathbb{N}_0;\ x \in \mathbb{C}). \tag{17}$$

The result (17) can be easily derived by using the moment generating functions of the corresponding random variables. For example, in the case of the Bernoulli polynomials, we find that

$$\mathbb{E}\left[\exp\left(zU_B\right)\right] = \frac{\exp(z) - 1}{z}$$

and

$$\mathbb{E}\left[\exp\left(z\left(\imath\mathscr{L}_B - \frac{1}{2}\right)\right)\right] = \frac{z}{\exp(z) - 1};$$

hence

$$\mathbb{E}\left[\exp\left(z\left(U_B + \imath\mathscr{L}_B - \frac{1}{2}\right)\right)\right] = 1,$$

which demonstrates the result asserted by Lemma (5).

Remark 4. *Lemma* (5) *expresses the fact that the independent random variables*

$$U_B \qquad \textit{and} \qquad \imath\mathscr{L}_B - \frac{1}{2}$$

or

$$U_E \qquad \textit{and} \qquad \imath\mathscr{L}_E - \frac{1}{2}$$

cancel each other in the sense that any non-zero moment of their sum equals 0*. We will also need a corollary of Lemma* (5) *in the following form.*

Lemma 6. *If, for all $x \in \mathbb{C}$,*

$$\mathbb{E}\left[\varphi(x + Z)\right] = \mathbb{E}\left[\psi(x + Z)\right]$$

with

$$Z = U_B,\ U_E,\ \imath\mathscr{L}_B - \frac{1}{2} \quad \textit{or} \quad \imath\mathscr{L}_E - \frac{1}{2},$$

then

$$\varphi(x) = \psi(x) \qquad (x \in \mathbb{C}).$$

Proof. If, for example,

$$\mathbb{E}\left[\varphi\left(x + U_B\right)\right] = \mathbb{E}\left[\psi\left(x + U_B\right)\right] \qquad (x \in \mathbb{C}),$$

then the result asserted by Lemma 6 follows upon setting

$$x \mapsto x + \imath\mathscr{L}_B - \frac{1}{2} \qquad (x \in \mathbb{C}).$$

Our demonstration of Lemma 6 is thus completed.

3. Probabilistic Proofs of Theorems 1 and 2

We now use the tools presented in the preceding section in order to prove the Srivastava-Pintér identities (5) and (6).

3.1. Proof of Theorem 1.

Assuming first that $\alpha \in \mathbb{N}$, let us replace the variables x and y in (5) by

$$x + U_E \qquad \text{and} \qquad y + \sum_{i=1}^{\alpha} U_B^{(i)},$$

respectively. The left-hand side of the Srivastava-Pintér identity (5) reads as follows:

$$\begin{aligned}\mathbb{E}\left[B_n^{(\alpha)}\left(x + U_E + y + \sum_{i=1}^{\alpha} U_B^{(i)}\right)\right] &= \mathbb{E}\left[\,(x+y+U_E)^n\,\right] \\ &= \frac{1}{2}(x+y+1)^n + \frac{1}{2}(x+y)^n. \end{aligned} \tag{18}$$

The same operation in the right-hand side of the Srivastava-Pintér identity (5) yields

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}\left(y + \sum_{i=1}^{\alpha} U_B^{(i)}\right)E_{n-k}\left(x+U_E\right)\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{n}\binom{n}{k}\left(y + \sum_{i=1}^{\alpha}\left(\imath\mathscr{L}_B^{(i)} - \frac{1}{2}\right) + \sum_{i=1}^{\alpha} U_B^{(i)}\right)^k x^{n-k}\right] \\ &= \sum_{k=0}^{n}\binom{n}{k} y^k x^{n-k} \\ &= (x+y)^n \end{aligned} \tag{19}$$

for the first term, and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dy}\sum_{k=0}^{n}\binom{n}{k}\mathbb{E}\left[\left(y + \sum_{i=0}^{\alpha} U_B^{(i)} + \sum_{i=1}^{\alpha-1}\left(\imath\mathscr{L}_B^{(i)} - \frac{1}{2}\right)\right)^k x^{n-k}\right] \\ &= \frac{1}{2}\frac{d}{dy}\left\{\mathbb{E}\left[\left(x+y+U_B^{(\alpha)}\right)^n\right]\right\} \\ &= \frac{1}{2}\left[(x+y+1)^n - (x+y)^n\right] \end{aligned} \tag{20}$$

for the second term. By applying the assertion of Lemma 6, the observations (19) and (20), together, conclude the proof of the Srivastava-Pintér identity (5).

3.2. Proof of Theorem 2.

In (6) we replace the variables x and y by

$$x + U_B \qquad \text{and} \qquad y + \sum_{i=1}^{\alpha} U_E^{(i)},$$

respectively. We thus obtain

$$\begin{aligned}
&\mathbb{E}\left[E_n^{(\alpha)}\left(x + U_B + y + \sum_{i=1}^{\alpha} U_E^{(i)} \right) \right] \\
&= \mathbb{E}\left[\left(x + y + U_B \right)^n \right] \\
&= \frac{1}{n+1}\left[\left(x + y + 1 \right)^{n+1} - \left(x + y \right)^{n+1} \right] \qquad (21)
\end{aligned}$$

for the left-hand side. For the right-hand side, we similarly obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(E_{k+1}^{(\alpha-1)}\left(y + \sum_{i=1}^{\alpha} U_E^{(i)} \right) - E_{k+1}^{(\alpha)}\left(y + \sum_{i=1}^{\alpha} U_E^{(i)} \right) \right) B_{n-k}\left(x + U_B \right) \right] \\
&= \left[\left(\frac{1}{2}\left(y + 1 \right)^{k+1} + \frac{1}{2} y^{k+1} \right) - y^{k+1} \right] x^{n-k} \\
&= \left(\frac{1}{2}\left(y + 1 \right)^{k+1} - \frac{1}{2} y^{k+1} \right) x^{n-k}. \qquad (22)
\end{aligned}$$

The derivative of the right-hand side sum in (22) is given by

$$\sum_{k=0}^{n} \binom{n}{k} \left(\left(y + 1 \right)^k - y^k \right) x^{n-k} = \left(x + y + 1 \right)^n - \left(x + y \right)^n,$$

which obviously coincides with the derivative of the left-hand side sum in (21). Applying the assertion of Lemma 6 once again, we are led to the Srivastava-Pintér identity (6).

4. Further Remarks and Observations

In this concluding section, we begin by presenting several further remarks and observations concerning (for example) the scope and prospects of our probabilistic and other approaches to the Srivastava-Pintér identities (5) and (6) asserted by Theorems 1 and 2, respectively.

Remark 5. *Although the probabilistic proofs of Theorems 1 and 2 were given in the preceding section only in the case when $\alpha \in \mathbb{N}_0$, yet they can be extended appropriately to any complex-valued parameter α, since the function $\alpha \mapsto B_n^{(\alpha)}(x)$ is a polynomial of degree n in α. For example, we have*

$$B_0^{(\alpha)}(x) = 1, \qquad B_1^{(\alpha)}(x) = x - \frac{\alpha}{2},$$

$$B_2^{(\alpha)}(x) = x^2 - x\alpha + \frac{\alpha}{6} + \frac{\alpha(\alpha-1)}{4},$$

and so on. Hence any identity that holds true for all $\alpha \in \mathbb{N}_0$ *extends also to the whole complex* α*-plane. Furthermore, the Bernoulli (or Nörlund) polynomials* $B_n^{(-\alpha)}(x)$ $(\alpha \in \mathbb{N}_0)$ *generated by*

$$\sum_{n=0}^{\infty} B_n^{(-\alpha)}(x)\frac{t^n}{n!} = \left(\frac{e^t-1}{t}\right)^{\alpha} \cdot e^{xt} \qquad (|t| < 2\pi;\ \alpha \in \mathbb{N}_0) \tag{23}$$

can be expressed as follows as moments:

$$B_n^{(-\alpha)}(x) = \mathbb{E}\left[\left(x + \sum_{i=1}^{\alpha} U_B^{(i)}\right)^n\right] \qquad (\alpha \in \mathbb{N}_0). \tag{24}$$

Similarly, for the Euler polynomials $E_n^{(-\alpha)}(x)$ $(\alpha \in \mathbb{N}_0)$ *generated by*

$$\sum_{n=0}^{\infty} E_n^{(-\alpha)}(x)\frac{t^n}{n!} = \left(\frac{e^t+1}{2}\right)^{\alpha} \cdot e^{xt} \qquad (|t| < \pi;\ \alpha \in \mathbb{N}_0), \tag{25}$$

we have

$$E_n^{(-\alpha)}(x) = \mathbb{E}\left[\left(x + \sum_{i=1}^{\alpha} U_E^{(i)}\right)^n\right] \qquad (\alpha \in \mathbb{N}_0). \tag{26}$$

Remark 6. *Another approach to the identities* (5) *and* (6) *for* $\alpha \in \mathbb{N}_0$ *consists in proving first their cases when* $\alpha = 0$*, namely*

$$B_n^{(0)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left[B_k^{(0)}(y) + \frac{k}{2} B_{k-1}^{(-1)}(y)\right] E_{n-k}(x) \tag{27}$$

for the identity (5).

The special identity (27) *can be checked easily, since*

$$B_n^{(0)}(x) = x^n \qquad \text{and} \qquad B_{k-1}^{(-1)}(y) = \mathbb{E}\Big((y+U_B)^{k-1}\Big)$$

so that the left-hand side of (27) *reads*

$$(x+y)^n,$$

while the right-hand side of (27) *is given by*

$$\mathbb{E}\left(\sum_{k=0}^{n} \binom{n}{k} \left[y^k + \frac{k}{2}(y+U_B)^{k-1}\right] E_{n-k}(x)\right)$$
$$= E_n(x+y) + \frac{1}{2}\left[E_n(x+y+1) - E_n(x+y)\right]$$

$$=\mathbb{E}\left(E_n\left(x+y+U_E\right)\right)=(x+y)^n.$$

Thus, upon replacing y by

$$y+\sum_{i=1}^{\alpha-1}\left(\imath\mathscr{L}_B^{(i)}-\frac{1}{2}\right),$$

we are led to the identity (5).

Remark 7. *The approach indicated in Remark 6 suggests a generalization of the identity* (5) *to the Bernoulli polynomials $B_n^{(\alpha)}(x|\mathbf{a})$ of order $\alpha\in\mathbb{N}$, degree n and parameter $\mathbf{a}\in\mathbb{R}^\alpha$ defined by the following generating function (see [2]):*

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x|\mathbf{a})\frac{t^n}{n!}=\exp(xt)\prod_{k=1}^{\alpha}\left(\frac{a_k t}{\exp(a_k t)-1}\right). \tag{28}$$

It can be easily verified that

$$B_n^{(\alpha)}(x|\mathbf{a})=\mathbb{E}\left[\left(x+\sum_{i=1}^{\alpha}a_i\left(\imath\mathscr{L}_B^{(i)}-\frac{1}{2}\right)\right)^n\right] \tag{29}$$

and that the case $\mathbf{a}=(1,\cdots,1)$ corresponds to the Bernoulli (or Nörlund) polynomials $B_n^{(\alpha)}(x)$ $(\alpha\in\mathbb{N}_0)$. The Euler case reads analogously as follows:

$$\sum_{n=0}^{\infty}E_n^{(\alpha)}(x|\mathbf{a})\frac{t^n}{n!}=\exp(xt)\prod_{k=1}^{\alpha}\left(\frac{2a_k}{\exp(a_k t)+1}\right) \tag{30}$$

and

$$E_n^{(\alpha)}(x|\mathbf{a})=\mathbb{E}\left[\left(x+\sum_{i=1}^{\alpha}a_i\left(\imath\mathscr{L}_E^{(i)}-\frac{1}{2}\right)\right)^n\right]. \tag{31}$$

Finally, we state and prove the following result.

Theorem 3. *For $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$, $\mathbf{a}\in\mathbb{C}^\alpha$ and any j $(1\leqq j\leqq\alpha)$ such that $a_j\neq 0$,*

$$B_n^{(\alpha)}(x+y|\mathbf{a})=\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(\alpha)}(y|\mathbf{a})+a_j\frac{k}{2}B_{k-1}^{(\alpha-1)}\left(y|\mathbf{a}\setminus a_j\right)\right]\cdot E_{n-k}\left(x|a_j\right) \tag{32}$$

and

$$E_n^{(\alpha)}(x+y|\mathbf{a})=\sum_{k=0}^{n}\frac{2}{k+1}\left[\frac{1}{a_j}E_{k+1}^{(\alpha-1)}\left(y|\mathbf{a}\setminus a_j\right)-\frac{1}{a_j}E_{k+1}^{(\alpha)}(y|\mathbf{a})\right]\cdot B_{n-k}^{(1)}\left(x|a_j\right), \tag{33}$$

where

$$\mathbf{a}\setminus a_j:=\left(a_1,\cdots,a_{j-1},a_{j+1},\cdots,a_\alpha\right).$$

Proof. Starting from the identity (5) with $\alpha = 1$, if we replace x and y by

$$\frac{x}{a_j} \quad \text{and} \quad \frac{y}{a_j},$$

respectively, we obtain

$$\mathbb{E}\left[\left(\frac{x}{a_j} + \frac{y}{a_j} + \left(\imath\mathscr{L}_B^{(j)} - \frac{1}{2}\right)\right)^n\right] = \mathbb{E}\left(\sum_{k=0}^{n}\binom{n}{k}\right.$$
$$\left.\cdot\left[\left(\frac{y}{a_j} + \left(\imath\mathscr{L}_B^{(j)} - \frac{1}{2}\right)\right)^k + \frac{k}{2}\left(\frac{y}{a_j}\right)^{k-1}\right]\left(\frac{x}{a_j} + \imath\mathscr{L}_E - \frac{1}{2}\right)^{n-k}\right),$$

which, when multiplied by a_j^n on both sides, yields

$$\mathbb{E}\left[\left(x + y + a_j\left(\imath\mathscr{L}_B^{(j)} - \frac{1}{2}\right)\right)^n\right] = \mathbb{E}\left(\sum_{k=0}^{n}\binom{n}{k}\right.$$
$$\left.\cdot\left[\left(y + a_j\left(\imath\mathscr{L}_B^{(j)} - \frac{1}{2}\right)\right)^k + \frac{k}{2}a_j y^{k-1}\right]\left[x + a_j\left(\imath\mathscr{L}_E - \frac{1}{2}\right)\right]^{n-k}\right). \tag{34}$$

Upon replacing y by

$$y + \sum_{i=1\ (i\neq j)}^{\alpha} a_i\left(\imath\mathscr{L}_B^{(i)} - \frac{1}{2}\right)$$

in (34), if we evaluate the resulting expectations, we get the first assertion (32) of Theorem 3. The second assertion (33) of Theorem 3 can indeed be proven similarly.

Remark 8. *The underlying principle of the approach involved in Remarks 6 and 7 (and leading to Theorem 3 above) is that any Bernoulli or Euler polynomial can be represented as a moment of a shifted monomial as (for example) in* (8) *and* (12)*. This can be related to the notion of polynomials of the binomial type which appears in the theory of operator calculus (see [6]).*

Remark 9. *Such other approaches as the umbral-calculus approach would allow an equally simple path to these proofs. In this connection, we refer the reader to the seminal paper by Rota and Taylor [7], where the notion of the cancellation properties exhibited by* (15) *and* (16) *corresponds to the notion of the inverse umbras. The paper by Gessel [3], too, provides simple derivations of several identities for the Bernoulli polynomials by using umbral calculus.*

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