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On Köthe-Toeplitz and Null Duals of Some Difference Sequence Spaces Defined by Orlicz Functions

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Abstract. The main aim of this paper is to compute Köthe-Toeplitz and Null duals of some difference sequence spaces, defined by means of a fixed sequence of multiplier and by an Orlicz function. Further the coincidence for three pairs of analogous spaces is established.

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1. Introduction and Preliminaries

Throughout this section w, ℓ_∞, ℓ_1, c and c_0 denote the spaces of all, $bounded$, $absolutely$ $summable$, $convergent$ and $null$ sequences $x = (x_k)$ with complex terms respectively.

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An Orlicz function is a function $M : [0,\infty) \longrightarrow [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that

$$M(2u) \leq KM(u)\ (u \geq 0).$$

The above Δ_2-condition implies $M(lu) \leq Kl^{log_2 K} M(u)$, for all $u > 0$, $l > 1$.

For details on integral representation of Orlicz function as well as on complementary Orlicz functions one may refer to [7, 12].

For an Orlicz function M, we have the following inequality:

$$M(\lambda x) < \lambda M(x), \text{ for all } x \geq 0 \text{ and } \lambda \text{ with } 0 < \lambda < 1.$$

Lindenstrauss and Tzafriri [9] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \{(x_k) \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{ for some } \rho > 0\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}.$$

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for E a sequence space, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [4] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and (k) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction.

The notion of difference sequence space was introduced by Kizmaz [6], who studied the difference sequence spaces $Z(\Delta)$, for $Z = \ell_\infty, c, c_0$ and defined as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in N$.

In this paper our aim is to investigate some important structures of some spaces which are defined using an Orlicz function and a multiplier sequence. These spaces generalize the spaces $Z(\Delta)$, for $Z = \ell_\infty, c, c_0$ introduced and studied by Kizmaz [6].

Let $\Lambda = (\lambda_k)$ be a non-zero sequence of scalars. Then we define the following sequence spaces for an Orlicz function M:

$$c_0(M, \Lambda, \Delta) = \{x = (x_k) : \lim_k M(\frac{|\Delta \lambda_k x_k|}{\rho}) = 0, \text{ for some } \rho > 0\},$$
$$c(M, \Lambda, \Delta) = \{x = (x_k) : \lim_k M(\frac{|\Delta \lambda_k x_k - L|}{\rho}) = 0, \text{ for some } L \text{ and } \rho > 0\},$$
$$\ell_\infty(M, \Lambda, \Delta) = \{x = (x_k) : \sup_k M(\frac{|\Delta \lambda_k x_k|}{\rho}) < \infty, \text{ for some } \rho > 0\},$$

where $\Delta \lambda_k x_k = \lambda_k x_k - \lambda_{k+1} x_{k+1}$, for all $k \in N$.

It is obvious that $c_0(M, \Lambda, \Delta) \subset c(M, \Lambda, \Delta) \subset \ell_\infty(M, \Lambda, \Delta)$.

Throughout the paper X will denote one of the sequence spaces c_0, c and ℓ_∞. The sequence spaces $X(M, \Lambda, \Delta)$ are Banach spaces normed by

$$\|x\|_\Delta = |\lambda_1 x_1| + \inf\{\rho > 0 : \sup_k M(\frac{|\Delta \lambda_k x_k|}{\rho}) \le 1\}.$$

Now we shall write $\Delta^{-1} x_k = x_k - x_{k-1}$, for all $k \in N$. It is trivial that $(\Delta \lambda_k x_k) \in X(M)$

if and only if $(\Delta^{-1}\lambda_k x_k) \in X(M)$. Now for $x \in X(M,\Lambda,\Delta^{-1})$, we define

$$\|x\|_{\Delta^{-1}} = \inf\{\rho > 0 : \sup_k M(\frac{|\Delta^{-1}\lambda_k x_k|}{\rho}) \le 1\}.$$

It can be shown that $X(M,\Lambda,\Delta)$ is a BK-space under the norms $\|.\|_\Delta$ and $\|.\|_{\Delta^{-1}}$ respectively and it is obvious that the norms $\|.\|_\Delta$ and $\|.\|_{\Delta^{-1}}$ are equivalent.

Obviously $\Delta^{-1} : X(M,\Lambda,\Delta^{-1}) \longrightarrow X(M)$, defined by $\Delta^{-1}x = y = (\Delta^{-1}\lambda_k x_k)$, is isometric isomorphism.

Hence $c_0(M,\Lambda,\Delta^{-1})$, $c(M,\Lambda,\Delta^{-1})$ and $\ell_\infty(M,\Lambda,\Delta^{-1})$ are isometrically isomorphic to $c_0(M)$, $c(M)$ and $\ell_\infty(M)$ respectively. From abstract point of view $X(M,\Lambda,\Delta^{-1})$ is identical with $X(M)$, for $X = c_0, c$ and ℓ_∞.

The results obtained in the next section also hold for the spaces $c_0(M,\Lambda,\Delta^{-1})$, $c(M,\Lambda,\Delta^{-1})$ and $\ell_\infty(M,\Lambda,\Delta^{-1})$ as well as for the spaces associated with these three spaces.

Now we define the spaces $\tilde{c}_0(M,\Lambda,\Delta)$, $\tilde{c}(M,\Lambda,\Delta)$ and $\tilde{\ell_\infty}(M,\Lambda,\Delta)$ as follows:

$\tilde{c}_0(M,\Lambda,\Delta)$ is a subspace of $c_0(M,\Lambda,\Delta)$ consisting of those $x \in c_0(M,\Lambda,\Delta)$ such that

$$\lim_k M(\frac{|\Delta\lambda_k x_k|}{d}) = 0 \ for \ each \ d > 0.$$

Similarly we can define $\tilde{c}(M,\Lambda,\Delta)$ and $\tilde{\ell_\infty}(M,\Lambda,\Delta)$ as subspace of $c(M,\Lambda,\Delta)$ and $\ell_\infty(M,\Lambda,\Delta)$ respectively.

It is obvious that $\tilde{c}(M,\Lambda,\Delta) \subset \tilde{c}(M,\Lambda,\Delta) \subset \tilde{\ell_\infty}(M,\Lambda,\Delta)$. Also as above we can show that $\tilde{c}_0(M,\Lambda,\Delta)$, $\tilde{c}(M,\Lambda,\Delta)$ and $\tilde{\ell_\infty}(M,\Lambda,\Delta)$ are isometrically isomorphic to $\tilde{c}_0(M)$, $\tilde{c}(M)$ and $\tilde{\ell_\infty}(M)$ respectively.

Moreover $X(M,\Lambda) \subset X(M,\Lambda,\Delta)$ and $\tilde{X}(M,\Lambda) \subset \tilde{X}(M,\Lambda,\Delta)$ which can be shown by using the following inequality:

$$M(\frac{|\Delta\lambda_k x_k|}{2\rho}) \le \frac{1}{2}M(\frac{|\lambda_k x_k|}{\rho}) + \frac{1}{2}M(\frac{|\lambda_{k+1} x_{k+1}|}{\rho}).$$

2. Köthe-Toeplitz and Null Dual Spaces

In this section we compute Köthe-Toeplitz or α-dual and Null or N- dual of some difference sequence spaces as described in the preceding section.

Let E and F be two sequence spaces. Then the F dual of E is defined as

$$E^F = \{(x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E\}.$$

For $F = \ell_1$ and c_0, the duals are termed as α-(or Köthe-Toeplitz) dual and N-(or Null) dual of E and denoted by E^α and E^N respectively. If $X \subset Y$, then $Y^z \subset X^z$ for $z = \alpha, N$.

Lemma 1. *$x \in \ell_\infty(M, \Lambda, \Delta)$ implies $\sup_k M(\frac{|k^{-1}\lambda_k x_k|}{\rho}) < \infty$, for some $\rho > 0$.*

Proof. Let $x \in \ell_\infty(M, \Lambda, \Delta)$, then

$$\sup_k M(\frac{|\lambda_k x_k - \lambda_{k+1} x_{k+1}|}{\rho}) < \infty, \text{ for some } \rho > 0.$$

Then there exists a $U > 0$ such that

$$M(\frac{|\lambda_k x_k - \lambda_{k+1} x_{k+1}|}{\rho}) < U, \text{ for all } k \in N.$$

Taking $\eta = k\rho$, for an arbitrary fixed positive integer k, by the subadditivity of modulus, the monotonicity and convexity of M:

$$M(\frac{|\lambda_1 x_1 - \lambda_{k+1} x_{k+1}|}{\eta}) < \frac{1}{k}\sum_{l=1}^{k} M(\frac{|\lambda_l x_l - \lambda_{l+1} x_{l+1}|}{\rho}) < U.$$

Then the above inequality, the inequality

$$\frac{|\lambda_{k+1} x_{k+1}|}{(k+1)\rho} \le \frac{1}{k+1}(\frac{|\lambda_1 x_1|}{\rho} + k\frac{|\lambda_1 x_1 - \lambda_{k+1} x_{k+1}|}{k\rho})$$

and the convexity of M imply

$$M(\frac{|\lambda_{k+1} x_{k+1}|}{(k+1)\rho}) \quad \le \quad \frac{1}{k+1}(M(\frac{|\lambda_1 x_1|}{\rho}) + kM(\frac{|\lambda_1 x_1 - \lambda_{k+1} x_{k+1}|}{k\rho}))$$

$$\leq \max\{M(\frac{|\lambda_1 x_1|}{\rho}), U\} < \infty$$

Hence we have the desired result.

Lemma 2. $x \in \ell_\infty(M, \Lambda, \Delta)$ *implies* $\sup_k k^{-1}|\lambda_k x_k| < \infty$.

Proof. Proof is obvious by using Lemma 1.

Remark 1. *Similar results as in Lemma 1 and Lemma 2 hold for* $\tilde{\ell_\infty}(M, \Lambda, \Delta)$ *also, where the statement 'for some* $\rho > 0$*' should be replaced by 'for every* $\rho > 0$*'.*

For the next theorem, let $D_1 = \{a = (a_k) : \sum_{k=1}^{\infty} k|\lambda_k^{-1} a_k| < \infty\}$, $D_2 = \{b = (b_k) : \sup_k k^{-1}|\lambda_k b_k| < \infty\}$.

Theorem 1. *Let M be an Orlicz function. Then*

(i) $[c(M, \Lambda, \Delta)]^\alpha = [\ell_\infty(M, \Lambda, \Delta)]^\alpha = D_1$,

(ii) $[\tilde{c}(M, \Lambda, \Delta)]^\alpha = [\tilde{\ell_\infty}(M, \Lambda, \Delta)]^\alpha = D_1$,

(iii) $D_1^\alpha = D_2$.

Proof. (i) Let $a \in D_1$, then $\sum_{k=1}^{\infty} |k\lambda_k^{-1} a_k| < \infty$. Now for any $x \in \ell_\infty(M, \Lambda, \Delta)$ we have $\sup_k |k^{-1}\lambda_k x_k| < \infty$. Then we have

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sup_k |k^{-1}\lambda_k x_k| \sum_{k=1}^{\infty} |k\lambda_k^{-1} a_k| < \infty.$$

Hence $a \in [\ell_\infty(M, \Lambda, \Delta)]^\alpha$.

Thus

$$D_1 \subseteq [\ell_\infty(M, \Lambda, \Delta)]^\alpha \tag{1}$$

Again we know

$$[\ell_\infty(M, \Lambda, \Delta)]^\alpha \subseteq [c(M, \Lambda, \Delta)]^\alpha \subseteq [c_0(M, \Lambda, \Delta)]^\alpha \tag{2}$$

Conversely suppose that $a \in [c(M,\Lambda,\Delta)]^{\alpha}$. Then $\sum\limits_{k=1}^{\infty} |a_k x_k| < \infty$, for each $x \in c(M,\Lambda,\Delta)$. So we take

$$x_k = \lambda_k^{-1} k, k \geq 1$$

then

$$\sum_{k=1}^{\infty} |k\lambda_k^{-1} a_k| = \sum_{k=1}^{\infty} |a_k x_k| < \infty.$$

This implies that $a \in D_1$. Thus

$$[c(M,\Lambda,\Delta)]^{\alpha} \subseteq D_1. \tag{3}$$

Combining (3) with (1), (2) it follows

$$[c(M,\Lambda,\Delta)]^{\alpha} = [\ell_{\infty}(M,\Lambda,\Delta)]^{\alpha} = D_1$$

This completes the proof of part(i).

(ii) Proof is similar to that of part (i).

(iii) The proof of the inclusion $D_1^{\alpha} \supseteq D_2$ is similar to that of $D_1 \subseteq [\ell_{\infty}(M,\Lambda,\Delta)]^{\alpha}$.

For the converse part suppose $a \in D_1^{\alpha}$ and $a \notin D_2$. Then we have

$$\sup_k |k^{-1}\lambda_k a_k| = \infty$$

Hence we can find a strictly increasing sequence (k_j) of positive integers k_j such that

$$|k_j^{-1}\lambda_{k_j} a_{k_j}| > j^2 \text{ for all } j \geq 1$$

We define the sequence x by

$$x_k = \begin{cases} |a_{k_j}^{-1}|, \text{ if } k = k_j \\ 0, \text{ otherwise} \end{cases}$$

Then $x \in D_1$, because

$$\sum_{k=1}^{\infty} |k\lambda_k^{-1} x_k| = \sum_{j=1}^{\infty} |k_j \lambda_{k_j}^{-1} a_{k_j}^{-1}| \leq \sum_{j=1}^{\infty} j^{-2} < \infty$$

Thus $x \in D_1$ but $\sum_{k=1}^{\infty} |a_k x_k| = \sum_{j=1}^{\infty} |a_{k_j} x_{k_j}| = \infty$. This is a contradiction to $a \in D_1^{\alpha}$. Hence $a \in D_2$. This completes the proof.

If we take $\lambda_k = 1$, for all $k \in N$ in Theorem 1, then we obtain the following corollary.

Corollary 1. *For $X = c$ and ℓ_{∞},*

(i) $[X(M,\Delta)]^{\alpha} = [\tilde{X}(M,\Delta)]^{\alpha} = H_1$,

(ii) $H_1^{\alpha} = H_2$,

where

$H_1 = \{a = (a_k) : \sum_{k=1}^{\infty} |k a_k| < \infty\}$

and

$H_2 = \{b = (b_k) : \sup_k |k^{-1} b_k| < \infty\}$.

For the next theorem, let $G_1 = \{a = (a_k) : \lim_k k\lambda_k^{-1} a_k = 0\}$.

Theorem 2. *Let M be an Orlicz function. Then*

(i) $[c(M,\Lambda,\Delta)]^N = [\ell_{\infty}(M,\Lambda,\Delta)]^N = G_1$,

(ii) $[\tilde{c}(M,\Lambda,\Delta)]^N = [\tilde{\ell}_{\infty}(M,\Lambda,\Delta)]^N = G_1$.

Proof. (i) Proof is immediate using Lemma 2.

(ii) Proof is similar to that of part (i).

If we take $\lambda_k = 1$, for all $k \in N$ in Theorem 2, then we obtain the following corollary.

Corollary 2. *For $X = c$ and ℓ_∞,*
(i) $[X(M,\Delta)]^N = [\tilde{X}(M,\Delta)]^N = L_1$,
where $L_1 = \{a = (a_k) : \lim_k k a_k = 0\}$.

Theorem 3. *If M satisfies the Δ_2-condition, then we have $X(M,\Lambda,\Delta) = \tilde{X}(M,\Lambda,\Delta)$, for every $X = c_0$, c and ℓ_∞.*

Proof. We give the proof for $X = \ell_\infty$ and for other spaces it will follow on applying similar arguments.

To prove the theorem, it is enough to show that $\ell_\infty(M,\Lambda,\Delta)$ is a subspace of $\tilde{\ell_\infty}(M,\Lambda,\Delta)$.
Let $x \in \ell_\infty(M,\Lambda,\Delta)$, then for some $\rho > 0$,

$$\sup_k M(\frac{|\Delta\lambda_k x_k|}{\rho}) < \infty$$

Therefore

$$M(\frac{|\Delta\lambda_k x_k|}{\rho}) < \infty, \text{ for every } k \in N.$$

Choose an arbitrary $\eta > 0$. If $\rho \le \eta$ then $M(\frac{|\Delta\lambda_k x_k|}{\eta}) < \infty$ for every $k \in N$. Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$.
Since M satisfies the Δ_2-condition, there exists a constant K such that

$$M(\frac{|\Delta\lambda_k x_k|}{\eta}) \le K(\frac{\rho}{\eta})^{\log_2 K} M(\frac{|\Delta\lambda_k x_k|}{\rho}) < \infty \text{ for every } k \in N.$$

Now let us denote

$$S = \sup_k M(\frac{|\Delta\lambda_k x_k|}{\rho}) < \infty, \text{ for the fixed } \rho > 0.$$

Then it follows that for every $\eta > 0$, we have

$$\sup_k M(\frac{|\Delta\lambda_k x_k|}{\eta}) \le K(\frac{\rho}{\eta})^{\log_2 K}.S < \infty.$$

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