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Cycles in the Chamber Homology for $SL(2, F)$

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Abstract. We emphasized finding the explicit cycles in the chamber homology groups and the K-theory groups in term of each representation for $SL(2, F)$. This led to an explicit computing of chamber homology and the K-theory groups. We have identified the base change effect on each of these cycles. The base change map on the homology group level works by sending a generator of the homology group of $SL(2, E)$ labeled by a character of $E^{\times}$ to the generator of the homology group of $SL(2, F)$ labeled by a character of $F^{\times}$ multiplied by the residue field degree. Whilst, it works by sending the K-theory group generator of the reduce C^{*}-algebra of $SL(2, E)$ labeled by the 1-cycle (resp. 0-cycle) to the multiplication of the residue field degree with a generator of the K-theory group of $SL(2, F)$ labeled by the base changed effect on 1-cycle (resp. 0-cycle).

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1. Introduction

Let F be a p-adic non-archimedean local field with $p \neq 2$ and $G = SL(2, F)$. We have $F^{\times} \cong \mathscr{U}_F \times \mathbb{Z}$, where $\mathscr{U}_F$ is the group of p-adic units, and the dual of $F^{\times}$ is $\widehat{F^{\times}} \cong \widehat{\mathscr{U}_F} \times \mathbb{T}$, where $\mathbb{T}$ is the circle group. In this paper we emphasized finding the explicit cycles in the chamber homology groups and the K-theory groups in term of each representation for $SL(2, F)$. This led to an explicit computing of chamber homology and the K-theory groups. We have identified the base change effect on each of these cycles. The base change map on the homology group level works by sending a generator of the homology group of $SL(2, E)$ labeled by a character of $E^{\times}$ to the generator of the homology group of $SL(2, F)$ labeled by a character of $F^{\times}$ multiplied by the residue field degree. Whilst, it works by sending the K-theory group generator of the reduce C^{*}-algebra of $SL(2, E)$ labeled by the 1-cycle (resp. 0-cycle) to the multiplication of the residue field degree with a generator of the K-theory group of $SL(2, F)$ labeled by the base changed effect on 1-cycle (resp. 0-cycle).

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Consequently, we showed that the base change of *Steinberg* is again a *Steinberg*, the base change of a *principal series* is always a principle series and the base change of a *cuspidal* can certainly be either another cuspidal or a principal series.

We have found that whilst the Baum-Connes correspondence takes the homology group generator of $SL(2,E)$ to a generator of the K-theory group of C^*-algebra of $SL(2,E)$ by induction, it takes the effect of the base change map on the homology side to the base change effect on the K-theory side by induction as well.

2. Local Langlands Correspondence and Base Change

Let F be a non-archimedean local field, and $G = SL(2,F)$. Let $\mathscr{L}_F$ be the local Langlands group:

$$\mathscr{L}_F := \mathscr{W}_F \times SL(2,\mathbb{C}).$$

A Langlands parameter is a continuous homomorphism

$$\phi : \mathscr{L}_F \to G^\vee = PGL(2,\mathbb{C}),$$

where $G^\vee = PGL(2,\mathbb{C})$ is the Langlands dual group. We say that two Langlands parameters are equivalent if they are conjugate under the group $PGL(2,\mathbb{C})$. Let $\Phi(G)$ be the set of equivalence classes of the Langlands parameters. Now, the Local Langlands correspondence is defined to be the surjective map

$$\begin{aligned} Irr(G) &\longrightarrow \Phi(G), \\ A_\phi &\longmapsto \phi \end{aligned}$$

where A_ϕ is the pre-image of ϕ which is called the L-packet. The base change map is defined by the restriction of L-parameter from $\mathscr{L}_F$ to $\mathscr{L}_E$, where E is a finite extension of F

$$\phi|_{\mathscr{W}_E} : \mathscr{W}_E \times SL(2,\mathbb{C}) \to PGL(2,\mathbb{C}).$$

Lemma 1. *Let* $\alpha_{_E} = \gamma_{_E} \circ \beta_{_E} : \mathscr{W}_E \to E^\times$, *where* $\gamma_{_E} : \mathscr{W}_E^{ab} \to E^\times$ *and* $\beta_{_E} : \mathscr{W}_E \to \mathscr{W}_E^{ab}$ *then we have:*

i) $N_{E/F}(\alpha_E(w)) = \alpha_F(w)$, $w \in \mathscr{W}_E \subset \mathscr{W}_F$.

ii) $f.val_E = val_F \circ N_{E/F}$.

iii) $d_E = -val_E \circ \alpha_E$.

iv) Let $w \in \mathscr{W}_E \subset \mathscr{W}_F$. *Then we have* $f.d_E(w) = d_F(w)$.

Proof. See [2, 1.2.2] for 1, [10, p. 139] for 2, and see [6] for 3 and 4. □

Now, an unramified character ψ of $\mathscr{W}_E$ is given by the following simple formula:

$$\psi(w) = z^{d_E(w)}, z \in \mathbb{C}^\times.$$

The base change formula for a character χ of $\mathscr{W}_F$ is given by

$$BC(\chi) = \chi \mid_{\mathscr{W}_E}.$$

Lemma 2. *Under base change we have*

$$BC(\psi)(w) = (z^f)^{d_E(w)}$$

for all $w \in \mathscr{W}_E$.

Proof. The result follows directly from part 4 of lemma 1. □

Lemma 3. *Let $\phi = \mathbb{1} \otimes \tau(2)$ and $\phi^{'} = \psi \otimes \tau(2)$ be two L-parameters, where ψ is an unramified character of $\mathscr{W}_F$. Then*

$$\phi = \phi^{'}$$

in $PGL(2,\mathbb{C})$.

Proof. Let

$$d_F : \mathscr{W}_F \longrightarrow \mathscr{W}_F^{ab} \simeq F^{\times} \xrightarrow{val_F} \mathbb{Z} .$$

We have $\psi(w) = z^{d(w)}$ where $z \in \mathbb{C}^{\times}$, ψ unitary character if and only if $z \in \mathbb{T}$. Let

$$\phi = \mathbb{1} \otimes \tau(2) : \mathscr{W}_F \times SL(2,\mathbb{C}) \to PGL_2(\mathbb{C})$$

and

$$\phi^{'} = \psi \otimes \tau(2) : \mathscr{W}_F \times SL(2,\mathbb{C}) \to PGL_2(\mathbb{C})$$

such that

$$\phi(w,A) = \mathbb{1} \cdot \tau(2)(A) = \tau(A)$$

and

$$\phi^{'}(w,A) = \psi(w) \cdot \tau(2)(A) = z^{d(w)} \cdot \tau(A).$$

We see that $\tau(A)$ and $(z^{d(w)} \cdot \tau(A))$ are both in the same group $PGL(2,\mathbb{C})$ and this means that $\phi = \phi^{'}$. □

Theorem 1. *Let $\phi = 1 \otimes \tau(2)$ be the L-parameter of the Steinberg representation, then we have $BC(St_G(F)) = St_G(E)$.*

Proof. Let

$$\mathscr{L}_F = \mathscr{W}_F \times SL(2,\mathbb{C}) \quad \text{and} \quad \mathscr{L}_E = \mathscr{W}_E \times SL(2,\mathbb{C})$$

be the local Langlands groups and let

$$\phi : \mathscr{L}_F \xrightarrow{\mathbb{1}_{\mathscr{W}_F} \otimes \tau(2)} PGL(2,\mathbb{C})$$

be the L-parameter, this parameter works as follows

$$(w, Y) \longmapsto [Y].$$

We know that $\mathscr{L}_E \subset \mathscr{L}_F$. The base change works by restriction the L-parameter to $\mathscr{W}_E$, in another words

$$\phi|_{\mathscr{W}_E} : \mathscr{L}_E \xrightarrow{\mathbb{1}_{\mathscr{W}_E} \otimes \tau(2)} PGL(2,\mathbb{C}) .$$

Since the restriction works only on the Weil group side which in our case is the trivial representation of $\mathscr{W}_F$ and since the restriction of the trivial representation of $\mathscr{W}_F$ is also the trivial representation of $\mathscr{W}_E$, then the resulting representation is also the Steinberg representation, i.e

$$BC(St_G(F)) = St_G(E).$$

$$\begin{array}{ccc} \phi : \mathscr{W}_F \times SL(2,\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) \\ \big\downarrow & & \big\downarrow \| \\ \phi|_{\mathscr{W}_E} : \mathscr{W}_E \times SL(2,\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) \end{array}$$

□

Theorem 2. *Let $\mathbb{T}$ be one of the circles in the unitary principal series of $SL(2,F)$, then we have $\mathbb{T} \to \mathbb{T}$, $z \mapsto z^f$, under base change E/F.*

i) At the level of the K-theory group K^1, BC induces the map $\mathbb{Z} \to \mathbb{Z}$, $\alpha_1 \mapsto f.\alpha_1$, where f is the residue field degree and α_1 denotes a generator of $K^1(\mathbb{T}) = \mathbb{Z}$.

ii) At the level of K-theory group K^0, BC induces the identity map $\mathbb{Z} \to \mathbb{Z}$, $\alpha_0 \mapsto \alpha_0$, where α_0 denotes a generator of $K^0(\mathbb{T}) = \mathbb{Z}$.

Proof. We know that the principal series of $SL(2,F)$ can be defined as follows:

$$Ind_B^{SL(2,F)}(\chi) \quad where \quad \chi\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} = \chi(x).$$

Now we have

$$\begin{array}{ccc} \mathscr{W}_F \times SL(2,\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) \\ \downarrow & & \downarrow \| \\ F^\times \times SL(2,\mathbb{C}) & \xrightarrow{\phi} & PGL_2(\mathbb{C}) \end{array}$$

This means the above map ϕ works as follows:

$$(x,\tau) \longmapsto \begin{bmatrix} \chi(x) & 0 \\ 0 & 1 \end{bmatrix}.$$

Here $\begin{bmatrix} \chi(x) & 0 \\ 0 & 1 \end{bmatrix}$ is the coset of $\begin{pmatrix} \chi(x) & 0 \\ 0 & 1 \end{pmatrix} \in PGL_2(\mathbb{C})$. If we twist χ by an unramified character we get a circle $\mathbb{T}$ embedded in $PGL_2(\mathbb{C})$. Also, the Weyl group $\mathbb{Z}/2\mathbb{Z}$ acts on character of $F^\times$, character of $F^\times = \mathscr{U}_F \times \langle\varpi_F\rangle$ splits into $\{$*ramified character of* $\mathscr{U}_F$ *say* $\chi_1\}$ and
$\{$*an unramified character of* $\langle\varpi_F\rangle$ *say* $\chi_0(\varpi) = z \in \mathbb{T}\}$. The generator w of $\mathbb{Z}/2\mathbb{Z}$ sends z to z^{-1}, it sends χ_1 to χ_1^{-1}. Suppose that

$$\chi_1 \neq \chi_1^{-1}, \quad \text{i.e.} \quad \chi_1^2 \neq 1.$$

For such χ, the representation $Ind_B^{SL(2,F)}\chi$ is irreducible. Define the L-parameter ϕ as follows: $\phi = \rho \otimes \mathbb{1}$ where ρ is a unitary character of $\mathscr{W}_F$ such that

$$\rho : \mathscr{W}_F \longrightarrow \mathscr{W}_F^{ab} \simeq F^\times \xrightarrow{\ \chi\ } \mathbb{T}\ .$$

Also, we have

$$\rho \longmapsto Ind_B^{SL(2,F)}\chi$$

The unitary characters ($\rho^2 \neq 1$) of $\mathscr{W}_F$ factor through $F^\times$ and we have

$$\widehat{F^\times} = \widehat{\langle\varpi\rangle} \times \widehat{\mathscr{U}_F},$$

ρ is a unitary character of $\widehat{\mathscr{U}_F}$. The group $\widehat{\mathscr{U}_F}$ admits countably many such characters ρ. Therefore, the compact orbit is the circle $\mathbb{T}$:

$$\mathfrak{O}^t(\phi) \cong \mathfrak{O}^t(BC(\phi)) \cong \mathbb{T}.$$

After restriction and using the local class functions theory we get that this map has degree f. Therefore, if $\chi^2 \neq 1$ this means by Lemma 1 and Theorem 2 in each circle the base change formula is $z \mapsto z^f$. □

3. Representatives in the Chamber Homology H_0

In this section we will investigate the case H_0. Since we have two types of representations for $SL(2,F)$ which are: the discrete series and the principal series representations, so we need to describe each case individually. The unitary principal series representation are as same as described in H_1. We need to deal with reducible principal series, the special representation and the discrete series. Let's start with the special representation. This means we are going to deal with the Steinberg representation. We recall the maximal compact subgroups J_0 and J_1, which were described in the previous section as the stabilizer subgroups of the vertices of the edge of the tree $\beta SL(2,F)$.

Theorem 3. *Let J_0 and J_1 be the two maximal compact open subgroups of $SL(2)$ and let I the Iwahori subgroup of $SL(2)$. There are only three generators for H_0 which are $\mathbb{1}_{J_0}$, $\mathbb{1}_{J_1}$, and the induced representation of $\mathbb{1}_I$ to J_0 or J_1.*

Proof. Let $\mathbb{1}_{J_0}$ (resp. $\mathbb{1}_{J_1}$) be a representation in $\mathfrak{R}(J_0)$ (resp. $\mathfrak{R}(J_1)$), so $[\mathbb{1}_{J_0},0]$ and $[0,\mathbb{1}_{J_1}] \in H_0$. We have

$$[Ind_I^{J_0}\mathbb{1}_I,0]=[0,Ind_I^{J_1}\mathbb{1}_I] \Longleftrightarrow \exists v \in \mathfrak{R}(I)$$

such that $(Ind_I^{J_0}\mathbb{1}_I,-Ind_I^{J_1}\mathbb{1}_I)=\partial(v)$.

This means we have only one possibility which is $v=\mathbb{1}_I$. Therefor three possibilities for H_0-generators are $\mathbb{1}_{J_0}$, $\mathbb{1}_{J_1}$, and $Ind_I^{J_0}\mathbb{1}_I$ (resp. $Ind_I^{J_1}\mathbb{1}_I$). □

The question here is which combination of these three generators correspond to the Steinberg representation St_G of $SL(2)$?

Theorem 4. *The 0-cycle corresponding to St_G of $SL(2)$ in K_0 is $(Ind_I^{J_0}\mathbb{1}_I-\mathbb{1}_{J_0},0)$.*

Proof. Let $G=SL(2,F)$ and $J_0=SL(2,\mathcal{O})$. According to the *Anh Reciprocity Theorem* in [5, p. 57], if $d\mu$ is a Haar measure then we have the following:

i) $Ind_I^G\mathbb{1}_I=\int_X \pi d\mu(\pi), X=\{\pi\in\widehat{G}_r:\pi|_I\supset\mathbb{1}_I\}$.

ii) $Ind_{J_0}^G\mathbb{1}_{J_0}=\int_Y \pi d\mu(\pi), Y=\{\pi\in\widehat{G}_r:\pi|_{J_0}\supset\mathbb{1}_{J_0}\}$.

Now,

$$\begin{aligned} Ind_I^G\mathbb{1}_I =& Ind_{J_0}^G\mathbb{1}_{J_0}\oplus St_G\\ \Longleftrightarrow & Ind_{J_0}^G(Ind_I^{J_0}\mathbb{1}_I)-Ind_{J_0}^G\mathbb{1}_{J_0}=St_G\\ \Longleftrightarrow & Ind_{J_0}^G(Ind_I^{J_0}\mathbb{1}_I-\mathbb{1}_{J_0})=St_G. \end{aligned}$$

Therefore the 0-cycle corresponding to St_G is $(Ind_I^{J_0}\mathbb{1}_I-\mathbb{1}_{J_0},0)$. We also see that the Baum-Connes conjecture (map) in this case is $Ind_{J_0}^G$. □

The proof of the above theorem shows that the map $Ind_{J_0}^G$ takes

$$[Ind_I^{J_0}\mathbb{1}_I-\mathbb{1}_{J_0},0]_F\mapsto[St_G]_F,$$

i.e. it takes the generator of H_0^F to the generator of K_0^F labeled by St_G. This means we have three independent elements. In the same way this map works on the E-sides by taking the

$$[Ind_I^{J_0}\mathbb{1}_I-\mathbb{1}_{J_0},0]_E\mapsto[St_G]_E.$$

From now on we will replace the notation of St_G by St_2^F and St_2^E to refer for the Steinberg representation of $SL(2,F)$ and $SL(2,E)$ respectively.

Theorem 5. *The base change on K_0-theory level takes the K_0-generator of the reduce C^*-algebra of $SL(2,E)$ labeled by St_2^E to the K_0-generator of the reduce C^*-algebra of $SL(2,F)$ labeled by St_2^F and the K-theory group $K_0\,C_r^*SL(2,F)=\mathbb{Z}^3$.*

Proof. From Theorems 3 and 4 we have only three generators and this implies that $K_0\,C_r^*SL(2,F)=\mathbb{Z}^3$. □

$$\begin{array}{ccc} \mathrm{H}_0(SL(2,E)) & \xrightarrow{\mu_0^E} & K_0C_r^*SL(2,E) \\ {\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_0(BC)} \\ \mathrm{H}_0(SL(2,F)) & \xrightarrow{\mu_0^F} & K_0C_r^*SL(2,F) \end{array}$$

Figure 1: The base change for $SL(2)$

On the chamber homology level, the base change map works by taking the generator of the group H_0 of $SL(2,E)$ to the generator of H_0 of $SL(2,F)$; see Figure 1.

On the other hand, if we deal with the reducible principal series (the intervals), this means we are going to induce the *Legendre* character to one of the maximal compact subgroups J_0, J_1 or both.

Theorem 6. *There are three generators for H_0 which they are constructed by inducing a representation of the* Legendre *character from I to the maximal subgroups J_0 and J_1.*

Proof. We know that if $\lambda^2=1$ then $Ind_B^{SL(2,\mathbb{F}_p)}\lambda=\lambda_B^+\oplus\lambda_B^-$. This means our induced representation can be written as decomposition of two representations. So if we induced to the maximal compact subgroups J_0, J_1 we would have three multiple choices. Let λ_I be any representation in $\mathfrak{R}(I)$, then $Ind_I^{J_0}\lambda_I$ (resp. $Ind_I^{J_1}\lambda_I$) is the induced representation of the *Legendre* character from I to J_0 (resp. J_1). Now, we have

$$Ind_I^{J_0}\lambda_I=\lambda_{J_0}^+\oplus\lambda_{J_0}^-$$

and

$$Ind_I^{J_1}\lambda_I=\lambda_{J_1}^+\oplus\lambda_{J_1}^-.$$

This means we have three generators for H_0 which are:

$$\lambda_{J_1}^+,\quad \lambda_{J_0}^+\quad\text{and}\quad \lambda_{J_0}^-\quad\text{or}\quad \lambda_{J_1}^-,\quad \lambda_{J_0}^+\quad\text{and}\quad \lambda_{J_0}^-.$$

This also shows that the assembly map $Ind_{J_0}^G$ works as follows

$$[Ind_I^{J_0}\lambda_I-\lambda_{J_0}^+,0]_F\mapsto[\lambda_{J_0}^-]_F$$

i.e. it takes the generator of H_0^F to the generator of K_0^F labeled by $\lambda_{J_0}^-$. In the same way this map works on the E-sides by taking the

$$[Ind_I^{J_0}\lambda_I-\lambda_{J_0}^+,0]_E\mapsto[\lambda_{J_0}^-]_E.$$

This means we have three independent elements. □

Theorem 7. *The base change on K_0-theory level takes the K_0-generator of the reduced C^*-algebra of $SL(2,E)$ labeled by $\lambda_{J_0(E)}^-$ to the K_0-generator of the reduce C^*-algebra of $SL(2,F)$ labeled by $\lambda_{J_0(F)}^-$. The K-theory group $K_0\,C_r^*SL(2,F)$ in this case is $\mathbb{Z}^3$.*

Proof. Theorem 6 shows that we have three generators for K_0 and this means $K_0 = \mathbb{Z}^3$. □

On the other hand, we introduce the cuspidal representations as follows:
let

$$\aleph = \rho \otimes \mathbb{1} : \mathscr{W}_F \times SL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$$

then we have

$$\aleph \mid_{\mathscr{W}_E} : \mathscr{W}_E \times SL(2,\mathbb{C}) \to PGL(2,\mathbb{C}).$$

Now let $\aleph$ be an irreducible representation.

i) If $\aleph \mid_{\mathscr{W}_E}$ remains irreducible after restriction, then this determines a cuspidal representation of $SL(2,E)$. Base change in this case, will send one cuspidal representation of $SL(2,F)$ to a cuspidal representation of $SL(2,E)$.

ii) If $\aleph \mid_{\mathscr{W}_E}$ is reducible, then this representation split into two 1-dimensional representations. i.e. $\aleph = \aleph_1 \oplus \aleph_2 = (\rho_1 \otimes \mathbb{1}) \oplus (\rho_2 \otimes \mathbb{1})$, where ρ_1 and ρ_2 are two characters of $\mathscr{W}_E$.

This means on the K-theory level there is one generator for each cuspidal representation and the $K_0 = \mathbb{Z}$.

4. Representatives in the Chamber Homology H_1

A description of the cycles in the group H_1 will be introduced in this section. Let
$G = SL(2,F)$ be the group of unimodular 2×2 matrices with entries in the field F. It is a locally compact totally disconnected topological group [8, 9].

Let $I = \begin{pmatrix} \mathscr{O} & \mathscr{O} \\ \varpi\mathscr{O} & \mathscr{O} \end{pmatrix} \cap SL(2)$. This is a compact open subgroup of G, called the Iwahori subgroup.

Let $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $w_1 = \begin{pmatrix} 0 & -\varpi^{-1} \\ \varpi & 0 \end{pmatrix}$. These elements appear in the Tits system associated to G, which plays an important role in what follows [3, 7].

Let

$$J_0 = I \cup I w_0 I = \begin{pmatrix} \mathscr{O} & \mathscr{O} \\ \mathscr{O} & \mathscr{O} \end{pmatrix} \cap SL(2)$$

and

$$J_1 = I \cup I w_1 I = \begin{pmatrix} \mathscr{O} & \varpi^{-1}\mathscr{O} \\ \varpi\mathscr{O} & \mathscr{O} \end{pmatrix} \cap SL(2),$$

these are compact open subgroups of G, we have $J_0 \cap J_1 = I$. The tree for $G = SL(2)$ is the graph βG, the group G acting on βG by multiplication on the left. We see that I is the stabilizer of the fundamental edge, and that J_0, J_1 are the stabilizer of the vertices of this edge, respectively.

Now if I, J_0 and J_1 are the compact subgroups of $G = SL(2)$ defined in the previous two paragraphs, then we have this chain complex

$$0 \longleftarrow \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \xleftarrow{Ind_I^{J_0} \oplus -Ind_I^{J_1} = \partial} \mathfrak{R}(I) \longleftarrow 0$$

So that

$$H_0 = \frac{\mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)}{\partial \mathfrak{R}(I)}$$

and

$$H_1 = \ker \partial.$$

For more details see [1].

Definition 1. *A character χ of $F^\times$ is called* tame *character if $\chi|_{\mathscr{U}_F^1}$ is trivial.*

Lemma 4. *Let χ be a tame character of I then $\chi - \chi^{-1} \in H_1$.*

Proof. Let χ be character of I, i.e.

$$I \xrightarrow{modp} \mathscr{B} \subset SL_2(\mathbb{F}_p)$$

$$\chi : SL(2) \cap \begin{pmatrix} \mathscr{O} & \mathscr{O} \\ p\mathscr{O} & \mathscr{O} \end{pmatrix} \longrightarrow \mathbb{T}, \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \longmapsto \chi(x).$$

Now let

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2), \quad w \in \mathbb{W}$$

where $\mathbb{W}$ is the Weyl group of $SL(2)$. We have $w\chi(x) = \chi(wxw^{-1})$. To prove that $\chi - w\chi \in H_1$ it is enough to show that $\chi - w\chi \in \mathfrak{R}(I)$, i.e. $\partial(\chi - w\chi) = 0$. In other words we need to show that

$$Ind_I^{J_0}(\chi - w\chi) = 0$$

and

$$Ind_I^{J_1}(\chi - w\chi) = 0.$$

Now choose $\chi \neq w\chi$, so

$$w\chi \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} := \chi \begin{pmatrix} x^{-1} & 0 \\ -y & x \end{pmatrix}.$$

This means $w\chi = \chi^{-1}$. Therefore we only need to prove $Ind_I^{J_0}(\chi - \chi^{-1}) = 0$ and $Ind_I^{J_1}(\chi - \chi^{-1}) = 0$. But $Ind_I^{J_0}(\chi - \chi^{-1}) = 0$ if and only if $Ind_I^{J_0}\chi \cong Ind_I^{J_0}\chi^{-1}$. Since χ and χ^{-1} are distinct, then they are determine the the same representation and this representation is irreducible if and only if $\chi^2 \neq 1$ [4]. This means $Ind_I^{J_0}\chi \cong Ind_I^{J_0}\chi^{-1}$. Therefore $Ind_I^{J_0}(\chi - \chi^{-1}) = 0$. Same results will be shown if we take J_1. This means we have $\chi - \chi^{-1} \in H_1$. □

Now let k be a positive integer, $I(k) = \begin{pmatrix} \mathscr{O} & \mathscr{O} \\ \varpi^k\mathscr{O} & \mathscr{O} \end{pmatrix} \cap SL(2, F)$. Consider now the subgroup $I(k)_w = I(k) \cap w^{-1}I(k)w = \begin{pmatrix} \mathscr{O} & \varpi^k\mathscr{O} \\ \varpi^k\mathscr{O} & \mathscr{O} \end{pmatrix} \cap SL(2)$. Let ψ be an invariant function on I and let

$$\alpha : SL(2) \to SL(2), \quad \alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & \varpi^{-1}c \\ \varpi b & a \end{pmatrix}.$$

Define $\psi^\alpha(g) = \psi(\alpha(g))$, then ψ induces to zero on J_1(resp. J_0) if and only if ψ^α induces to zero on J_0(resp. J_1). Therefore $\psi \in H_1$ if and only if $\psi^\alpha \in H_1$. Fix a character $\chi : \mathscr{U}_F \to \mathbb{T}$ not of order two, and let k be the least positive integer such that $\chi[1 + \varpi^k\mathscr{O}] = 1$. The character χ extends to the group $I(k)$ using the formula

$$\chi : \begin{pmatrix} a & b \\ \varpi^k c & d \end{pmatrix} \mapsto \chi(a).$$

Lemma 5. *Let $\psi_\chi = Ind_{I(k)}^I \chi$ then*

i) ψ_χ is an irreducible character.

ii) $\psi_{\chi}^{\alpha} = \psi_{\overline{\chi}}$, *where* α *is the automorphism of* $SL(2)$.

Now let $c_\chi = \psi_{\chi} - \psi_{\overline{\chi}}$. Then $c_\chi \in H_1$, and the cycle c_χ, one selected from each pair of characters $\{\chi, \overline{\chi}\}$, constitute a basis for H_1. Therefore all cycles will be of this form.

Theorem 8.

i) The base change on K_1*-theory level works as follows:*

$$K_1C_r^*SL(2,E) \xrightarrow{K_1(BC)} K_1C_r^*SL(2,F)\ , \quad \alpha_{\sigma\circ N_{E/F}} \longmapsto f\cdot\alpha_\sigma$$

ii) The base change on H_1*-level works as follows:*

$$(\beta SL(2,\underset{1}{E})) \xrightarrow{BC} H_1(\beta SL(2,F))\ , \quad c_{\sigma\circ N_{E/F}} \longmapsto f\cdot c_\sigma$$

where

$$\alpha_{\sigma\circ N_{E/F}}(resp.\alpha_{\sigma}) \quad and \quad c_{\sigma\circ N_{E/F}}(resp.c_\sigma)$$

are the K_1 *and* H_1 *generator of* $SL(2,E)\big(SL(2,F)\big)$ *respectively.*

Proof. By Theorem 2 this map has degree f. The base change on the K_1-theory level takes the K_1-generator of the reduce C^*-algebra of $SL(2,E)$ to the K_1-generator of the reduce C^*-algebra of $SL(2,F)$ multiplying by the residue field degree f, so (1) has been proved. We also know that the base change on the chamber homology side works by sending each unramified unitary character of the Iwahori subgroup of $SL(2,F)$ to itself composed with the norm map. So, the base change map on the chamber homology side takes the generator of the chamber homology group of $SL(2,E)$ (labeled by this composite) to the generator of the chamber homology group of $SL(2,F)$ (labeled by unramified unitary character) multiplying by the residue field degree f. □

Corollary 1. *The assembly map* $\mathrm{H}_1(SL(2,F)) \xrightarrow{\mu_1^F} K_1C_r^*SL(2,F)$ *under the base change works as follows:*

$$f\cdot c_\sigma \longmapsto f\cdot\alpha_\sigma$$

where c_σ *and* α_σ *are* H_1 *and* K_1 *generators for* $SL(2,F)$ *respectively.*

This corollary shows that the Baum-Connes conjecture under base change takes the base change's effect on the homology group side to the base change's effect on the K-theory side, i.e. the multiplication between the generator of the chamber homology group for $SL(2,F)$ and the residue field degree to the multiplication between the K_1-generator of the reduce C^*-algebra for $SL(2,F)$ by the residue field's degree:

$$\begin{array}{ccccccc}
\mathrm{H}_1(SL(2,E)) & \xrightarrow{\mu_1^E} & K_1C_r^*SL(2,E) & \quad & c_{\sigma\circ N_{E/F}} & \xrightarrow{\mu_1^E} & \alpha_{\sigma\circ N_{E/F}} \\
{\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_1(BC)} & & {\scriptstyle BC}\downarrow & & \downarrow{\scriptstyle K_1(BC)} \\
\mathrm{H}_1(SL(2,F)) & \xrightarrow{\mu_1^F} & K_1C_r^*SL(2,F) & & f\cdot c_\sigma & \xrightarrow{\mu_1^F} & f\cdot\alpha_\sigma
\end{array}\ .$$

References

[1] P. Baum, N. Higson, and R. Plymen. Equivariant homology for $SL(2)$ of a p-adic field. In *Index theory and operator algebras*, volume 148 of *Contemporary Mathematics*, pages 1–18. American Mathematical Society, Providence, Rhode Island, 1993.

[2] A. Borel and W. Casselman, editors. *Automorphic forms, representations, and L-functions. Part 2*. Proceedings of Symposia in Pure Mathematics, XXXIII. American Mathematical Society, Providence, Rhode Island, 1979.

[3] K. Brown. *Buildings*. Springer-Verlag, New York, New York, 1989.

[4] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[5] R.L. Lipsman. *Group representations*, volume 388 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Germany, 1974. A survey of some current topics.

[6] S. Mendes and R. Plymen. Base change and K-theory for $GL(n)$. *Journal of Noncommutative Geometry*, 1(3):311–331, 2007.

[7] J.P. Serre. *Trees*. Springer-Verlag, New York, March 2003.

[8] A.J. Silberger. *Introduction to harmonic analysis on reductive p-adic groups*, volume 23 of *Mathematical Notes*. Princeton University Press, Princeton, New Jersey, 1979. Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971-1973.

[9] G. van Dijk. *Harish-Chandra Harmonic analysis on reductive p-adic groups*. Lecture Notes in Mathematics, Vol. 162. Springer-Verlag, Berlin, Germany, 1970.

[10] A. Weil. *Basic number theory*. Springer, London, United Kingdom, 1995.