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Finiteness Conditions for Unions of Two Semigroups and Ranks of $B(G,n)$

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Abstract. In this paper we try to find the finiteness conditions for union of two finite semigroups with a specially defined binary equation. Moreover we find the ranks of the semigroup $B(G,n)$.

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1. Introduction

Finiteness conditions of semigroups (the properties of semigroups which all finite semigroups have) have been considered for certain classes of semigroup constructions. (for examples see [1, 2]). In this paper periodicity, residual finiteness and solvability of word problem of union of two finite semigroups are determined.

Let S and T be two finite semigroups with empty intersection. We define a binary equation on $S \cup T$ as follows:
If $s_1 \in S$ and $s_2 \in S$ then $s_1.s_2$ is considered as the same operation defined on S. If $t_1 \in T$ and $t_2 \in T$ then $t_1.t_2$ is considered as the same operation defined on T. If $s \in S$ and $t \in T$ then $st = ts = t$. In [3] it is shown that any finitely presented semigroup S is embedded into an inefficient semigroup, namely, the semigroup $S \cup SL_n$ where SL_n is the free semilattice of rank n.

Let S be a finite semigroup. A subset U of S is called *independent* if, for every u in U, the element u does not belong to the semigroup $< U \setminus \{u\} >$ generated by the remaining elements of U (see [4]). In [5] Howie and Ribeiro introduced $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$ defined as follows:

- $r_1(S) = max\{k :$ every subset U of S of cardinality k is independent$\}$
- $r_2(S) = min\{k :$ there exists a subset U of S of cardinality k which generates $S\}$

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- $r_3(S) = max\{k :$ there exists a subset U of S of cardinality k which is independent and which generates $S\}$
- $r_4(S) = max\{k :$ there exists a subset U of S of cardinality k which is independent$\}$
- $r_5(S) = min\{k :$ every subset U of S of cardinality k generates $S\}$

Generally in [5], $r_1(S)$ is small rank, $r_2(S)$ is lower rank, $r_3(S)$ is intermediate rank, $r_4(S)$ is upper rank and $r_5(S)$ is large rank. In [5] $r_5(C_n)$, $r_5(T_n)$ and $r_5(B(G,n))$ are given. Here C_n is the cyclic group of order n, T_n is the full transformation semigroup and $B(G,n)$ is a Brandt semigroup. In [5] it is also shown that all five ranks of the aperiodic Brandt semigroup B_n are different. In this paper we examine $r_1(B(G,n))$, $r_2(B(G,n))$, $r_3(B(G,n))$ and $r_4(B(G,n))$.

2. Periodicity

Recall that a semigroup S is periodic if, for each $s \in S$ the monogenic semigroup generated by s is finite, or equivalently there exists positive integers m and n (depending on S) such that $s^m = s^n$.

Theorem 1. *Let S and T be finite semigroups. Then S and T are periodic if and only if $S \cup T$ is periodic.*

Proof. ($\Rightarrow$) Let S and T be periodic. Let $x \in S \cup T$. Then $x \in S$ or $x \in T$. If $x \in S$, since S is periodic there exists $\exists m, n \in N$ such that $x^m = x^n$. If $x \in T$, since T is periodic there exists $\exists k, l \in N$ such that $x^k = x^l$. So $S \cup T$ is periodic.
($\Leftarrow$) Let $S \cup T$ be periodic. Let $x \in S$. Since $S \subseteq S \cup T$ we have $x \in S \cup T$. Since $S \cup T$ is periodic there exists $\exists k_1, k_2 \in N$ such that $x^{k_1} = x^{k_2}$. We obtain S is periodic. Let $y \in T$. Since $T \subseteq S \cup T$ we have $y \in S \cup T$. Since $S \cup T$ is periodic there exists $\exists k_3, k_4 \in N$ such that $y^{k_3} = y^{k_4}$. Thus T is also periodic. $\square$

3. Residual Finiteness

We call a semigroup residually finite if, for each pair $s \neq t \in S$ there exists a homomorphism ϕ from S onto a finite semigroup such that $\phi(s) \neq \phi(t)$, or equivalently, there exists a congruance ρ with finite index (that is ρ has finitely many equivalence classes) such that $(s,t) \notin \rho$. (Residual finiteness of completely (0)-simple semigroups, which are Rees matrix semigroups $M[G; I, J, P]$ over groups was investigated in [2].)

Theorem 2. *$S \cup T$ is residually finite if and only if S and T are residually finite.*

Proof. ($\Rightarrow$) Assume that $S \cup T$ is residually finite. Since S and T are subsemigroups of $S \cup T$ then S and T are residually finite.
($\Leftarrow$) Assume that S and T are residually finite semigroups. We will show that $S \cup T$ is residually finite. Let $s_1, s_2 \in S \cup T$ and $s_1 \neq s_2$. Since S is residually finite there is a finite semigroup K and an onto homomorphism $\phi : S \to K$ such that $\phi(s_1) \neq \phi(s_2)$. Let $\Psi : S \cup T \to K \cup \{0\}$. If $x \in S$

let $\Psi(x) = \phi(x)$ and if $x \in T$ let $\Psi(x) = 0$. Then $\Psi(s_1) = \phi(s_1) \neq \phi(s_2) = \Psi(s_2)$. If $s_1, s_2 \in S$ then $\Psi(s_1 s_2) = \phi(s_1 s_2) = \phi(s_1).\phi(s_2)$. If $t_1, t_2 \in T$ then $\Psi(t_1 t_2) = 0 = \psi(t_1).\psi(t_2) = 0.0$. If $s \in S$ and $t \in T$ then $\Psi(st) = \Psi(t) = 0 = \Psi(s)\Psi(t)$. So Ψ is an onto homomorphism.

Let $t_1, t_2 \in S \cup T$ and $t_1 \neq t_2$. Since T is residually finite there is a finite semigroup L and an onto homomorphism $\theta : T \to L$ such that $\theta(t_1) \neq \theta(t_2)$. We define $\alpha : S \cup T \to L \cup \{1\}$ as follows. If $x \in S$ let $\alpha(x) = 1$ and if $x \in T$ let $\alpha(x) = \theta(x)$. It is clear that $\alpha(t_1) = \theta(t_1) \neq \theta(t_2) = \alpha(t_2)$. If $s_1, s_2 \in S$ then $\alpha(s_1 s_2) = \alpha(s_1).\alpha(s_2) = 1.1 = 1$. If $t_1, t_2 \in T$ then $\alpha(t_1 t_2) = \theta(t_1 t_2) = \theta(t_1).\theta(t_2)$. If $s \in S$ and $t \in T$ then $\alpha(st) = \alpha(t) = \theta(t) = \alpha(s).\alpha(t) = 1.\theta(t)$. So α is an onto homomorphism.

Let $s, t \in S \cup T$ and $s \neq t$. We define $\mu : S \cup T \to R_2 = \{a, b\}$. Here $R_2 = \{a, b\}$ is the right zero semigroup with 2 elements and $ab = b$, $ba = a$. If $s \in S$ let $\mu(s) = a$ and if $t \in T$ let $\mu(t) = b$. We have $\mu(s) = a \neq \mu(t) = b$. If $s_1, s_2 \in S$ then $\mu(s_1 s_2) = a = \mu(s_1).\mu(s_2) = a.a = a$. If $t_1, t_2 \in T$ then $\mu(t_1 t_2) = b = \mu(t_1).\mu(t_2) = b.b = b$. If $s \in S$ and $t \in T$ then $\mu(st) = \mu(t) = b = \mu(s)\mu(t) = a.b = b$. Thus μ is an onto homomorphism. □

4. Solvable Word Problem

A semigroup S is said to have a solvable word problem with respect to a generating set A if there exists a algorithm which, for any two words $u, v \in A^+$, decides whether the relation $u = v$ holds in S or not. It is a well-known fact that, for a finitely generated semigroup S, the solvability of the word problem does not depend on the choice of the finite generating set for S. Thus we say that a semigroup S has a solvable word problem with respect to any finite generating set.

Theorem 3. *$S \cup T$ has solvable word problem if and only if S and T have solvable word problem.*

Proof. ($\Rightarrow$) Let $S \cup T$ have solvable word problem. Since S and T are finitely generated, let Y_1 be generating set of S and Y_2 be generating set of T. Then $Y_1 \cup Y_2$ is a generating set for $S \cup T$. Let $w_1, w_2 \in Y_1^+$. Since $w_1, w_2 \in Y_1^+ \subseteq (Y_1 \cup Y_2)^+$ and since $S \cup T$ has solvable word problem there exists an algorithm which decides whether $w_1 = w_2$ holds in $S \cup T$. Since $w_1, w_2 \in Y_1^+$ and Y_1 is a generating set for S, the algorithm decides whether $w_1 = w_2$ holds in S. So S has a solvable word problem. Similarly it is shown that T has a solvable word problem.

($\Leftarrow$) Assume that S and T have solvable word problem. Let X be a finite generating set for $S \cup T$. Then $X_1 = X \cap S$ and $X_2 = X \cap T$ are generating sets for S and T. The set $Z = \{x_1 x_2 = x_2, x_2 x_1 = x_2 \mid x_1 \in X_1, x_2 \in X_2\}$ is finite. For $w_1, w_2 \in X^+$, if we apply some necessary relations from Z we obtain $w_1, w_2 \in X^+$ such that $w_1 = w_1'$ and $w_2 = w_2'$ holds in T. $w_i' \in X_1^+ (i = 1, 2)$ or $w_i' \in X_2^+ (i = 1, 2)$. If w_1' and w_2' are not elements of the same free semigroup $X_i^+ (i = 1, 2)$ then $w_1' = w_2'$ does not hold in $S \cup T$. If w_1' and w_2' are in the same free semigroup $X_i^+ (i = 1, 2)$ there exists an algorithm which decides whether the relation $w_1' = w_2'$ holds in S or T. Because S and T have solvable word problem. So $S \cup T$ has solvable word problem. □

5. Ranks of $B(G,n)$

The semigroup $B(G,n)=\{1,2,\dots,n\}\times G\times\{1,2,\dots,n\}\cup\{0\}$ is the Brandt semigroup. The binary operation on $B(G,n)$ is defined as follows

$$(i,a,j).(k,b,l)=(i,ab,l) \text{ if } j=k$$
$$0 \text{ if } j\neq k$$
$$0.(i,a,j)=(i,a,j).0=0.0=0$$

In [5] $r_5(B(G,n))$ is given. Now we define other ranks of $B(G,n)$.

Lemma 1. *Let $B(G,n)$ be the Brandt semigroup. Let A be the minimum generating set of G. Then $r_1(B(G,n))=1$, $r_2(B(G,n))=2n.|A|$ and $r_3(B(G,n))=2n.|A|$.*

Proof. Let A be the minimum generating set of G. We show the set

$$B=\{(1,a,j),(i,a,1)|a\in A,1\leq i\leq n,1\leq j\leq n\}$$

is the minimum generating set for $B(G,n)$. For $(i,g,j)\in B(G,n)$ we have

$$(i,g,j)=(i,a_1,1).(1,a_2,1).(1,a_3,1)\dots(1,a_m,j),\quad (a_i\in A,i=1,2,\dots m).$$

So B is a generating set for $B(G,n)$. Let C be a generating set for $B(G,n)$. Since $(i,a,1)=(i,a,1).(1,1,1)(a\in A)$ and $(1,1,j)=(1,1,1).(1,1,j)$. So we have $B\subseteq C$. Thus B is the minimum generating set for $B(G,n)$. We have $r_2(B(G,n))=2n.|A|$.

Let D be a generating set for $B(G,n)$ and assume that D is independent. Since D is a generating set and B is the minimum generating set then $B\subseteq D$. Let $(i',g,j')\in D\ B$. Let $g=a'_1a'_2\dots a'_l(a'_i\in A)$. Then $(i',g,j')=(i',a'_1,1).(1,a'_2,1)(1,a'_3,1)\dots(1,a'_l,j')$. This contradicts with the assumption of D to be independent. So B is the unique independent generating subset of $B(G,n)$. Thus $r_3(B(G,n))=2n.|A|$.

Let $(i,g,j)\in B(G,n)$. If $i=j$ then $(i,g,j).(i,g,j)=(i,g^2,j)\neq(i,g,j)$ unless $g^2=g$. If $i\neq j$ then $(i,g,j).(i,g,j)=0$. So $B(G,n)$ is not a band. Since $r_2(B(G,n))\neq|B(G,n)|$ then $B(G,n)$ is not royal. (see [5]) So $r_1(B(G,n))=1$. □

In the following theorem we determine $r_4(B(G,n))$.

Theorem 4. *Let $B(G,n)$ be a Brandt semigroup. Let $r_4(G)=k$. Then $r_4(B(G,n))=k+1$.*

Proof. Let U be the maximum independent subset of G. Since $r_4(G)=k$ then $|U|=k$. We will show that $U'=\{(1,u,1)|u\in U\}\cup\{0\}$ is the maximum independent subset of $B(G,n)$. Let $(1,u,1)=(1,u_1,1).(1,u_2,1)(u_1,u_2\in U)$. Then $u=u_1.u_2$. Since U is independent then $u=u_1$ or $u=u_2$. We obtain U' is independent. Let $U''\subseteq B(G,n)$ be another independent set. We have $U'\cup(S\backslash U')=S=B(G,n)$. Let $s\in S\backslash U'$. Let

$$s=(i,g,j)(1\leq i\leq n,g\in G\backslash U,1\leq j\leq n).$$

Since $g\in G\backslash U$ then $g=g_1.g_2(g_1,g_2\in G,g_1\neq g,g_2\neq g)$. We have $(i,g,j)=(i,g_1,j).(j,g_2,j)$ and $U''=((U'\cup\{0\})\cap U'')\cup(U''\cap(S\backslash U'))$. Since the elements of $S\backslash(U'\cup\{0\})$ can be written

as a product of two elements. So $U'' \cap (S \backslash (U' \cup \{0\}) = \emptyset$. Then $U'' = (U' \cup \{0\}) \cap U''$. So $U'' \subseteq (U' \cup \{0\})$. We obtain $U' \cup \{0\}$ is the maximum independent set.
So $r_4(B(G,n)) = k+1$. □

The studies on finiteness conditions of semigroups and ranks of semigroups may be expanded to different classes of semigroups as future work.

References

[1] H. Ayık. *Presentations and Efficiency of Semigroups.* PhD thesis, 1998.

[2] H. Ayık. On Finiteness Conditions for Rees Matrix Semigroups. *Czechoslovak Mathematical Journal*, 55, 2005.

[3] H. Ayık, M. Minisker, and B. Vatansever. Minimal Presentations and Embedding Into Inefficient Semigroups. *Algebra Colloquium*, 12:59–65, 2005.

[4] E. Giraldes and J.M. Howie. Semigroups of High Rank. *Proceedings of the Edinburgh Mathematical Society*, 28:13–34, 1985.

[5] J.M. Howie and M.I.M. Ribeiro. Rank Properties of Semigroups II: the small rank and the large rank. *Southeast Asian Bulletin of Mathematics*, 24:231–237, 2000.