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Local Solvability for the 2-Coupled System of Nonlinear Schrödinger Equations in a Banach Algebra $E^0_{2,1}$

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Abstract. This paper is concerned with initial value problem of the nonlinear coupled Schrödinger equations. We study local well posedness in the Banach algebra $E^0_{2,1}(R^n)$ which is the extension of $H^s(R^n)$ when $s \geq \frac{n}{2}$. The method we use is similar to the method of semigroup.

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1. Introduction

It is well-known that $H^s(R^n)$ is an algebra when $s > \frac{n}{2}$ and the Schrödinger operator generate an unitary group in $H^s(R^n)$. The well-posedness in $H^s(s \geq \frac{n}{2})$ for the Cauchy problem of the cubic semi-linear Schrödinger equation

$$iu_t + \Delta u = a|u|^2 u, \quad x \in R^n, t \in R.$$

were treated by using the method of semigroup [we refer to 6]. Recently, Wang et al in [10] introduce a new Banach algebra $E^0_{2,1}(R^n)$ which is the extension of $H^s(R^n)$ when $s \geq \frac{n}{2}$ and investigated the Cauchy problem of semi-linear Schrödinger equation with nonlinear term $|u|^{2k}u, k \in N$. We shall study a coupled system by Wang's approach in this paper.

As a natural extension of the single cubic nonlinear Schrödinger equation, the 2-coupled nonlinear Schrödinger equations:

$$\begin{cases} iu_t + \Delta u = a|u|^2 u + |v|^2 u, & x \in R,\ t \in R, \\ iv_t + \Delta v = |u|^2 v + a|v|^2 v, & x \in R,\ t \in R, \end{cases} \tag{1}$$

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have many applications including, for example, nonlinear optics [cf. 2, 4, 5, 9] and geophysical fluid dynamics [cf. 7, 8]. In the above equations, $a \in R$, the unknowns $u(x,t)$, $v(x,t)$ are the envelopes of wave packets in two different degrees of freedom of the underlying physical systems which we shall call 'modes'. The system is derived as an approximation to a more complex set of equations by singular perturbation theory.

Instead of studying (1), this paper is concerned with the following general Schrödinger system:

$$\begin{cases} iu_t + \Delta u = a|u|^\alpha u + |v|^\alpha u, & x \in R^n,\ t \in R, \\ iv_t + \Delta v = |u|^\alpha v + a|v|^\alpha v, & x \in R^n,\ t \in R, \\ u(0,x) = \phi(x), & x \in R^n, \\ v(0,x) = \psi(x), & x \in R^n. \end{cases} \tag{2}$$

As in [10], for technical reason, the restriction $\alpha = 2k$, $k \in N$ or $|u|^\alpha = u^\alpha($ or $\bar{u}^\alpha)$ and $|v|^\alpha = v^\alpha($ or $\bar{v}^\alpha)$, $\alpha \in N$ is required in the nonlinear coupled terms. By denoting $U = \binom{u}{v}$, $F(U) = \binom{a|u|^\alpha u + |v|^\alpha u}{u|^\alpha v + a|v|^\alpha v}$ and $\Phi = \binom{\phi}{\psi}$, we see readily that (2) take the following form:

$$\begin{cases} \partial_t U + \Delta U = F(U) & x \in R^n,\ t \in R. \\ U(0,x) = \Phi(x) & x \in R^n. \end{cases} \tag{3}$$

Let

$$\Lambda(t) = \begin{pmatrix} S(t) & 0 \\ 0 & S(t) \end{pmatrix},$$

where $S(t) = e^{it\Delta}$ is the fundamental solution operator of the Schrödinger equation and is given by

$$S(t)\phi = \int_{R^n} e^{-it|\xi|^2 + ix\xi} \hat{\phi} d\xi,\ \forall\ \phi \in S(R^n).$$

Then by the Duhanmel principle we see that the Cauchy problem (3) is equivalent to the following integral equation:

$$U(t) = \Lambda(t)\Phi - i\int_0^t \Lambda(t-\tau)F(U(\tau))d\tau.$$

Thus, in the sequel we shall solve this integral equation.

We shall use the notation $|||\cdot|||$ to denote the norm of 2-dimensional vector functions, and use $\|\cdot\|$ to denote the norm of scale functions, so that $|||U||| = \|u\| + \|v\|$ if $U = \binom{u}{v}$.

The main result of this paper is

Theorem 1. *Let $\Phi \in E^0_{2,1}(R^n)$. Then there exists $T^* \equiv T^*(|||\Phi|||_{E^0_{2,1}(R^n)} > 0$ such that the Cauchy problem (3) has a unique solution*

$$U \in C([0,T^*), E^0_{2,1}(R^n)).$$

Moreover, if $T^* < \infty$ then

$$\lim_{t \to T^*} \sup |||U(t)|||_{E^0_{2,1}(R^n)} = \infty.$$

$E^0_{2,1}(R^n)$ *will be introduced in the next section.*

In the sequel, C will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters. For $p \geq 1$ we set $p' = \frac{p}{p-1}$.

2. Preliminaries

2.1. The Banach Algebra $E^0_{2,1}$

We denote by $S(R^n)$ and $S'(R^n)$ the Schwartz space and its dual space, respectively. Let $\rho \in S(R^n)$ and $\rho : R^n \to [0,1]$ be a smooth radial bump function adapted to the ball $B(0, \sqrt{2n})$, say $\rho(\xi) = 1$ as $0 \leq |\xi| \leq \sqrt{\frac{n}{2}}$, and $\rho(\xi) = 0$ as $|\xi| \geq \sqrt{2n}$. Let ρ_k be a translation of ρ :

$$\rho_k(\xi) = \rho(\xi - k), k \in Z^n,$$

where $k \in Z^n$ means that $k = (k_1, k_2, \cdots, k_n)$, and $k_1, k_2, \cdots, k_n$ are all integers. Since $\rho(\xi) = 1$ in the unit closed cube Q_k with center k and $\{Q_k\}_{k \in Z^n}$ is a covering of R^n, one has that $\sum_{k \in Z^n} \rho_k(\xi) \geq 1$ for all $\xi \in R^n$. We write

$$\sigma_k(\xi) = \rho_k(\xi) \Big(\sum_{k \in Z^n} \rho_k(\xi) \Big)^{-1}, \quad k \in Z^n.$$

It is easy to see that

$$\begin{cases} |\sigma_k(\xi)| \geq C, & \forall\ \xi \in Q_k; \\ suup \sigma_k(\xi) \subset \{\xi : |\xi - k| \leq \sqrt{2n}\}; & \\ \sum_{k \in Z^n} \sigma_k(\xi) = 1, & \forall\ \xi \in R^n; \\ |\sigma_k^{(m)}(\xi)| \leq C_m, & \forall\ \xi \in R^n. \end{cases} \tag{4}$$

Hence, the set

$$\Upsilon = \{\{\sigma_k\}_{k \in Z^n} : \{\sigma_k\}_{k \in Z^n} \text{ satisfies (4)}\}$$

is non-void. Let $\{\sigma_k\}_{k \in Z^n} \in \Upsilon$ be a function sequence. Define operator:

$$\Box_k \equiv \mathscr{F}^{-1} \sigma_k \mathscr{F},\ k \in Z^n,$$

where the operator $\mathscr{F}$ means Fourier transformation.

For any $k \in Z^n$, we write $|k| = |k_1| + |k_2| + \cdots + |k_n|$. Let $0 \leq \lambda < \infty$, $0 < p,\ q \leq \infty$, we introduce the following function space

$$E^\lambda_{p,q}(R^n) = \Big\{ f \in S'(R^n) : \|f\|_{E^\lambda_{p,q}} \equiv \Big(\sum_{k \in Z^n} [2^{\lambda |k|} \|\Box_k f\|_{L^p(R^n)}]^q \Big)^{\frac{1}{q}} < \infty \Big\}.$$

Obviously, the function space $E^{\lambda}_{p,q}(R^n)$ is modified from the Besov space $B^{s}_{p,q}(R^n)$ [see 1]. Since the relation between $E^{\lambda}_{p,q}(R^n)$ and the Besov space $B^{s}_{p,q}(R^n)$ have nothing to do with our result, we omit it here [for the details, we refer to 10].

The algebra property of $E^0_{2,1}$ may deduce from the following embedding property and bilinear estimate.

Lemma 1. *Let* $0 \le \lambda < \infty$, $0 < p_1 \le p_2 \le \infty$, $0 < q_1 \le q_2 \le \infty$. *Then we have*

$$E^{\lambda}_{p_1,q_1}(R^n) \subset E^{\lambda}_{p_2,q_2}(R^n).$$

Proof. See the proof of Proposition 3.5 in [10].

Lemma 2. *Let* $0 \le \lambda < \infty$, $0 < p \le p_1,\ p_2 \le \infty$, $0 < q \le \infty$. *If* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, *then we have*

$$\|uv\|_{E^{\lambda}_{p,q}} \le C2^{Cq\lambda}\|u\|_{E^{\lambda}_{p_1,q\wedge 1}}\|v\|_{E^{\lambda}_{p_2,q\wedge 1}},$$

where $a \wedge b = \min\{a, b\}$. C *is independent of* λ, q *and if* p *is fixed, then* C *is also independent of* p_1, p_2.

Proof. See the proof of Lemma 4.1 in [10].

As a matter of fact, by Lemma 1 and Lemma 2, we have

$$\|uv\|_{E^0_{2,1}} \le C\|uv\|_{E^0_{1,1}} \le C\|u\|_{E^0_{2,1}}\|v\|_{E^0_{2,1}}. \tag{5}$$

Which suggest that $E^0_{2,1}$ is a Banach algebra.

From the comparison between $E^0_{2,q}(R^n)$ and $H^s(R^n)$, we find that $E^0_{2,1}(R^n)$ is the extension of $H^s(R^n)$:

$$H^s(R^n) \subset E^0_{2,1}(R^n) \text{ for } s > \frac{n}{2}, \text{ and } H^s(R^n) \subset E^0_{2,1} \text{ fails, for } s \le \frac{n}{2}.$$

Indeed, we have

Lemma 3. *We have*

$$H^s(R^n) \subset E^0_{2,q}(R^n),\ s > n(\frac{1}{q} - \frac{1}{2}),\ 0 < q < 2,$$

$$L^2(R^n) = E^0_{2,2}(R^n) \ \textit{(equivalent norm)},$$

$$E^0_{2,q}(R^n) \subset H^s(R^n),\ s < n(\frac{1}{q} - \frac{1}{2}),\ 2 < q \le \infty.$$

Furthermore, $E^0_{2,1}$ *is the intermediate space of* $H^s(R^n)$ *and* $L^\infty(R^n)$, *that is*

$$H^s(R^n) \subset E^0_{2,1} \subset L^\infty(R^n), \quad s > n/2.$$

[(3.43) in 10].

Proof. See the proof of Proposition 3.8 in [10].

2.2. Some Preliminary Lemmas

Estimate for the Schrödinger group

Lemma 4. *Let $0 < r \leq 2 \leq p \leq \infty$, $0 < q \leq \infty$. Then for the Schrödinger group $S(t) = e^{it\Delta}$ we have the estimate*

$$\|S(t)\phi\|_{E^0_{p,q}} \leq C\|\phi\|_{E^0_{r,q}}.$$

In particular,

$$\|S(t)\phi\|_{E^0_{2,1}} \leq C\|\phi\|_{E^0_{2,1}}.$$

Proof. See the proof of Proposition 5.5 in [10].

From Lemma 4, we deduce that

Lemma 5. *Let $0 < r \leq 2 \leq p \leq \infty$, $0 < q \leq \infty$. Then for the group $\Lambda(t)$ we have the estimate*

$$|||\Lambda(t)\Phi|||_{E^0_{p,q}} \leq C|||\Phi|||_{E^0_{r,q}}.$$

In particular,

$$|||\Lambda(t)\Phi|||_{E^0_{2,1}} \leq C|||\Phi|||_{E^0_{2,1}}.$$

With the algebra property, we have the Estimates for the nonlinear coupled terms

Lemma 6.

$$|||F(U)|||_{E^0_{2,1}} \leq C|||U|||^{\alpha+1}_{E^0_{2,1}},$$

and

$$|||F(U_1) - F(U_2)|||_{E^0_{2,1}} \leq C|||U_1 - U_2|||_{E^0_{2,1}}\left[|||U_1|||^{\alpha}_{E^0_{2,1}} + |||U_2|||^{\alpha}_{E^0_{2,1}}\right],$$

where $U = \binom{u}{v}$, $U_1 = \binom{u_1}{v_1}$, $U_2 = \binom{u_2}{v_2}$.

Proof. By (5), we have

$$\begin{aligned}
|||F(U)|||_{E^0_{2,1}} &= \|a|u|^{\alpha}u + |v|^{\alpha}u\|_{E^0_{2,1}} + \||u|^{\alpha}v + a|v|^{\alpha}v\|_{E^0_{2,1}} \\
&\leq |a|\||u|^{\alpha}u\|_{E^0_{2,1}} + \||v|^{\alpha}u\|_{E^0_{2,1}} + \||u|^{\alpha}v\|_{E^0_{2,1}} + |a|\||v|^{\alpha}v\|_{E^0_{2,1}} \\
&\leq |a|\|u\|^{\alpha+1}_{E^0_{2,1}} + \||v\|^{\alpha}_{E^0_{2,1}}\|u\|_{E^0_{2,1}} + \|u\|^{\alpha}_{E^0_{2,1}}\|v\|_{E^0_{2,1}} + |a|\|v\|^{\alpha+1}_{E^0_{2,1}} \\
&\leq \|u\|^{\alpha}_{E^0_{2,1}}\left(\|u\|_{E^0_{2,1}} + \|v\|_{E^0_{2,1}}\right) + C\|v\|^{\alpha}_{E^0_{2,1}}\left(\|u\|_{E^0_{2,1}} + \|v\|_{E^0_{2,1}}\right) \\
&\leq C|||U|||^{\alpha+1}_{E^0_{2,1}}
\end{aligned}$$

By mean value theorem we obtain $x^{\alpha} - y^{\alpha} = \alpha(x-y)(x-\eta y)^{\alpha-1}$, $(0 \leq \eta \leq 1)$. Using this fact and (5), Young's inequality (since $\frac{\alpha-1}{\alpha} + \frac{1}{\alpha} = 1$), we have

$$\begin{aligned}
&|||F(U_1) - F(U_2)|||_{E^0_{2,1}} \\
&= \|a|u_1|^{\alpha}u_1 + |v_1|^{\alpha}u_1 - (a|u_2|^{\alpha}u_2 + |v_2|^{\alpha}u_2)\|_{E^0_{2,1}}
\end{aligned}$$

$$
\begin{aligned}
&+\||u_1|^\alpha v_1 + a|v_1|^\alpha v_1 - (|u_2|^\alpha v_2 + a|v_2|^\alpha v_2)\|_{E^0_{2,1}} \\
&= \|(a|u_1|^\alpha + |v_1|^\alpha)(u_1 - u_2) + (a(|u_1|^\alpha - |u_2|^\alpha) + (|v_1|^\alpha - |v_2|^\alpha))u_2\|_{E^0_{2,1}} \\
&+\|(|u_1|^\alpha + a|v_1|^\alpha)(v_1 - v_2) + ((|u_1|^\alpha - |u_2|^\alpha) + a(|v_1|^\alpha - |v_2|^\alpha))v_2\|_{E^0_{2,1}} \\
&\leq \|(a|u_1|^\alpha + |v_1|^\alpha)(u_1 - u_2)\|_{E^0_{2,1}} + \|(a(|u_1|^\alpha - |u_2|^\alpha) + \|(|v_1|^\alpha - |v_2|^\alpha))u_2\|_{E^0_{2,1}} \\
&+\|(|u_1|^\alpha + a|v_1|^\alpha)(v_1 - v_2)\|_{E^0_{2,1}} + \|((|u_1|^\alpha - |u_2|^\alpha) + a(|v_1|^\alpha - |v_2|^\alpha))v_2\|_{E^0_{2,1}} \\
&\leq C\left[\|u_1\|^\alpha_{E^0_{2,1}} + \|u_2\|^\alpha_{E^0_{2,1}} + \|v_1\|^\alpha_{E^0_{2,1}} + \|v_2\|^\alpha_{E^0_{2,1}}\right](\|u_1 - u_2\|_{E^0_{2,1}} + \|v_1 - v_2\|_{E^0_{2,1}}) \\
&\leq C(|||U_1|||^\alpha_{E^0_{2,1}} + |||U_2|||^\alpha_{E^0_{2,1}})|||U_1 - U_2|||_{E^0_{2,1}}.
\end{aligned}
$$

3. Proof of the Main Result

We shall make use of the fixed point Theorem to solve the integral equation

$$U = \mathscr{T}(U) = \Lambda(t)\Phi - i\int_0^t \Lambda(t-\tau)F(U(\tau))d\tau. \tag{6}$$

Define a metric space as follows:

$$D = \{U : |||U|||_{C(0,T;E^0_{2,1})} \leq M\},$$

$$d(U,V) = |||U - V|||_{C(0,T;E^0_{2,1})}.$$

By Lemma 5, we have

$$|||\Lambda(t)\Phi|||_{C(0,T;E^0_{2,1})} \leq C|||\Phi|||_{E^0_{2,1}}. \tag{7}$$

By Lemma 5 and the first inequality of Lemma 6, we obtain

$$\left|\left|\left|\int_0^t \Lambda(t-\tau)F(U(\tau))d\tau\right|\right|\right|_{C(0,T;E^0_{2,1})} \leq CT|||U|||^{\alpha+1}_{C(0,T;E^0_{2,1})}. \tag{8}$$

Let us consider the mapping $\mathscr{T} : U \to \Lambda(t)\Phi - i\int_0^t \Lambda(t-\tau)F(U(\tau))d\tau$. We show that $\mathscr{T} : (D,d) \to (D,d)$ is a contraction mapping. Indeed, for any $U \in D$, by (7) and (8) we have

$$|||\mathscr{T}(U)|||_{C(0,T;E^0_{2,1})} \leq C|||\Phi|||_{E^0_{2,1}} + CT|||U|||^{\alpha+1}_{C(0,T;E^0_{2,1})}.$$

Put $M = 2C|||\Phi|||_{E^0_{2,1}}$, we have

$$|||\mathscr{T}(U)|||_{C(0,T;E^0_{2,1})} \leq \frac{M}{2 + CTM^{\alpha+1}}. \tag{9}$$

Let T be small enough to satisfies $CTM^\alpha \leq \frac{1}{4}$. It follows from (9) that $\mathscr{T}(U) \in D$.

Similarly, we have

$$|||\mathscr{T}(U)-\mathscr{T}(V)|||_{C(0,T;E^0_{2,1})} \le \frac{1}{2}|||U-V|||_{C(0,T;E^0_{2,1})}.$$

Indeed, $\forall\, U,\ V \in (D,d)$, by Lemma 5 and the second inequality of Lemma 6, we obtain

$$\begin{aligned}
|||\mathscr{T}(U)-\mathscr{T}(V)|||_{C(0,T;E^0_{2,1})} &\le \left|\left|\left|\int_0^t \Lambda(t-\tau)[F(U(\tau))-F(V)d\tau\right|\right|\right|_{C(0,T;E^0_{2,1})}\\
&\le CT|||F(U)-F(V)|||_{C(0,T;E^0_{2,1})}\\
&\le CT\left[|||U|||^{\alpha}_{C(0,T;E^0_{2,1})}+|||V|||^{\alpha}_{C(0,T;E^0_{2,1})}\right]|||U-V|||_{C(0,T;E^0_{2,1})}\\
&\le 2CTM^{\alpha}|||U-V|||_{C(0,T;E^0_{2,1})}\\
&\le \frac{1}{2}|||U-V|||_{C(0,T;E^0_{2,1})}.
\end{aligned}$$

Hence, by Banach fixed point theorem, we see that $\mathscr{T}$ has a fixed point $U \in D$ which is a solution of integral equation (6). We can extend this solution step by step and finally find a maximal $T^* > 0$ such that $U \in C([0,T^*), E^0_{2,1}(R^n))$ and $\lim_{t\to T^*}\sup|||U(t)|||_{E^0_{2,1}(R^n)} = \infty$. The uniqueness of such solutions can also be shown in a standard way. This finishes the proof of the Theorem.

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