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Modeling the Prey Predator Problem by a Graph Differential Equation

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Abstract. In this paper, we introduce new concepts like a pseudo simple graph, product of two graphs and obtain a sufficient condition which will guarantee that the solution of the IVP of a graph differential equation has the same nature as its graph of initial conditions. Further, we model the well known pre-predator problem by graph differential equations and show that the nonlinearity is naturally preserved.

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1. Introduction

Any natural or a man made system involves interconnections between its constituents, thus forming a network, which can be expressed by a graph [2, 3]. Graphs arise naturally when trying to model organizational structures in social sciences. It has been noted that a graph which is static in nature is not suitable for social phenomena whose changes with time are natural. This led to the introduction of a dynamic graph and a Graph Differential Equation (GDE) in [3]. The introduced concepts were successfully utilized to study stability of complex dynamic systems through its associated adjacency matrix [3].

In [2] we have utilized the concepts defined in [3] including a graph linear space and its associated matrix linear space. Using the notion of a dynamic graph and the graph differential equations we observed that the study of GDEs falls into the realm of differential equations in abstract spaces. This study, through highly mathematical, would be of little use for practical problems. On the other hand, if we consider the associated matrix differential equation(MDE) then the approach appeared more reasonable and practical for the study of GDEs. Hence in [2], we considered a weighted directed simple graph as the basic element and developed

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the theory. We have obtained existence and uniqueness of solutions of a GDE through its associated MDE using the monotone iterative technique.

In [2] through we have developed significant results, the basic concept involved was weighted directed simple graph. Since a simple graph has no loops, this fact when translated into differential equations frame work states that there is no way to accommodate the rate of change of an edge e_{ii} and its relation with other edges including the edge e_{ii}. This is a drawback that had to be handled to model physical phenomena using graph differential equations, which called for a new concept that we plan to introduce in this paper.

Further, since there exists an isomorphism between graphs and their adjacency matrices, we successfully exploited it and defined the product of two graphs. A good example , will go a long way in support of the theory, we have considered the prey predator problem and developed the corresponding matrix differential equation and showed how the nonlinearity is preserved in this set up.

The rest of the paper is as follows. In section two, we introduce the concepts of pseudo simple graph and product of two graphs and have obtained a result, that can be of practical importance in this set up.

In section three we obtained the matrix differential equation for prey predator problem and extended it to three species and further generalized it. In section four we conclude our work.

2. Main Results

In this section, we begin with the concept of a pseudo simple and later introduce the product of two graphs.

Definition 1 (Pseudo simple graph)**.** *A simple graph having loops is called as a pseudo simple graph.*

Parallel to the definitions and theory developed in [2] we proceed to state the results in this set up. We avoid the details for fear of repetition.

Let $v_1, v_2, \ldots, v_N$ be N verticess, N fixed. Let D_N be the set of all weighted directed pseudo simple graphs $D = (V, E)$. Then $(D_N, +, .)$ is a linear space with the definitions given in [3] and [2].

Let the set of all corresponding adjacency matrices be E_N. Then $(E_N, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' indicates scalar multiplication. With this basic structures defined, the comparison theorems, existence and uniqueness results of solutions of MDE and the corresponding GDE follow as in [2].

Taking cue from matrix multiplication we define the product of two graphs as follows.

Product of graphs: Let G_1 and G_2 be two graphs with edges $(e_{ij})_{N\times N}$ and $(d_{ij})_{N\times N}$ respectively. Then the product of the two graphs G_1 and G_2 is the graph G in which the weight g_{ij} of the edge from v_j to v_i is the dot product of the vectors one having the weights of the edges inwards to v_i and the other having weights of the edges outwards from v_j.

We now proceed to develop a result on the nature of solutions of a graph differential equation.

Let

$$D' = g(t, D) \tag{1}$$

be a graph differential equation.

Now if possible suppose $g(t, D)$ can be written as a product of two graphs CD where C is a graph having constant weights.

Then the GDE (1) can be written in the form

$$\begin{aligned} D' &= CD \\ D_{t_0} &= D_0 \end{aligned} \tag{2}$$

where C is a graph called a coefficient graph and D_0 is the initial graph.

Let

$$\begin{aligned} E' &= AE \\ E_{t_0} &= E_0 \end{aligned} \tag{3}$$

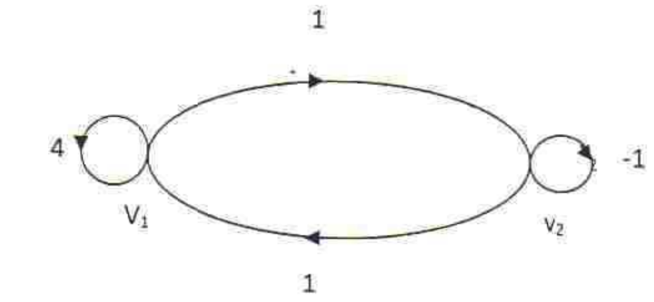
be the corresponding IVP of the MDE where E_0 is the adjacency matrix corresponding to the initial graph D_0. Then we have the following result relating to the solutions of MDE and hence to that of GDEs.

Theorem 1. *Let $E(t)$ be a solution of the IVP* (3)*. Suppose there exists a non singular matrix P such that $P^{-1}AP = H$ is a diagonal matrix. Then the solution $E(t)$ has the same nature as that of E_0. In other words, the solution of the IVP of the GDE* (2) *has the same nature as that of the initial graph D_0.*

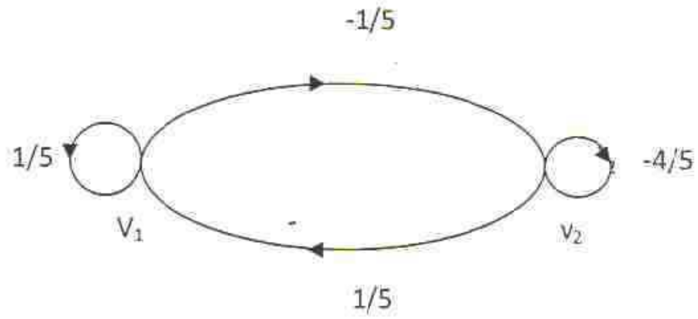
Proof. Suppose there exist a matrix P such that $P^{-1}AP = H$. Then we know from the theory of linear algebra that A and H have the same eigen values. Further we know that the solution of the IVP of MDE (3) is same as the solution of the MDE $E' = HE$, $E_{t_0} = E_0$. The solution of the MDE (3) is given by $E(t) = e^{Ht}\ E_0$, where e^{Ht} is a diagonal matrix.

Thus it is clear that $E(t)$ enjoys the same character as of E_0. Using the fact that there exists an isomorphism between matrices and graphs. We can easily conclude that the IVP of GDE (2) has a solution $D(t)$ having the same nature as that of D_0. □

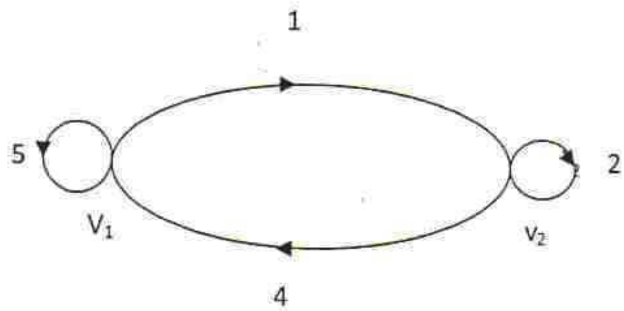
Remark 1. *Corresponding to the matrices P^{-1} and P we can find two graphs $G_{P^{-1}}$ and G_p such that $G_{P^{-1}}G_AG_P = G_H$ is a graph having only loops. Please see Figures 1a through 1d.*



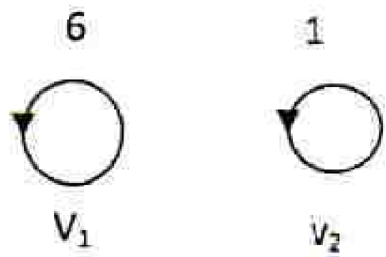
(a) G_p



(b) ${G_p}^{-1}$



(c) G_A



(d) ${G_p}^{-1}G_AG$

Figure 1: Graphs

3. Modeling of the Prey-Predator Problem

In this section, we formulate a matrix differential equation for the famous prey predator model and later extend it to three species and N-species.

Let x denote the prey population and y denote the predator population, Then the rate of change of prey and that of predator gives rise to a system of nonlinear differential equations given by

$$\frac{dx}{dt} = ax + bxy, \quad a > 0, \ b < 0, \tag{4}$$

$$\frac{dy}{dt} = cyx + dy, \quad c > 0, \ d < 0. \tag{5}$$

It is well known that the above differential equations are linearized and solved as a linear system of differential equations.

We now express the above system as a Graph Differential Equation and consider the corresponding Matrix Differential Equation. We show that the nonlinearity is preserved in this set up.

Let the vertex v_1 denote the prey and v_2 denote the predator. Set $e_{11} = x$ as population of the prey and $e_{22} = y$ as the population of the predator. It can be seen that e_{12} is the edge going outward from v_2 and is incident on v_1. This means that e_{12} denotes the interaction between predator and prey. Actually, e_{12} gives the status of predators finding the prey. Similarly e_{21} denotes the edge outward from v_1 and incident on v_2. In terms of our model, this edge indicates the status of prey that fall prey to predators.

Now the graph of the prey predator model is of the form

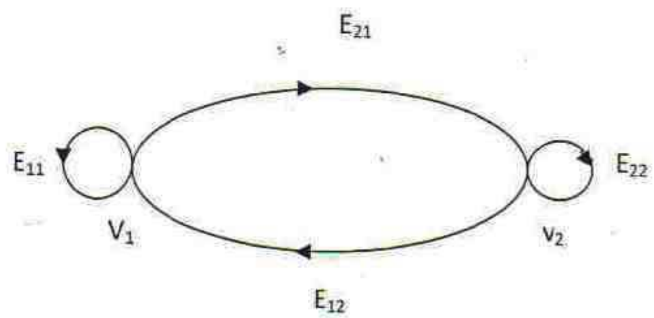


Figure 2: The Graph of the prey-predator problem

and its adjacency matrix is given by

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}.$$

The equations (4) and (5) reduce to the form

$$e'_{11} = ae_{11} + be_{21}, \tag{6}$$

$$e'_{22} = ae_{12} + be_{22}. \tag{7}$$

Our aim is to obtain a Matrix Differential Equations of the form

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}' = A \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

where $A_{2\times 2}$ is the coefficient matrix.

It can be easily seen that

$$\begin{bmatrix} e_{11} \\ e_{22} \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \end{bmatrix}$$

and hence we propose to choose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and obtain matrix differential equation of the form

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \tag{8}$$

The system (8) yields the equations (6), (7) and the following two differential equations given by

$$e'_{12} = ae_{12} + be_{22}, \tag{9}$$

$$e'_{21} = ce_{11} + de_{21}. \tag{10}$$

The equation (9) describe the rate of change of predator finding prey and it is positively proportional to the predator finding prey and negatively proportional to the predator population.

The equation (10) gives the rate of change prey coming in way of predator and this is positively proportional to prey available and negatively proportional to prey falling to predator.

Hence it can be seen that all the four equations given by (6), (7), (9) and (10) are consistent with the standard prey predator problem.

The beauty in this set up is that the nonlinearity is preserved and effectively used. The system obtained reduces to a Matrix linear differential equation and the solution is immediately given by

$$\begin{bmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{bmatrix} = e^{A(t-t_0)}\, E_0$$

where E_0 is the given matrix of initial conditions at $t = t_0$, see [1]. Observe that $e^{A(t-t_0)}$ is a matrix. If A is diagonalizable then $e^{A(t-t_0)}$ can be replaced by the diagonal matrix $e^{H(t-t_0)}$, where $H = diag[\lambda_1, \lambda_2]$ where λ_1 and λ_2 are the eigen values of A and the matrix has the form

$$e^{H(t-t_0)} = \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{bmatrix}.$$

Thus it has been effectively shown that a physical phenomena can be described through a graph and using the standard models we can preserve the nonlinearity and obtain more information using its associated matrix differential equation.

Next we consider a three species model given by

$$\frac{dx}{dt} = ax + bxy + cxz,$$
$$\frac{dy}{dt} = dyx + ey + fyz,$$
$$\frac{dz}{dt} = gzx + hzy + kz.$$

Working parallel to the prey predator problem, we consider three vertices v_1, v_2 and v_3 representing x, y and z respectively. Proceeding as in the prey predator problem, we arrive at the linear matrix differential equation of the form

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}' = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33}. \end{bmatrix}$$

It can be observed that we will get six additional equations and hence more information is known. The solution for this system can be immediately given by $E(t) = e^{A(t-t_0)}C$, where C is the matrix of initial conditions. Clearly, this approach can be extended suitably to a N-species model.

Remark 2. *It can be observed that if the rate of change of an edge e_{ij} is proportional only to the edges that are incident outward from v_j then we obtain a matrix differential equation.*

On the other hand, if the rate of change of edge e_{ij} is proportional to all the $N \times N$ edges or to some of them (without any structure) then we can treat the $N \times N$ edges as an N^2 vector and consider a vector differential equation of the $X' = AX$ where A is an $N \times N$ matrix.

4. Conclusion

In this paper we have introduced the notions of a pseudo simple graph and the product of two graphs we have given sufficient conditions under which a solution of a Graph Differential Equation has the same nature as its graph of initial conditions. Further, we have obtained a matrix differential equation for a prey predator probleml and explicitly gave its solutions preserving the nonlinearity. From the model, it is clear that the nonlinearity in the prey predator problem is preserved by using a graph differential equation.

As long as the rate of change of the species X_i is proportional to linear interactions between itself and X_j species it will be possible to obtain a linear graph differential equation. In other words, in the above discussed prey-predator problem if the rate of change of prey population X w.r.t time is proportional to the interaction of the square of prey population (X^2) and square of product population (Y^2) and similarly with predator population then the problem reduces to the form

$$\frac{dx}{dt} = ax + bx^2y^2,$$
$$\frac{dy}{dt} = cy^2x^2 + dy.$$

This cannot be immediately modeled by a linear graph differential equation and needs further investigations.

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