EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 1, No. 1, 2008, (82-98)
ISSN 1307-5543 – www.ejpam.com

Honorary Invited Paper

Some Forms of C-continuity for Multifunctions

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Abstract. Lipski [14] introduced the notion of c-quasi-continuous multifunctions as a generalization of c-continuous multifunctions [20] and quasi-continuous multifunctions [26]. In this paper we obtain the unified theory of multifunctions containing upper/lower c-quasi-continuous multifunctions and upper/lower c-continuous multifunctions.

AMS subject classifications: 54C08, 54C60.

Key words: c-continuous, c-quasi-continuous, upper/lower C-m -continuous, multifunction.

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1. Introduction

Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play an important role in researching of generalizations of continuity in topological spaces. By using these sets many authors introduced and investigated various types of noncontinuous functions and multifunctions. In 1970, Gentry and Hoyle III [9] defined a function $f : X \to Y$ to be c-continuous at a point $x \in X$ if for each open set V of Y containing $f(x)$ and having compact complement, there exists an open set U of X containing x such that $f(U) \subset V$. Some properties of c-continuous functions are studied in [15], [16], [24] and other papers. Neubrunn [20] and Holá et al. [11] extended this notion to the setting of multifunctions. In [14], Lipski introduced the notion of C -quasicontinuous multifunctions as a generalization of C-continuous multifunctions and quasi-continuous multifunctions [26]. Some properties of C-quasi-continuous multifunctions are studied in [36].

In this paper we introduce upper/lower C-m-continuous multifunctions as multifunctions defined on a set satisfying some minimal conditions. We obtain some characterizations and several properties of such multifunctions which turn out unify some results established in [11], [14] and [36]. In the last section, we recall some types of modifications of open sets and point out the possibility for new forms of C -continuous multifunctions.

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2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

α-open [22] (resp. *semi-open* [13], *preopen* [18], *β-open* [1] or *semi-preopen* [4], *b-open* [5]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).

The family of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets in X is denoted by SO(X) (resp. PO(X), $\alpha(X)$, $\beta(X)$, SPO(X), BO(X)).

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, β-open

, semi-preopen, b-open) set is said to be *semi-closed* [7] (resp. *preclosed* [8], *α-closed* [19], *β-closed* [1], *semi-preclosed* [4], *b-closed* [5]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-preclosed, b-closed) sets of X containing A is called the *semi-closure* [7] (resp. *preclosure* [8], *α-closure* [19], *β-closure* [2], *semi-preclosure* [4], *b-closure* [5]) of A and is denoted by sCl(A) (resp. pCl(A), αCl(A), ${}_\beta$Cl(A), spCl(A), bCl(A).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, semi-preopen, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *semi-preinterior*, *b-interior*) of A and is denoted by sInt(A) (resp. pInt(A), αInt(A), ${}_\beta$Int(A), spInt(A), bInt(A)).

Throughout the present paper, (X, τ) and (Y, σ) (briefly X and Y) always denote topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper semi-continuous* (briefly *u.s.c.*) at a point $x \in X$ if for each open set V containing $F(x)$, there exists an open set U of X containing x such that $F(U) \subset V$,

(2) *lower semi-continuous* (briefly *l.s.c.*) at a point $x \in X$ if for each open set V meeting $F(x)$, there exists an open set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower semi-continuous* on X if it has this property at each point of X.

Definition 2.6. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(a) *upper C-continuous* (briefly *u.c.c.*) [20] (resp. *upper C-quasi-continuous* (briefly *u.c.q.c.*) [14], [36]) if for each open set V containing $F(x)$ and having compact complement, there exists an open (resp. semi-open) set U of X containing x such that $F(U) \subset V$,

(2) *lower C-continuous* (briefly *l.c.c.*) [20] (resp. *lower C-quasi-continuous* (briefly *l.c.q.c.*) [14], [36]) at a point $x \in X$ if for each open set V meeting $F(x)$ and having compact complement, there exists an open (resp. semi-open) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower C-continuous* (resp. *upper/lower C-quasi-continuous*) on X if it has this property at each point of X.

Remark 2.1. For the multifunctions defined above, the following implications hold:

u.s.c. $\Rightarrow$ *u.c.c.* $\Rightarrow$ *u.c.q.c.*; *l.s.c.* $\Rightarrow$ *l.c.c.* $\Rightarrow$ *l.c.q.c.*

3. C-m-continuous multifunctions

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [31], [33] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, BO(X) and SPO(X) are all m-structures on X.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [17] as follows:
(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, BO(X), SPO(X)), then we have

(a) m_X-Cl(A) = Cl(A) (resp. sCl(A), pCl(A), αCl(A), bCl(A), spCl(A)),

(b) m_X-Int(A) = Int(A) (resp. sInt(A), pInt(A), αInt(A), bInt(A), spInt(A)).

Lemma 3.1. (Maki et al. [17]).
Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:

(1) m_X-Cl$(X - A) = X - m_X$-Int(A) *and* m_X-Int$(X - A) = X - m_X$-Cl(A),

(2) If $(X - A) \in m_X$, *then* m_X-Cl$(A) = A$ *and*
if $A \in m_X$, *then* m_X-Int$(A) = A$,

(3) m_X-Cl$(\emptyset) = \emptyset, m_X$-Cl$(X) = X$,
m_X-Int$(\emptyset) = \emptyset$ *and* m_X-Int$(X) = X$,

(4) If $A \subset B$, *then* m_X-Cl$(A) \subset m_X$-Cl(B) *and*
m_X-Int$(A) \subset m_X$-Int(B),

(5) $A \subset m_X$-Cl(A) *and* m_X-Int$(A) \subset A$,

(6) m_X-Cl$(m_X$-Cl$(A)) = m_X$-Cl(A) *and* m_X-Int$(m_X$-Int$(A)) = m_X$-Int(A).

Lemma 3.2. (Popa and Noiri [31]).
Let (X, m_X) *be an m-space and A a subset of X. Then* $x \in m_X$-Cl(A) *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m_X$ *containing x.*

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [17] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3. (Popa and Noiri [33]).
For an m-structure m_X *on a nonempty set* X, *the following properties are equivalent:*

(1) m_X *has property* $\mathcal{B}$;

(2) If m_X*-Int(*A*)* $= A$, *then* $A \in m_X$;

(3) If m_X-Cl$(A) = A$, *then* A *is* m_X*-closed.*

Definition 3.4. *Let* (X, m_X) *be an m-space and* (Y, σ) *a topological space.* A multifunction $F : (X, m_X) \to (Y, \sigma)$ is said to be

(1) *upper C-m-continuous* (briefly *u.C.m.c.*) at a point $x \in X$ if for each open set V containing $F(x)$ and having compact complement,
there exists an m_X-open set U containing x such that $F(U) \subset V$,

(2) *lower C-m-continuous* (briefly *l.C.m.c.*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having compact complement, there exists an m_X-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower C-m-continuous* on X if it has this property at every point of X.

Remark 3.3. Let (X, τ) and (Y, σ) be topological spaces.

(1) If $m_X = \tau$ (resp. SO(X)) and is upper/lower C-m-continuous, then F is upper/lower C-continuous (resp. upper/lower C-quasi-continuous).

(2) For m_X = $\alpha(X)$, PO(X), SPO(X) or BO(X), we can define new types of modifications of upper/lower C-continuous multifunctions. The definitions will be given in the last section.

Theorem 3.1. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.C.m.c. at $x \in X$;

(2) $x \in m_X$-Int$(F^+(V))$ for each open set V containing F(x) and having compact complement;

(3) $x \in F^-(\mathrm{Cl}(B))$ for each subset B of Y having the compact closure such that $x \in m_X$-Cl$(F^-(B))$;

(4) $x \in m_X$-Int$(F^+(B))$ for each subset B of Y such that $Y - \mathrm{Int}(B)$ is compact and $x \in F^+(\mathrm{Int}(B))$.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y containing $F(x)$ and having compact complement. There exists an m_X-open set U containing x such that $F(U) \subset V$. Thus $x \in U \subset F^+(V)$. Since $U \in m_X$, we have $x \in m_X$-Int$(F^+(V))$.

(2) $\Rightarrow$ (3): Suppose that B is any subset of Y having the compact closure. Then Cl(B) is closed and Y-Cl(B) is an open set having compact complement. Let $x \notin F^-(\mathrm{Cl}(B))$. Then $x \in X - F^-(\mathrm{Cl}(B)) = F^+(Y - \mathrm{Cl}(B))$. This implies $F(x) \subset Y - \mathrm{Cl}(B)$. Since $Y - \mathrm{Cl}(B)$ is an open set having compact complement, by (2) we have

$$x \in m_X\text{-Int}(F^+(Y - \mathrm{Cl}(B))) = m_X\text{-Int}(X - F^-(\mathrm{Cl}(B)) \\ = X - m_X\text{-Cl}(F^-(\mathrm{Cl}(B))) \subset X - m_X\text{-Cl}(F^-(B)).$$

Hence $x \notin m_X$-Cl$(F^-(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y such that Y -Int(B) is compact and let $x \notin m_X$-Int$(F^+(B))$. Then we have $x \in X - m_X\text{-Int}(F^+(B)) = m_X\text{-Cl}(X - F^+(B)) = m_X\text{-Cl}(F^-(Y - B))$. By (3), we have $x \in F^-(\mathrm{Cl}(Y - B)) = F^-(Y - \mathrm{Int}(B)) = X - F^+(\mathrm{Int}(B))$. Hence $x \notin F^+(\mathrm{Int}(B))$.

(4) $\Rightarrow$ (1): Let V be any open set of Y containing $F(x)$ and having compact complement. We have $F^+(V) = F^+(\mathrm{Int}(V))$. Then $Y - \mathrm{Int}(V) = Y - V$ which is compact and by (4) $x \in m_X$-Int($F^+(V)$). Therefore, there exists an m_X-open set U containing x such that $x \in U \subset F^+(V)$. Thus $F(U) \subset V$. This shows that F is *u.C.m.c.* at x.

Theorem 3.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is l.C.m.c. at $x \in X$;

(2) $x \in m_X$-Int$(F^-(V))$ for each open set V containing F(x) and having compact complement;

(3) $x \in F^+(\mathrm{Cl}(B))$ for each subset B of Y having the compact closure such that $x \in m_X$-Cl$(F^+(B))$;

(4) $x \in m_X$-Int$(F^-(B))$ for each B of Y such that $Y - \mathrm{Int}(B)$ is compact and $x \in F^-(\mathrm{Int}(B))$.

Proof. The proof is similar to that of Theorem 3.1

Theorem 3.3. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is u.C.m.c.;

(2) $F^+(V) = m_X$-Int$(F^+(V))$ for each open set V of Y having compact complement;

(3) $F^-(K) = m_X$-Cl$(F^-(K))$ for every compact closed set K of Y;

(4) m_X-Cl$(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y having the compact closure;

(5) $F^+(\mathrm{Int}(B)) \subset m_X$-Int$(F^+(B))$ for every subset B of Y such that $Y - \mathrm{Int}(B)$ is compact.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y having compact complement and $x \in F^+(V)$. Then $F(x) \subset V$ and by Theorem 3.1, $x \in m_X$-Int$(F^+(V))$. By Lemma 3.1, we have m_X-Int$(F^+(V)) \subset F^+(V)$. Therefore, we obtain $F^+(V) = m_X$-Int$(F^+(V))$.

(2) $\Rightarrow$ (3): Let K be any compact closed set of Y. Then, by Lemma 3.1 we have $X - F^-(K) = F^+(Y - K) = m_X$-Int$(F^+(Y - K)) = m_X$ -Int$(X - F^-(K)) = X - m_X$-Cl$(F^-(K))$. Therefore, we obtain $F^-(K) = m_X$-Cl$(F^-(K))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y having the compact closure. By Lemma 3.1, we have $F^-(B) \subset F^-(\mathrm{Cl}(B)) = m_X$-Cl$(F^-(\mathrm{Cl}(B)))$. Hence m_X-Cl$(F^-(B)) \subset m_X$-Cl$(F^-(\mathrm{Cl}(B))) = F^-(\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be a subset of Y such that $Y - \mathrm{Int}(B)$ is compact. Then by Lemma 3.1 we have

$$\begin{aligned} X - m_X\text{-Int}(F^+(B)) = m_X\text{-Cl}(X - F^+(B)) = m_X\text{-Cl}(F^-(Y - B)) \subset \\ \subset m_X\text{-Cl}(F^-(Y - \mathrm{Int}(B))) \subset F^-(Y - \mathrm{Int}(B)) = X - F^+(\mathrm{Int}(B)). \end{aligned}$$

Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset m_X$-Int$(F^+(B))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any open set of Y containing $F(x)$ and having compact complement. Then $x \in F^+(V) = F^+(\text{Int}(V)) \subset m_X\text{-Int}(F^+(V))$. By Theorem 3.1, F is *u.C.m.c.* at x.

Theorem 3.4. *For a multifunction , the following properties are equivalent:*

(1) F is l.C.m.c.;

(2) $F^-(V) = m_X\text{-Int}(F^-V))$ *for each open set V of Y having compact complement;*

(3) $F^+(K) = m_X\text{-Cl}(F^+(K))$ *is for every compact closed set K of Y;*

(4) $m_X\text{-Cl}(F^+(B)) \subset F^+(\text{Cl}(B))$ *for every subset B of Y having the compact closure;*

(5) $F^-(\text{Int}(B)) \subset m_X\text{-Int}(F^-(B))$ *for every subset B of Y such that* $Y - \text{Int}(B)$ *is compact.*

Proof. The proof is similar to that of Theorem 3.3.

Corollary 3.1. *Let* (X, m_X) *be an m-space and* m_X *have property* $\mathcal{B}$. *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is u.C.m.c. (resp. l.C.m.c.);

(2) $F^+(V)$ *(resp.* $F^-(V)$*) is* m_X*-open for each open set V of Y having compact complement;*

(3) $F^-(K)$ *(resp.* $F^+(K)$*) is* m_X*-closed for every compact closed set K of Y.*

Proof. This is an immediate consequence of Theorems 3.3 and 3.4 and Lemma 3.3.

Remark 3.4. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$ (resp. SO(X)) and is upper/lower C-m-continuous, then by Theorems 3.3 and 3.4 and Corollary 3.1 we obtain the results established in Proposition 1 of [11] (resp. Theorem 1 of [14], Theorems 3.3 and 3.4 of [36]).

Definition 3.5. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be
c-m-continuous if for each point $x \in X$ and each open set V containing $f(x)$ and having compact complement, there exists an m_X-open set U containing x such that $f(U) \subset V$.

Corollary 3.2. *For a function* $f : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) f is c-m-continuous;

(2) $f^{-1}(V) = m_X\text{-Int}(f^{-1}(V))$ *for each open set V of Y having compact complement;*

(3) $f^{-1}(K) = m_X\text{-Cl}(f^{-1}(K))$ *for every compact closed set K of Y;*

(4) m_X-Cl$(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset B of Y having the compact closure;*

(5) $f^{-1}(\mathrm{Int}(B)) \subset m_X$-Int$(f^{-1}(B))$ *for every subset B of Y such that* $Y - \mathrm{Int}(B)$ *is compact.*

Remark 3.5. Let (X, τ) and (Y, σ) be topological spaces. If $m_X = \tau$ and $f : (X, m_X) \to (Y, \sigma)$ is c-m-continuous, then by Corollary 3.2 we obtain the results established in Theorem 1 of [9] and Theorems 2 of [15].

Corollary 3.3. *A multifunction is u.C.m.c. (resp. l.C.m.c.) if* $F^-(K) = m_X$-Cl$(F^-(K))$ *(resp.* $F^+(K) = m_X$-Cl$(F^+(K))$*) for every compact set K of Y.*

Proof. Let G be any open set of Y having compact complement. Then $Y - G$ is a compact closed set. By the hypothesis, $X - F^+(G) = F^-(Y - G) = m_X$-Cl$(F^-(Y - G)) = m_X$-Cl$(X - F^+(G)) = X - m_X$-Int$(F^+(G))$ and hence, $F^+(G) = m_X$-Int$(F^+(G))$. It follows from Theorem 3.3 that F is *u.C.m.c.* The proof of lower C-m-continuity is entirely similar.

Remark 3.6. (1) Let $m_X = \tau$ (resp. SO(X)), then by Corollary 3.3 we obtain the results established in Proposition 2 of [20] (resp. Corollary 3.3 of [36]).

(2) It is shown in Remark 4 of [11] that the converse of Corollary 3.3 is not true.

Definition 3.6. A subset A of a topological space (X, τ) is said to be

(1) *α-paracompact* [40] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,

(2) *α-regular* [12] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

Lemma 3.4. (Kovačević [12])
If A is an α-regular α-paracompact set of a topological space X and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset \mathrm{Cl}(G) \subset U$.

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, by Cl$F : (X, m_X) \to (Y, \sigma)$ we denote a multifunction defined as follows: (ClF)(x) = Cl$(F(x))$ for each point $x \in X$. Similarly, we can define αClF, sClF, pClF, spClF, bClF.

Lemma 3.5. *If is a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$, then for each open set V of Y* $F^+(V) = G^+(V)$*, where G denotes* ClF*,* αClF*,* sClF*,* pClF*,* bClF *or* spClF*.*

Proof. The proof is similar to that of Lemma 3.3 of [30].

Theorem 3.5. *Let be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then the following properties are equivalent: (1) F is u.C.m.c.; (2)* Cl*F is u.C.m.c.; (3)* αCl*F is u.C.m.c.; (4)* sCl*F is u.C.m.c.; (5)* pCl*F is u.C.m.c.; (6)* bCl*F is u.C.m.c.; (7)* spCl*F is u.C.m.c.*

Proof. We set G = ClF, αClF, sClF, pClF, bClF or spClF. Suppose that F is *u.C.m.c.* Let V be any open set of Y containing $G(x)$ and having compact complement. By Lemma 3.5, we have $x \in G^+(V) = F^+(V)$ and by Theorem 3.1 there exists $U \in m_X$ containing x such that $F(U) \subset V$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, by Lemma 3.4 there exists an open set H such that $F(u) \subset H \subset \mathrm{Cl}(H) \subset V$; hence $G(u) \subset \mathrm{Cl}(H) \subset V$ for every $u \in U$. Therefore, we obtain $G(U) \subset V$. This shows that G is *u.C.m.c.*

Conversely, suppose that G is *u.C.m.c.* Let $x \in X$ and V be any open set of Y containing $F(x)$ and having compact complement. By Lemma 3.5, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. By Theorem 3.1, there exists $U \in m_X$ containing x such that $G(U) \subset V$. Therefore, we obtain $U \subset G^+(V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is *u.C.m.c.*

Lemma 3.6. *If is a multifunction, then for each open set V of Y $G^-(V) = F^-(V)$, where G =* Cl*F,* αCl*F,* sCl*F,* pCl*F,* bCl*F or* spCl*F.*

Proof. The proof is similar to that of Lemma 3.4 of [30].

Theorem 3.6. *For a multifunction , the following properties are equivalent: (1) F is l.C.m.c.; (2)* Cl*F is l.C.m.c.; (3)* αCl*F is l.C.m.c.; (4)* sCl*F is l.C.m.c.; (5)* pCl*F is l.C.m.c.; (6)* bCl*F is l.C.m.c.; (7)* spCl*F is l.C.m.c.*

Proof. By using Lemma 3.6 this is shown similarly as in Theorem 3.5.

Remark 3.7. Let (X, τ) and (Y, σ) be topological spaces and m_X = SO(X). By Theorems 3.5 and 3.6, we obtain the results established in Theorems 3.5 and 3.6 of [36].

4. The set of points of m-c-discontinuity

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the sets $D^+_{mc}(F)$ and $D^-_{mc}(F)$ are defined as follows:

$D^+_{mc}(F) = \{x \in X : F$ is not upper C-m-continuous at $x\}$,
$D^-_{mc}(F) = \{x \in X : F$ is not lower C-m-continuous at $x\}$.

Theorem 4.1. *For a multifunction , the following properties hold:*

$$\begin{aligned} D^+_{mc}(F) &= \bigcup_{G\in c\sigma}\{F^+(G) - [m_X\text{-Int}(F^+(G))]\} \\ &= \bigcup_{B\in\, i\mathcal{P}(Y)} \{F^+(\mathrm{Int}(B)) - [m_X\text{-Int}(F^+(B))]\} \\ &= \bigcup_{B\in\, c\mathcal{P}(Y)} \{m_X\text{-Cl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\, c\mathcal{F}} \{m_X\text{-Cl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where

$c\sigma$ is the family of open set G having compact complement,
$i\mathcal{P}(Y)$ is the family of subset B of Y such that $Y - \mathrm{Int}(B)$ is compact,
$c\mathcal{P}(Y)$ is the family of subset B of Y with the compact closure and
$c\mathcal{F}$ is the family of closed compact subsets of Y.

Proof. We shall show only the first equality and the last since the proof of any other equality is similar to the first.

Let $x \in D^+_{mc}(F)$. By Theorem 3.1, there exists an open set V of Y having compact complement such that $x \in F^+(V)$ and $x \notin m_X$- $\mathrm{Int}(F^+(V))$. Therefore, we obtain $x \in F^+(V) - [m_X$-$\mathrm{Int}(F^+(V))] \subset \bigcup_{G\in c\sigma}\{F^+(G) - [m_X\text{-Int}(F^+(G))]\}$. Conversely, let $x \in \bigcup_{G\in c\sigma}\{F^+(G) - [m_X\text{ -Int}(F^+(G))]\}$. There exists $V \in c\sigma$ such that $x \in F^+(V) - [m_X\text{-Int}(F^+(V))]$. By Theorem 3.1, we obtain $x \in D^+_{mc}(F)$.

We prove the last equality.
$\bigcup_{H\in\, c\mathcal{F}} \{m_X\text{-Cl}(F^-(H)) - F^-(H)\} \subset \bigcup_{B\in\, c\mathcal{P}(Y)} \{m_X\text{-Cl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} = D^+_{mc}(F)$.
Conversely, by Lemma 3.1 we have

$$\begin{aligned} D^+_{mc}(F) &= \bigcup_{B\in\, c\mathcal{P}(Y)} \{m_X\text{- Cl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &\subset \bigcup_{H\in\, c\mathcal{F}} \{m_X\text{-Cl}(F^-(H)) - F^-(H)\}. \end{aligned}$$

Theorem 4.2. *For a multifunction , the following properties hold:*

$$\begin{aligned} D^-_{mc}(F) &= \bigcup_{G\in c\sigma}\{F^-(G) - [m_X\text{-Int}(F^-(G))]\} \\ &= \bigcup_{B\in\, i\mathcal{P}(Y)} \{F^-(\mathrm{Int}(B)) - [m_X\text{-Int}(F^-(B))]\} \\ &= \bigcup_{B\in\, c\mathcal{P}(Y)} \{m_X\text{-Cl}(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \bigcup_{H\in\, c\mathcal{F}} \{m_X\text{-Cl}(F^+(H)) - F^+(H)\}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 4.1

Remark 4.1. If is a multifunction and $m_X = \tau$ (resp. SO(X)), then the set of points of upper/lower C-discontinuity (resp. c-quasi-discontinuity) is obtained.

Definition 4.1. Let (X, m_X) be an m-space and A a subset of X. The m_X-$frontier$ of A [35], denoted by m_X-Fr(A), is defined as follows:

$$m_X\text{-Fr}(A) = m_X\text{-Cl}(A) \cap m_X\text{-Cl}(X - A) = m_X\text{-Cl}(A) - m_X\text{-Int}(A).$$

Theorem 4.3. *The set of all points $x \in X$ at which a function is not u.C.m.c. (resp. l.C.m.c.) is identical with the union of the m_X-frontiers of the u.C.m.c. (resp. l.C.m.c.) inverse images of open sets containing (resp. meeting) F(x) and having compact complement.*

Proof. Suppose that F is not *u.C.m.c.* at $x \in X$. Then, there exists an open set V of Y containing $F(x)$ and having compact complement such that $U \cap (X - F^+(V)) \neq \emptyset$ for every m_X-open set U containing x. Hence, by Lemma 3.2 we have $x \in m_X$-Cl$(X - F^+(V))$. On the other hand, we have $x \in F^+(V) \subset m_X$-Cl$(F^+(V))$ and hence $x \in m_X$-Fr$(F^+(V))$.

Conversely, suppose that V is an open set of Y containing $F(x)$ and having compact complement such that $x \in m_X$-Fr$(F^+(V))$. If F is *u.C.m.c.* at $x \in X$, then there exists $U \in m_X$ containing x such that $U \subset F^+(V)$ and hence, $x \in m_X$-Int$(F^+(V))$. This is a contradiction and hence, F is not *u.C.m.c.* The proof for *l.C.m.c.* is similar.

5. m-continuity and C-m-continuity

Definition 5.1. A multifunction is said to be

(1) *upper m-continuous* (briefly *u.m.c.*) at $x \in X$ [34] if for each open set V containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower m-continuous* (briefly *l.m.c.*) at $x \in X$ [34] if for each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) *upper/lower m-continuous* on X if it has the properties at each point of X.

Remark 5.1. Let (X, τ) be a topological space and $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, SPO(X), BO(X)). If a multifunction is upper/lower m-continuous, then F is upper/lower continuous (resp. upper/lower semi-continuous [27] or upper/lower quasi-continuous [28], upper/lower precontinuous [29], upper/lower α-continuous [21], upper/lower β-continuous [30], upper/lower b-continuous or upper/lowerγ-continuous [3]).

A topological space (Y, σ) is called a *KC-space* [39] if every compact set of Y is closed.

Definition 5.2. A multifunction is said to be *m-bounded* at the point $p \in X$ if there exists $U \in m_X$ containing p and a compact set C of Y such that $F(x) \subset C$ for each $x \in U$.

Theorem 5.1. *Let (Y, σ) be a KC space and X a nonempty set with two minimal structures m^1_X and m^2_X such that $U \cap V \in m^2_X$ for every $U \in m^1_X$ and $V \in m^2_X$. Then $F : (X, m^2_X) \to (Y, \sigma)$ is u.m.c. (resp. l.m.c.) at $p \in X$ if the following conditions satisfy:*

(1) $F : (X, m^1_X) \to (Y, \sigma)$ is m-bounded at $p \in X$,

(2) $F : (X, m^2_X) \to (Y, \sigma)$ is u.C.m.c. (resp. l.C.m.c.) at $p \in X$.

Proof. We prove only the first case, the proof of the second being entirely analogous. Let $U \in m^1_X$ containing p and C be a compact set of Y such that $F(x) \subset C$ for each $x \in U$. Let V be any open set of Y such that $F(p) \subset V$. Put $G = V \cup (Y - C)$. Then G is open and $Y - G$ is compact. By the condition (2), there exists $W \in m^2_X$ containing p such that $F(x) \subset G$ for every $x \in W$. Put $H = W \cap U$, then $H \in m^2_X$ containing p and $F(x) \subset G \cap C$ for any $x \in H$. Then $F(x) \subset V$ for any $x \in H$. Therefore, $F : (X, m^2_X) \to (Y, \sigma)$ is *u.m.c.* at $p \in X$.

Remark 5.2. If $m^1_X = m^2_X = \tau$, then by Theorem 5.1 we obtain the result established in Proposition 5 of [11].

Definition 5.3. An m-space (X, m_X) is said to be *m-saturated* if for any $x \in X$ the intersection of all m_X-open sets containing x is m_X-open.

Theorem 5.2. *Let (X, m_X) be an m-saturated m-space and (Y, σ) a T_1-space. If is u.C.m.c., then F is u.m.c.*

Proof. Suppose that F is not *u.m.c.* at some point $x_0 \in X$. There exists an open set V of Y such that $F(x_0) \subset V$ and $F(U) \cap (Y - V) \neq \emptyset$ for every $U \in m_X$ containing x_0. Let U_0 be the intersection of all m_X-open sets containing x_0. Then $U_0 \in m_X$ and there exists $z_1 \in U_0$ such that $F(z_1) \cap (Y - V) \neq \emptyset$. Hence there exists $y \in F(z_1) \cap (Y - V)$. The set $Y - \{y\}$ is an open set with compact complement. Since $F(x_0) \subset Y - \{y\}$ and F is *u.C.m.c.* at x_0, there exists $G \in m_X$ containing x_0 such that for any $x \in G$ we have $F(x) \subset Y - \{y\}$. This is a contradiction. Since $U_0 \subset G$, $z_1 \in G$ and $F(z_1) \subset Y - \{y\}$. This contradicts that $y \in F(z_1)$.

Remark 5.3. If $m_X = \tau$, then by Theorem 5.2 we obtain the result established in Proposition 8 of [11].

Theorem 5.3. *Let (X, m_X) be an m-saturated m-space and (Y, σ) a locally compact Hausdorff space. If is an u.C.m.c. and closed valued multifunction, then F is u.m.c.*

Proof. Suppose that F is not *u.m.c.* at $x_0 \in X$. Then, there exists an open set V of Y such that $F(x_0) \subset V$ and $F(U) \cap (Y - V) \neq \emptyset$ for every $U \in m_X$ containing x_0. Let U_0 be the intersection of all m_X-open sets containing x_0. Then $U_0 \in m_X$ and there exists $z_1 \in U_0$ such that $F(z_1) \cap (Y - V) \neq \emptyset$. Hence there exists $y \in F(z_1) \cap (Y - V)$. Since (Y, σ) is locally compact Hausdorff, (Y, σ) is regular. Since $F(x_0)$ is a closed set and $y \notin F(x_0)$, there exists an open set W containing y such that $\mathrm{Cl}(W)$ is a compact set and $\mathrm{Cl}(W) \subset Y - F(x_0)$. Since $F(x_0) \subset Y - \mathrm{Cl}(W)$ and F is *u.C.m.c.* at x_0, there exists an m_X-open set G containing x_0 and $F(x) \subset Y - \mathrm{Cl}(W)$ for each $x \in G$. This is a contradiction. Since $z_1 \in U_0 \subset G$, $F(z_1) \subset Y - \mathrm{Cl}(W)$. This contradicts that $F(z_1) \cap \mathrm{Cl}(W) \neq \emptyset$.

Remark 5.4. If $m_X = \tau$, then by Theorem 5.3 we obtain the result established in Proposition 10 of [11].

Theorem 5.4. *Let (X, m_X) be an m-saturated m-space and (Y, σ) a KC space. If is l.C.m.c. and for each $x \in X$ there exists a compact set C_x such that $F(x) \subset C_x$, then F is l.m.c.*

Proof. Suppose that F is not *l.m.c.* at $x_0 \in X$. Then, there exists an open set V of Y such that $F(x_0) \cap V \neq \emptyset$ and for each $U \in m_X$ containing x_0 there exists $u \in U$ such that $F(u) \cap V = \emptyset$. Let U_0 be the intersection of all m_X-open sets containing x_0. Then $U_0 \in m_X$ and there exists $x \in U_0$ such that $F(x) \cap V = \emptyset$. By the hypothesis, there exists a compact set C_x such that $F(x) \subset C_x$. Therefore, we have $F(x) \subset C_x - V$ and $C_x - V$ is a compact set.The set $Y - (C_x - V)$ is open and $F(x_0) \cap (Y - (C_x - V)) \neq \emptyset$. Since F is *l.C.m.c.* at x_0, there exists an m_X-open set G containing x_0 such that for any $z \in G$ we have $F(z) \cap (Y - (C_x - V)) \neq \emptyset$. This is a contradiction because $x \in U_0 \subset G$ and $F(x) \subset C_x - V$.

Remark 5.5. If $m_X = \tau$, then by Theorem 5.4 we obtain the result established in Proposition 11 of [11].

6. Some properties

Definition 6.1. A multifunction is said to be *upper C-m-rarely continuous* at a point $x \in X$ if for each open set G of Y containing $F(x)$ and having compact complement, there exists a rare set R_G with $\mathrm{Cl}(R_G) \cap G = \emptyset$ and an m_X-open set U containing x such that $F(U) \subset G \cup R_G$. A multifunction is said to be *upper C-m-rarely continuous* if it has this property at each point $x \in X$.

Theorem 6.1. *Let X be a nonempty set with two minimal structures m_X^1 and m_X^2 such that $U \cap V \in m_X^2$ for every $U \in m_X^1$ and $V \in m_X^2$. Then $F : (X, m_X^2) \to (Y, \sigma)$ is u.C.m.c. if the following conditions satisfy:*

(1) $F : (X, m_X^1) \to (Y, \sigma)$ is upper C-m-rarely continuous and

(2) for each open set G containing F(x) and having compact complement, $F^-(\mathrm{Cl}(R_G))$ is an m_X^2-closed set of X, where R_G is the rare set of Definition 6.1.

Proof. Let $x \in X$ and G be any open set of Y containing $F(x)$ and having compact complement. By the condition (1), there exists $V \in m_X^1$ containing x and a rare set R_G with $\mathrm{Cl}(R_G) \cap G = \emptyset$ such that $F(V) \subset G \cup R_G$. If we suppose that $x \in F^-(\mathrm{Cl}(R_G))$, then $\mathrm{Cl}(R_G) \cap G \neq \emptyset$. This is a contradiction. Thus $x \notin F^-(\mathrm{Cl}(R_G))$. Put $U = V \cap (X - F^-(\mathrm{Cl}(R_G)))$. Then $U \in m_X^2$ and $x \in U$ since $x \in V$ and $x \in X - F^-(\mathrm{Cl}(R_G))$. Let $u \in U$, then $F(u) \subset G \cup R_G$ and $F(u) \cap \mathrm{Cl}(R_G) = \emptyset$. Therefore, we have $F(u) \cap R_G = \emptyset$ and hence, $F(u) \subset G$ for each $u \in U$. Since $U \in m_X^2$ containing x, it follows that $F : (X, m_X^2) \to (Y, \sigma)$ is *u.C.m.c.*

Definition 6.2. For a multifunction , the graph $\mathrm{G}(F) = \{(x, F(x)) : x \in X\}$ is said to be *strongly m-closed* [32] if for each $(x, y) \in (X \times Y) - \mathrm{G}(F)$, there exist an m_X-open set U containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(F) = \emptyset$.

Lemma 6.1. *A multifunction has a strongly m-closed graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exist an m_X-open set U containing x and an open set V of Y containing y such that $F(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Theorem 6.2. *Let (Y, σ) be a locally compact Hausdorff space. If a multifunction is u.C.m.c. and $F(x)$ is closed for each $x \in X$, then $\mathrm{G}(F)$ is strongly m-closed.*

Proof. Let $(x, y) \in (X \times Y) - \mathrm{G}(F)$. Then $y \notin F(x)$. Since Y is locally compact Hausdorff, Y is regular. Since $F(x)$ is a closed set and $y \notin F(x)$, there exists an open set V in Y containing y such that $\mathrm{Cl}(V)$ is a compact set and $\mathrm{Cl}(V) \subset X - F(x)$ and hence, $F(x) \subset Y - \mathrm{Cl}(V)$. Since F is *u.C.m.c.* at x and $Y - \mathrm{Cl}(V)$ is an open set having compact complement, there exists $U \in m_X$ containing x such that $F(U) \subset Y - \mathrm{Cl}(V)$. This implies that $F(U) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 6.1 $\mathrm{G}(F)$ is strongly m-closed.

7. New modifications of C-continuous multifunctions

For modifications of open sets defined in Definition 2.1, the following relationships are known:

$$\begin{array}{ccccc} \text{open} \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} & \\ & \Downarrow & & \Downarrow & \\ & \text{semi-open} & \Rightarrow & b\text{-open} & \Rightarrow \text{semi-preopen} \end{array}$$

First, we can define the following modifications of upper/lower C -continuous multifunctions.

Definition 7.1. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper C-α-continuous* (resp. *upper C-precontinuous*, *upper C-b-continuous*, *upper C-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having compact complement, there exists an α-open (resp. preopen, b-open, semi-preopen) set U containing x such that $F(U) \subset V$,

(2) *lower C-α-continuous* (resp. *lower C-precontinuous*, *lower C-b-continuous*, *lower C-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having compact complement, there exists an α-open (resp. preopen, b-open, semi-preopen) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower C-α-continuous* (resp. *upper/lower C-precontinuous*, *upper/lower C-b-continuous*, *upper/lower C-sp-continuous*) on X if it has this property at each $x \in X$.

For multifunctions defined in Definition 7.1, the following relationships hold:

$$\begin{array}{ccccc} \text{upper } C\text{-con.} \Rightarrow & \text{upper } C\text{-}\alpha\text{-con.} & \Rightarrow & \text{upper } C\text{-precon.} & \\ & \Downarrow & & \Downarrow & \\ \text{upper } C\text{-quasi-con.} \Rightarrow & \text{upper } C\text{-}b\text{-con.} & \Rightarrow & \text{upper } C\text{-}sp\text{-con.} & \end{array}$$

Remark 7.1. In the diagram above, "con." means continuity and the analogous diagram holds for the case "lower".

Let define the further modifications of upper/lower C-continuous multifunctions. For the purpose, we recall the definitions of the θ -closure and the δ-closure due to Veličko [38]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$) [38]. A subset A is said to be *θ-closed* (resp. *δ-closed*) if $\mathrm{Cl}_\theta(A) = A$ (resp. $\mathrm{Cl}_\delta(A) = A$). The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*). The union of all θ-open (resp. δ-open) sets contained in the subset A is called the θ-interior (resp. δ-interior) of A and is denoted by $\mathrm{Int}_\theta(A)$ (resp. $\mathrm{Int}_\delta(A)$).

Definition 7.2. A subset A of a topological space (X, τ) is said to be
(1) *δ-semiopen* [25] (resp. *θ-semiopen* [6]) if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}_\theta(A))$),
(2) *δ-preopen* [37] (resp. *θ-preopen* [23]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),
(3) *δ-sp-open* [10] (resp. *θ-sp-open* [23]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$).

By $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{PO}(X)$, $\delta\mathrm{SPO}(X)$, $\theta\ \mathrm{SO}(X)$, $\theta\mathrm{PO}(X)$, $\theta\mathrm{SPO}(X)$), we denote the collection of all δ-semiopen (resp. δ-preopen, δ-sp-open, θ -semiopen, θ-preopen, θ-sp-open) sets of a topological space (X, τ). These six collections are all m-structures with property $\mathcal{B}$. It is known that the families of all θ-open sets and δ-open sets of (X, τ) are topologies for X, respectively. In [23] and [6], the following relationships are known:

$$\begin{array}{ccccccccccc} \theta\text{-open} & \Rightarrow & \delta\text{-open} & \Rightarrow & \text{open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \theta\text{-semiopen} & \Rightarrow & \delta\text{-semiopen} & \Rightarrow & \text{semi-open} & \Rightarrow & sp\text{-open} & \Rightarrow & \delta\text{-}sp\text{-open} & \Rightarrow & \theta\text{-}sp\text{-open} \end{array}$$

Definition 7.3. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper C-θ-continuous* (resp. *upper C-θ-precontinuous*, *upper C-θ-semi-continuous*, *upper C-θ-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having compact complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-*sp*-open) set U containing x such that $F(U) \subset V$,

(2) *lower C-θ-continuous* (resp. *lower C-θ-precontinuous*, *lower C-θ-semi-continuous*, *lower C-θ-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having compact complement, there exists a θ-open (resp. θ-preopen, θ-semiopen, θ-*sp*-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower C-θ-continuous* (resp. *upper/lower C-θ-precontinuous*, *upper/lower C-θ-semi-continuous*, *upper/lower C-θ-sp-continuous*) on X if it has this property at each $x \in X$.

Definition 7.4. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper C-δ-continuous* (resp. *upper C-δ-precontinuous*, *upper C-δ-semi-continuous*, *upper C-δ-sp-continuous*) at a point $x \in X$ if for each open set V containing $F(x)$ and having compact complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-*sp*-open) set U containing x such that $F(U) \subset V$,

(2) *lower C-δ-continuous* (resp. *lower C-δ-precontinuous*, *lower C-δ-semi-continuous*, *lower C-δ-sp-continuous*) at a point $x \in X$ if for each open set V meeting $F(x)$ and having compact complement, there exists a δ-open (resp. δ-preopen, δ-semiopen, δ-*sp*-open) set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$,

(3) *upper/lower C-δ-continuous* (resp. *upper/lower C-δ-precontinuous*, *upper/lower C-δ-semi-continuous*, *upper/lower C-δ-sp-continuous*) on X if it has this property at each $x \in X$.

For the multifunctions defined above, the following diagram hold, where c. means continuity.

$$\begin{array}{ccccccccccc} \text{u/l C.}\theta\text{-c.} & \Rightarrow & \text{u/l C. }\delta\text{-c.} & \Rightarrow & \text{u/l C.c.} & \Rightarrow & \text{u/l C.p.c.} & \Rightarrow & \text{u/l C.}\delta\text{-p.c.} & \Rightarrow & \text{u/l C.}\theta\text{ -p.c.} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{u/l C.}\theta\text{-s.c.} & \Rightarrow & \text{u/l C.}\delta\text{-s.c.} & \Rightarrow & \text{u/l C.q.c.} & \Rightarrow & \text{u/l C.sp.c.} & \Rightarrow & \text{u/l C.}\delta\text{-sp.c.} & \Rightarrow & \text{u/l C.}\theta\text{-sp.c.} \end{array}$$

Conclusion. We can apply the results established in Sections 3 - 6 to all multifunctions defined in Definitions 7.1, 7.2 and 7.3.

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