



Sequentially Complete S -acts and Baer Type Criteria over Semigroups

H. Barzegar

Department of Mathematics, Tafresh University, Tafresh, Iran.

Abstract. Dedicated to Professor M. Mehdi Ebrahimi on his 65th Birthday

Although the Baer Criterion for injectivity is true for modules over a ring with an identity, it is an open problem for acts over a semigroup S (with or without identity). In this work, we study a kind of Baer Criterion for injectivity of acts over a semigroup S . We consider a kind of weak injectivity which we call s -completeness and give some conditions under which s -completeness coincides with injectivity.

2010 Mathematics Subject Classifications: 20M30, 08A60; 08B30

Key Words and Phrases: Sequentially pure, Injectivity, sequentially complete

1. Introduction

Throughout this paper S will denote a given semigroup and Recall that, for a semigroup S , a set A is a right S -act (or an S -act) if there is a, so called, action $\mu : A \times S \rightarrow A$ such that, denoting $\mu(a, s) := as$, $a(st) = (as)t$ and if S is a monoid with 1 , $a1 = a$. A morphism $f : A \rightarrow B$ between S -acts A, B is called a *homomorphism* if, for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

The category of all (right) S -acts and homomorphisms between them is denoted by **Act- S** . Sequentially complete S -acts are special objects of this category which will be studied here.

an S -act A is called *pure* in an extension B of A if any system of finitely many equations over A has a solution in A whenever this is the case for B . An S -act A is called *absolutely pure* if it is pure in all of which extensions. A monoid S is said to be *completely right pure* if all its right S -acts are absolutely pure. Completely right pure monoids have further been studied in literature [see e.g. 4, 7, 8].

A characterization of completely right pure monoids was given in [7], but clearly not satisfactory. Gould in [8] attempts to remedy this by giving a characterization of completely right pure monoids in terms of right ideals and right congruences that is a closer analogue of Proposition 2.1 of [7].

Email address: h56bar@tafreshu.ac.ir

Proposition 2.1 of [8] shows that the monoid S is completely right pure if and only if every S -act A is pure with one variable in any extension. Ebrahimi and Mahmoudi [4] has introduced the concept of s -pure monomorphism in the category of *Projection Algebras* by a system of equations such as $xs = a_s (s \in S, a_s \in A)$. So we are persuaded to study a kind of completely right purity for semigroups for these equations in general in the Category **Act-S**.

The sense of s -complete is equivalent to a kind of injectivity which is called s -injectivity and studied in [12]. Here in, we characterize the semigroups S over which all S -acts are s -complete. Also, every injective S -act is s -complete but the converse is not true in general. The Baer Criterion for injectivity (weak injectivity implies injectivity) of S -acts, which is true for modules over a ring with an identity, is an open problem for acts over a semigroup S (with or without identity).

Furthermore in this paper a weaker kind of Baer Criterion (s -completeness implies injectivity) is investigated and some semigroups over all of which every s -complete S -act is injective is introduced. These conclusions are the main part of this article, which appear in section 4.

2. Preliminary

In this section we briefly recall the definition and the categorical and algebraic ingredients of the category **Act-S** of (right) S -acts over a semigroup S and recall sequentially pure monomorphisms in this category. For more information and the notions not mentioned here about this category see, for example, [9].

Recall that an element $a \in A (t \in S)$ is said to be a *fixed element* (*left zero element*) if $as = a (ts = t)$ for all $s \in S$. The S -act $A \cup \{0\}$ with a fixed element adjoined to A is denoted by A^0 .

Since the class of S -acts is an equational class, the category **Act-S** is complete (has all products and equalizers) and cocomplete (has all coproducts and coequalizers). In fact, limits and colimits in this category are computed as in the category **Set** of sets and equipped with a natural action. In particular, for a family $\{A_i\}$ of S -acts their cartesian product $\prod A_i$ with the S -action defined by $(a_i)s = (a_i \cdot s)$ is the product of a family $\{A_i\}$ in **Act-S**. the coproduct of a family $\{A_i\}$ in **Act-S** is their disjoint union $\coprod A_i = \cup(A_i \times \{i\})$ with the action of S defined by $(a, i)s = (as, i)$ for $s \in S, a \in A_i$. Recall that for a family $\{A_i : i \in I\}$ of S -acts with a unique fixed element 0 , the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the subact of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

We use $\bigoplus_{i \in I} A_i$ only for S -acts with unique fixed element.

An S -act A is said to be *injective* if for any monomorphism $g : B \rightarrow C$ and any homomorphism $f : B \rightarrow A$ there exists a homomorphism $h : C \rightarrow A$ such that $hg = f$. An S -act A is said to be *weakly injective* if it is injective with respect to right ideals of S .

In this section we recall the notion of *sequentially pure monomorphisms* mainly from [1, 2]. For simplicity, we let the letter “ s ” stand for the prefix “sequentially”.

Definition 1. We say that A is s -pure in an extension B of A if every sequential system of equations with constants from A such as $\Sigma_A = \{xs = a_s : s \in S, a_s \in A\}$ has a solution in A whenever it has a solution in B . The system Σ_A is said to be *consistent* if it has a solution in some extension B of A .

Note that there is a one to one correspondence between the set of all systems of equations Σ_A of the above form on an S -act A and the set of all functions $k : S \rightarrow A$. For any S -act B and $b \in B$, let us denote the homomorphism $\lambda_b : S \rightarrow B$, defined by $\lambda_b(s) := bs$, by λ_b . In these notations, we have

Lemma 1 ([1]). *A map $k : S \rightarrow A$ is a homomorphism if and only if there exists an extension B of A and $b \in B$ such that $k = \lambda_b$.*

Remark 1 ([1]). *For a subact A of B , the following are equivalent:*

- (i) *A is s -pure in B .*
- (ii) *For every $b \in B$ with $bS \subseteq A$ there is an element $a \in A$ with $\lambda_b = \lambda_a$.*
- (iii) *Every homomorphism $k : S \rightarrow A$ is of the form λ_a for some $a \in A$ whenever it is of the form λ_b for some $b \in B$.*

Throughout the paper we will opt for one of the three equivalents above for s -purity. The above remark also shows that if one defines $\tilde{A} := \{b \in B : \exists a \in A, \lambda_b = \lambda_a\}$ and $\bar{A} := \{b \in B : bS \subseteq A\}$, then A is s -pure in B if and only if $\tilde{A} = \bar{A}$. For more details, see [1].

3. Sequentially Complete S -acts

In this section we study some algebraic and categorical properties of s -complete S -acts and characterize the semigroups S over which all acts are s -complete. The main result of this section is Theorem 1, which shows that s -completeness is equivalent to s -injectivity that is defined and studied in [11]. Also it is equivalent to *absolutely s -pure*. Some of the following results will be used in the next section.

Definition 2. *An S -act A is called sequentially complete or (s -complete) if every consistent system Σ_A has a solution in A .*

Theorem 1. *For an S -act A , the following are equivalent:*

- (i) *A is s -complete.*
- (ii) *A is absolutely s -pure (that is, it is s -pure in each of its extension).*
- (iii) *A is s -pure in its injective hull $E(A)$.*
- (iv) *A is s -injective (i.e, every homomorphism $k : S \rightarrow A$ is of the form λ_a for some $a \in A$).*
- (v) *Every homomorphism $f : S \rightarrow A$ can be extended to a homomorphism $\bar{f} : S^1 \rightarrow A$.*

Proof. (i) \Rightarrow (ii) Let B be an extension of A and for $b \in B$, $bS \subseteq A$. So $\Sigma_A = \{xs = bs \mid s \in S\}$ is a consistent system which has a solution a in A . Thus A is s -pure in B .

(ii) \Rightarrow (i) Let the system Σ_A has a solution in an s -pure extension B of A . Since A is absolutely s -pure, Σ_A has a solution a in A .

The equivalency of (ii), (iii) and (iv) is obtained from Theorem 2.2 of [10].

The equivalency of (iv) and (v) is obtained from [11], Theorem 2.7.

From here to the end of paper we use some parts of Theorem 1 for s -completeness.

Remark 2. Having parts (iv) and (v) of the above theorem, one can easily get some relations between s -completeness and injectivity or any types of weak injectivity [see 9]. For example, we have the following:

Every injective S -act is s -complete.

But the converse is not in generally true, indeed, the monoid $S = \mathbb{Z}$ with the usual multiplication, is s -complete as an S -act but it is not divisible. So it is not injective S -act.

Also weak injectivity does not imply s -completeness. To show this fact, consider $S = (\mathbb{N}, \min)$. Since identity homomorphism on \mathbb{N} is not of the form λ_a , then $\mathbb{N}_{\mathbb{N}}$ is not s -complete. Now let I be a right ideal of S and $f : I \rightarrow \mathbb{N}$ be a homomorphism. We want to extend f to S . The case $I = S$ is obvious. Otherwise, I is of the form $mS = \{ms \mid s \in S\}$. The homomorphism $g : \mathbb{N} \rightarrow \mathbb{N}$ defined by $g(n) = f(mn)$ is an extension of f . Thus \mathbb{N} is a weakly injective as an S -act.

Theorem 2. An S -act A is s -complete and weakly injective if and only if for every right ideal I of S , every homomorphism $f : I \rightarrow A$ is of the form λ_a for some $a \in A$.

Proof. By using Theorem 1 the Sufficiency is clear. We show only the Necessity. Let A be an s -complete and weakly injective S -act. Take a homomorphism $f : I \rightarrow A$ from a right ideal I of S . Since A is weakly injective, f can be extended to a homomorphism $\bar{f} : S \rightarrow A$. Now, since A is s -complete, $\bar{f} = \lambda_a$ for some $a \in A$, and hence so is f .

Lemma 2. If A is s -pure in B and B is s -complete, then A is s -complete.

Proof. Let $f : S \rightarrow A$ be a homomorphism. Since B is s -complete, f is of the form λ_b for some $b \in B$ and since A is s -pure in B , it is of the form λ_a for some $a \in A$.

As a result of Lemma 2, we have the following corollary. But first recall that a subact A of B is a retract of B if there exists a homomorphism, so called retraction, $g : B \rightarrow A$ such that $g|_A = id_A$.

Corollary 1. A retract of an s -complete S -act is an s -complete S -act.

Proof. [By [1, Lemma 2.4]] If an S -act A is a retract of B then it is s -pure in B . Now we are done by applying Lemma 2.

In the next theorem, we mention a characterization of semigroup S over which all acts are s -complete.

Definition 3. An S -act A is said to be principally s -complete if A is s -pure in each of its cyclic extension.

Theorem 3. For a semigroup S the following are equivalent:

- (i) All right S -acts are s -complete.
- (ii) All right S -acts are principally s -complete.
- (iii) S is an s -complete S -act.
- (iv) S is s -pure in S^1 .
- (v) S has a left identity element.
- (vi) pullbacks preserve s -pure monomorphisms.

Proof. The implications (i) \implies (ii) \implies (iv), (i) \implies (iii) and (iii) \implies (iv) are obtained by using Theorem 1. By [1, Theorem 3.1], (iv) and (v) are equivalent.

(v) \implies (i) For every S -act A every homomorphism $k : S \rightarrow A$ is of the form $k = \lambda_{k(1)}$. So by Theorem 1.(iv) the result is true.

(v) \iff (vi) Apply [1, Lemma 3.3].

The following lemma will be used in Corollary 2 and in Section 4.

Lemma 3. Let $A_i (i \in I)$ be a family of S -acts and S be a finitely generated as an S -act. Then $\bigoplus A_i$ is s -pure in $\prod A_i$.

Proof. Consider $S \xrightarrow{k} \bigoplus A_i \hookrightarrow \prod A_i$ such that $k = \lambda_{\{a_i\}}(\{a_i\} \in \prod A_i)$, and $S = \bigcup_{i=1}^n t_i S^1$. So there exists a finite subset $J \subset I$ such that for every $s \in S$, $k(s)_i = 0 (i \notin J)$. Thus $k = \lambda_{\{b_i\}}$ for

$$b_i = \begin{cases} a_i & \text{for } i \in J \\ 0 & \text{for } i \notin J \end{cases}.$$

Now Theorem 1.(iv) completes the proof.

Theorem 4. For a semigroup S , the following statements are equivalent:

- (i) Every direct sum of s -complete S -acts is s -complete.
- (ii) Every direct sum of s -complete S -acts is s -pure in their direct.

Proof. (i) \implies (ii) Let $\{A_i\}$ be a family of s -complete S -acts. Then $\bigoplus A_i$ is s -complete and by Theorem 1 it is s -pure in $\prod A_i$.

(ii) \implies (i) This implication is obtained by applying [11, Theorem 3.1], Theorem 1 and Lemma 2.

Corollary 2. If the semigroup S is a finitely generated as an S -act, then every direct sum of s -complete S -acts is s -complete.

Lemma 4. Let $g : S \rightarrow T$ be an epimorphism. If right T -act A be s -complete as an S -act, then it is s -complete as a T -act.

Proof. Let $f : T \rightarrow A$ be a T-homomorphism. Since A is s -complete as an S -act, then $f g = \lambda_a$ for some $a \in A$ which implies that for every $t \in T$,
 $f(t) = f(g(s_t)) = a_s s_t = a_T g(s_t) = a_t t = \lambda_a(t)$.

Remark 3. As we saw in Theorem 1, s -completeness is equivalent to s -injectivity which defined in [11]. Some categorical properties such as product, coproduct and direct sum of s -injective S -acts were checked in [11].

4. Some Baer Type Criteria for Injectivity of S -Acts

Although the Baer Criterion for injectivity (weak injectivity implies injectivity) is true for modules over a ring (with an identity), it is an open problem for acts over a semigroup S (with or without identity). In fact, we are not aware of any type of weak injectivity implying injectivity of S -acts, in general, other than Skornjakov-Baer Criterion, which says that injectivity with respect to subacts of cyclic acts implies injectivity with respect to all monomorphisms.

One line of study in this regard is to investigate the relation between \mathcal{M}_1 -injectivity and injectivity with respect to another subclass \mathcal{M}_2 of monomorphisms, the results of which may be called the **Baer type criteria**. Note that if $\mathcal{M}_2 \subseteq \mathcal{M}_1$, then \mathcal{M}_1 -injectivity implies \mathcal{M}_2 -injectivity. The Baer type problem is about the converse of this fact.

By using Theorem 1 in this paper in fact we use injectivity only with respect to a monomorphism $S \rightarrow S^1$ which is s -completeness.

As we saw in Remark 2, every injective S -acts is s -complete but the converse is not true in general. In this final section, we use s -completeness to give some Baer type results about injectivity of S -acts. We introduce some semigroups over all of which every s -complete S -acts is injective.

First recall the following definition from the closure operator given in [6].

Definition 4. For a subact A of B , let $\bar{A} := \{b \in B : bS \subseteq A\}$. Then, A is said to be s -dense in B if $\bar{A} = B$.

The following definition is a well known definition in the literature as we use here.

Definition 5. For a subclass \mathcal{M} of monomorphisms we say that the \mathcal{M} -morphism $f : A \rightarrow B$ is \mathcal{M} -essential if every $g : B \rightarrow C$ is a monomorphism whenever gf is a monomorphism. For simplicity, we say essential when \mathcal{M} is the class of all monomorphism.

We have the following result from [3].

Theorem 5. An S -act A is injective if and only if it has no proper essential extension.

Henceforth in this section, we have some conditions for which every s -complete S -act is injective. But first recall the following lemma:

Lemma 5. [[1]] Any s -dense, s -pure monomorphism has a retraction.

Theorem 6. *If every essential extension is s -dense, then every nonempty right ideal of S has a left zero element. In particular, S has a left zero element and every S -act A has a fixed element.*

Proof. Let I be a nonempty right ideal of S that does not have a left zero element. By [2], Lemma 4.1, I^0 is an essential extension and hence s -dense extension of I . Thus for every $s \in S$, $0 = 0s \in I$ which is a contradiction.

Similar to the proof of [2, Proposition 3.6(ii) and Lemma 3.9] one gets:

Lemma 6. *Let A be a subact of B :*

- (i) *If C be a subact of B that $|C| \geq 2$ and $|C \cap A| \leq 1$, then B is not an essential extension of A .*
- (ii) *If A and $B \setminus A$ have fixed element, then B is not an essential extension of A .*

For a subact A of an S -act B and $b \in B$, we use the notation $I_b = \{s \in S \mid bs \in A\}$. Also the set of all fixed elements of an S -act B and the set of all left zero elements of a semigroup S are denoted respectively by $Fix(B)$ and $Z(S)$.

Corollary 3. *Let A have at least one fixed element and B be an s -pure essential (essential) extension of A . Then:*

- (i) $Fix(B) \subseteq A$.
- (ii) *For every $b \in B$, $I_b \neq \emptyset$.*

Corollary 4. *If S has a left zero element and S is an essential extension of a right ideal I , then $Z(S) \subseteq I$. If S is a left zero semigroup, then $I = S$.*

Theorem 7. *If every essential extension is s -dense, then every S -act has an s -dense injective (which is injective with respect to s -dense monomorphisms) s -dense extension.*

Proof. Let $A \in \mathbf{Act-S}$ and $\iota : A \rightarrow E(A)$ be an injective hull of A (which exists as proved in [3]). So ι is s -dense and $E(A)$ is an s -dense injective s -dense essential extension of A .

Lemma 7. *Let A have at least one fixed element and B be an s -pure essential extension of A . Then*

- (i) $\bar{A} = A$.
- (ii) *For each $b \in B \setminus A$, $\emptyset \neq I_b \neq S$*

Proof. (i) By [2, Lemma 4.7.(v)], A is s -pure essential in \bar{A} . Since A is s -dense in \bar{A} , by Lemma 5, A is a retract of \bar{A} which is an isomorphism by essentiality. Thus $A = \bar{A}$.

(ii) By part (i), $\bar{A} = A$, so $I_b \neq S$ and by Corollary 3, $\emptyset \neq I_b$.

Theorem 8. *If A is s -complete S -act and every s -pure essential extension of A is s -dense, then A is injective.*

Proof. Let $E(A)$ be an injective hull of A . Since A is an s -complete and $E(A)$ is an essential extension of A , then $E(A)$ is an s -pure essential extension of A . Thus A is a retract of $E(A)$ by Lemma 5, which implies A is an injective S -act.

Theorem 9. *If $S^2 = S$, then the following are equivalent:*

- (i) *Every s -complete S -act is injective.*
- (ii) *Every essential extension is s -dense.*
- (iii) *Every s -pure essential extension is s -dense.*
- (iv) *Every s -pure essential extension is isomorphism.*

Proof. (i) \Rightarrow (ii) Let B be an essential extension of A . By [5, Theorem 3.10], A has an s -dense injective hull such as $\iota : A \rightarrow E_d(A)$. By Theorem 1, $E_d(A)$ is s -complete and hence it is injective. So there exists $g : B \rightarrow E_d(A)$ such that $g|_A = \iota$ which implies g is a monomorphism. Since $g|_A = \iota$ is an s -dense monomorphism, it is clear that B is an s -dense extension of A .

(iii) \Rightarrow (iv) Let B be an s -pure essential extension of A . Then A is s -dense in B and by Lemma 5, it is a retract of B . So there is a homomorphism $g : B \rightarrow A$ such that $g|_A$ is a monomorphism, which implies g is an isomorphism.

(iv) \Rightarrow (i) It is concluded from Theorem 8.

Theorem 10. *If for every nontrivial right ideal I of S , $\bar{I} \neq I$, then every s -complete S -act with at least one fixed element is injective.*

Proof. Let B be an s -pure essential extension of an s -complete S -act A and $b \in B \setminus A$. By Lemma 7, $\emptyset \neq I_b \neq S$. By hypothesis $\bar{I}_b \neq I_b$ and there exists $x \in \bar{I}_b \setminus I_b$. Since $A \rightarrow B$ is an s -pure essential extension, the inclusion map $\iota : A \rightarrow A \cup \{bx\}$ is also s -pure essential and s -dense which is a retraction by Lemma 5. Essentiality of ι implies that it is an isomorphism. So $bx \in A$ and $x \in I_b$ which is a contradiction. Thus $A = B$ and by Theorem 8, A is an injective S -act.

Corollary 5. *If S is an infinite monogenic semigroup, then every s -complete S -act with at least one fixed element is injective.*

Proof. First recall that every infinite cyclic semigroup is isomorphic to $(\mathbb{N}, +)$ [see 9]. Let I be a nontrivial right ideal of S . So there exists $1 \leq n_0 \in \mathbb{N}$ such that $I = \{n \in \mathbb{N} \mid n_0 \leq n\}$ and $S \setminus I = \{1, 2, \dots, n_0 - 1\}$. Thus $\bar{I} = I \cup \{n_0 - 1\}$. Now we get the result by using Theorem 10.

Corollary 6. *If S is a finite monogenic semigroup, then every s -complete S -act with at least one fixed element is injective.*

Proof. The proof is similar to the proof of Theorem 5.

Corollary 7. *Let S be a semigroup and $s_0 \in S$, such that for all $s, t \in S$, $st = s_0$. Then every s -complete S -act is injective.*

Proof: It is clear that every nonempty right ideal of S is a subset of S containing an element s_0 and for every nonempty proper right ideal I of S , $I \neq \bar{I} = S$. So the proof is complete by Theorem 10.

Theorem 11. *If every nonempty proper right ideal of S generates by a central idempotent element, then every s -complete S -act with at least one fixed element is injective.*

Proof: Let A be an s -complete S -act with at least one fixed element and B be an essential extension of A . By using Theorem 1 and Lemma 7, for every $b \in B \setminus A$, $\emptyset \neq I_b \neq S$. So $I_b = e_b S^1$ such that e_b is a central idempotent element. Consider the map $g : B \rightarrow A$ defined by

$$g(b) = \begin{cases} b & , \text{ if } b \in A \\ be_b & , \text{ if } b \notin A \end{cases}$$

Let $b \in B$ and $s \in S$. If $b \in A$, $g(bs) = bs = g(b)s$. If $b \notin A$, $g(b)s = be_b s$. Two cases may happen:

Case (1): $s \in I_b$. So $s = e_b s_1 = e_b(e_b s_1) = e_b s$ and $g(bs) = bs = b(e_b s) = g(b)s$.

Case (2): $s \notin I_b$. Then $bs \notin A$ and $I_{bs} = e_{bs} S^1$.

Since $(bs)e_b = (be_b)s \in A$, $e_b \in I_{bs}$ and hence $I_b \subseteq I_{bs}$. So $e_b s = e_{bs} t$ for some $t \in S$. On the other hand $se_{bs} \in I_b$ implies $se_{bs} = e_b t_1 (t_1 \in S) = e_b(e_b t_1) = e_b(se_{bs})$. Thus

$$g(bs) = (bs)e_{bs} = be_b(se_{bs}) = b(e_b s)e_{bs} = b(e_{bs} t)e_{bs} = be_{bs}(e_{bs} t) = be_{bs} t = be_b s = g(b)s.$$

Therefore g is a homomorphism and since B is an essential extension of A , g is an isomorphism. So we get the result by Theorem 9.

As usual S is a Clifford semigroup [see 9] if each $\alpha \in S$ has an inverse element (i.e, there exists $\alpha^{-1} \in S$ such that $\alpha\alpha^{-1}\alpha = \alpha^{-1}\alpha\alpha^{-1}$) and the set of all idempotents is equal to the set of all central elements.

It is easy to check that for every $\alpha \in S$, $\alpha\alpha^{-1}$ is an idempotent element and $\alpha S^1 = \alpha\alpha^{-1} S^1$. So we conclude the following corollary;

Corollary 8. *Let S be a Clifford semigroup. If each of proper nonempty right ideals of S is principal, then every s -complete S -act with at least one fixed element is injective.*

The following two corollaries are special cases of clifford semigroup.

Corollary 9. *Let the semigroup S be a commutative chain with the relation $(x \leq y \Leftrightarrow xy = x)$ or $(x \leq y \Leftrightarrow xy = y)$. Then every s -complete S -act with at least one fixed element is injective.*

Corollary 10. *Let the semigroup S be a commutative band. Then every s -complete S -act with at least one fixed element is injective.*

An important semigroup satisfying to these corollaries is $S = (\mathbb{N}, \min)$. The category of all S -acts for this semigroup, so called *projection algebras*, has been studied by Ebrahimi and Mahmoud [4] and Giuli [6]. These types of S -acts are mostly used in computer science.

Lemma 8. *Let A have a fixed element and B be a proper s -pure essential extension of A . Then for every $b \in B$ and every nonempty right ideal J of S , $I_b \cap J \neq \emptyset$.*

Proof. For $b \in A$, $I_b = S$ and the result is obvious. For $b \notin A$, let J be a nonempty right ideal of S and $I_b \cap J = \emptyset$. By Corollary 3, $I_b \neq \emptyset$ and $\text{Fix}(B) \subseteq A$. It is clear that $B' = \{bs | s \in J\}$ is a subact of B and $B' \cap A = \emptyset$. By Lemma 6, $|B'| = 1$ and hence for every $s \in J$, $bs = b_0$ for some $b_0 \in B$. Consider $s_0 \in J$. Then for every $t \in S$, $b_0t = (bs_0)t = b(s_0t) = b_0$. So $b_0 \in \text{Fix}(B) \subseteq A$, which is impossible.

Theorem 12. *Suppose that every nonempty nontrivial right ideal I of S is maximal. Then every s -complete S -act with at least one fixed element is injective.*

Proof. Let A be an s -complete S -act and B be an s -pure essential extension of A . Let there exist an element $b \in B \setminus A$. By using Lemma 7, $\emptyset \neq I_b \neq S$. Consider $t \in S \setminus I_b$, so $I_b \cap tS^1 \neq \emptyset$. If $tS^1 \neq S$, $I_b = I_b \cap tS^1 = tS^1$. Thus $t \in I_b$, which is a contradiction. If $tS^1 = S$, then $I_b \subseteq tS^1$ and since I_b is maximal right ideal, $I_b = tS$. Since $A \rightarrow B$ is an s -pure essential extension, the inclusion map $\iota : A \rightarrow A \cup \{bt\}$ is also s -pure essential and s -dense which is a retraction by Lemma 5. Essentiality of ι implies that it is an isomorphism. So $bt \in A$ and $t \in I_b$ which is a contradiction. Thus $A = B$ and by Theorem 8, A is an injective S -act.

Corollary 11. *If S is a simple semigroup, then every s -complete S -act with at least one fixed element is injective.*

Corollary 12. *Let S be a semigroup with one zero element s_0 , such that the set of whose ideals be $\{\emptyset, \{s_0\}, S\}$. Then every s -complete S -act is injective.*

Theorem 13. *Assume that for every proper nonempty right ideal I of S there exists a nonempty right ideal J of S such that $I \cap J = \emptyset$. Then every s -complete S -act with at least one fixed element is injective.*

Proof. we begin by proving $S^2 = S$. Let $S^2 \neq S$. Then there exists a right ideal j of S such that $S^2 \cap j = \emptyset$. Consider $x \in j$. for every $s \in S$, $xs \in j \cap S^2$, which is impossible. Now the proof is straightforward by using Theorem 9 and Lemma 8.

Corollary 13. *Let S be a Boolean Algebra on ideals (i.e compliment of every right ideal is a right ideal). Then every s -complete S -act with at least one fixed element is injective.*

Corollary 14. *If S is a left zero semigroup, then every s -complete S -act is injective.*

There is still an open question concerning s -complete:

Is there a necessary and sufficient condition on S such that all s -complete S -acts A with at least one fixed element is injective?

ACKNOWLEDGEMENTS I would like to express my appreciation to the referee for carefully reading the paper.

References

- [1] H. Barzegar and M.M. Ebrahimi. Sequentially pure monomorphism of acts over semigroups. *European journal of pure and applied mathematics*, 1(4):41–55, 2008.
- [2] H. Barzegar, M.M. Ebrahimi, and M. Mahmoudi. Essentiality and injectivity relative to sequential purity of acts. *Semigroup Forum*, 79:128–144, 2009.
- [3] P. Berthiaume. The injective envelope of s -sets. *Canad. Math. Bull.*, 10(2):261–273, 1967.
- [4] M.M. Ebrahimi and M. Mahmoudi. Purity and equational compactness of projection algebras. *Applied Categorical Structure*, 9:381–394, 2001.
- [5] M.M. Ebrahimi, M. Mahmoudi, and L. shahbaz. Proper behaviour of sequential injectivity of acts over semigroups. *Communication in Algebra*, 37(7):2511–2521, 2009.
- [6] E. Giuli. On m -separated projection spaces. *Applied Categorical Structure*, 2:91–99, 1994.
- [7] V. Gould. The characterisation of monoids by properties of their s -systems. *Semigroup Forum*, 32(3):251–265, 1985.
- [8] V. Gould. Completely right pure monoids. *Proc.Roy. Irish Acad, Section A*, 87:73–82, 1987.
- [9] M. Kilp, U. Knauer, and A. Mikhalev. *Monoids, Acts and Categories*. Walter de Gruyter, Berlin, New York, 2000.
- [10] M. Mahmoudi and Gh. Moghaddasi. Sequential purity and injectivity of acts over some classes of semigroups. *Taiwanese journal of mathematics*, 15(2):733–744, 2011.
- [11] M. Mahmoudi and L. Shahbaz. Characterizing semigroups by sequentially dense injective acts. *Semigroup Forum*, 75(1):116–128, 2007.
- [12] M. Mahmoudi and L. Shahbaz. Sequential dense essential monomorphisms of acts over semigroups. *Applied Categorical Structure*, 18:461–471, 2010.