Fractional Helmholtz and Fractional Wave Equations with Riesz-Feller and Generalized Riemann-Liouville Fractional Derivatives

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Abstract. The objective of this paper is to derive analytical solutions of fractional order Laplace, Poisson and Helmholtz equations in two variables derived from the corresponding standard equations in two dimensions by replacing the integer order partial derivatives with fractional Riesz-Feller derivative and generalized Riemann-Liouville fractional derivative recently defined by Hilfer. The Fourier-Laplace transform method is employed to obtain the solutions in terms of Mittag-Leffler functions, Fox H-function and an integral operator containing a Mittag-Leffler function in the kernel. Results for fractional wave equation are presented as well. Some interesting special cases of these equations are considered. Asymptotic behavior and series representation of solutions are analyzed in detail. Many previously obtained results can be derived as special cases of those presented in this paper.

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1. Introduction

Fractional differential equations have been used in different fields of science. To mention a few examples: fractional relaxation equations have applications in the non-exponential relaxation theory [11, 13, 23–25]; fractional diffusion [31, 32] and fractional Fokker-Planck equations [30], as well as fractional master equations [10, 16, 33, 41], in the description of anomalous diffusive processes; fractional wave equations has been used in the theory of vibrations of smart materials in media where the memory effects can not be neglected [20, 21]; etc.

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In the present paper, we introduce a new generalization of time-independent diffusion/wave equations, i.e. fractional Laplace, fractional Poisson and fractional Helmholtz equations in two variables in which both space variables x and y are of fractional orders. We use fractional Riesz-Feller space derivative [6] for the first variable, and generalized Riemann-Liouville (R-L) fractional derivative [11, 14, 49] for the second variable. Space-time fractional wave equation with Riesz-Feller space derivative and generalized R-L fractional time derivative is considered as well. Similar time-dependent models are discussed earlier by many authors, such as Haubold *et al*. [28], Saxena [41], Saxena *et al*. [42–44], Tomovski *et al*. [49, 51–53], etc.

Such generalized R-L time fractional derivative (or so-called Hilfer-composite fractional time derivative in [7, 15, 26, 38, 50, 54]) was used by Hilfer [11, 12], Sandev *et al*. [38] and Tomovski *et al*. [54] in the analysis of fractional diffusion equations, obtaining that such models may be used in context of glass relaxation and aquifer problems. Hilfer-composite time fractional derivative was also used by Saxena *et al*. [45] and Garg *et al*. [8] in the theory of fractional reaction-diffusion equations, where the obtained results are presented through Mittag-Leffler (M-L) and Fox H-functions. Furthermore, an operational method for solving differential equations with the Hilfer-composite fractional derivative is presented in [14, 19]. From the other side, Riesz-Feller fractional derivative has been used in analysis of space-time fractional diffusion equations by Mainardi, Pagnini and Saxena [27] and Tomovski *et al*. [54], where they expressed the solutions in terms of Fox H-function. It is shown that space fractional diffusion equation with fractional Riesz-Feller space derivative [3] gives same results as those obtained from the continuous time random walk theory for Lévy flights [31, 32]. A numerical scheme for solving fractional diffusion equation with Hilfer-composite fractional time derivative and Riesz-Feller space fractional derivative is elaborated in [54]. Furthermore, the quantum fractional Riesz-Feller derivative has been used by Luchko *et al*. [22, 40] in the Schrödinger equation for a free particle and a particle in an infinite potential well. Local fractional derivative operators have been used as well [9] in Helmholtz and diffusion equations.

The paper is organized as following. In Section II we give an introduction to the fractional derivatives and integrals used in the paper. Fractional form of the Laplace and Poisson equations in two variables are considered in Section III. We give analytical results for different forms of the boundary conditions and for the source term. Asymptotic behavior and series representation of solutions are given. We also give remarks on the general space-time fractional wave equation for a vibrating string with fractional Riesz-Feller space derivative and Hilfer-composite fractional time derivative. In Section IV we analyze the fractional Helmholtz equation for different forms of the boundary conditions and source term. The obtained results are of general character and include those recently given by Thomas [48]. Conclusions are given in Section V. At the end of the paper in an Appendix we give definitions, relations, and some properties of M-L functions and Fox H-function.

2. Fractional Derivatives and Integrals

The Riesz-Feller fractional derivative of order α and skewness θ is defined by the following Fourier transform formula [6]

$$\mathscr{F}\left[{}_xD_\theta^\alpha f(x)\right](\kappa) = -\psi_\alpha^\theta(\kappa)\mathscr{F}\left[f(x)\right](\kappa), \tag{1}$$

where

$$\mathscr{F}\left[f(x)\right](\kappa) = \hat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{\iota\kappa x}dx, \text{ and}$$

$$\mathscr{F}^{-1}\left[\hat{f}(\kappa)\right](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\kappa)e^{-\iota\kappa x}d\kappa, \tag{2}$$

are Fourier transform and inverse Fourier transform, respectively, and $\psi_\alpha^\theta(\kappa)$ is given by

$$\psi_\alpha^\theta(\kappa) = |\kappa|^\alpha \exp\left[\iota\,\mathrm{sign}(\kappa)\frac{\theta\pi}{2}\right], \quad 0 < \alpha \le 2, \quad |\theta| \le \min\{\alpha, 2-\alpha\}. \tag{3}$$

Riesz-Feller fractional derivative is a pseudo-differential operator whose symbol $-\psi_\alpha^\theta(\kappa)$ is the logarithm of the characteristic function of a general Lévy strictly stable probability density with stability index α and asymmetry parameter θ (for details, see Mainardi, Pagnini and Saxena [27]). For $\theta = 0$ one obtains Riesz fractional derivative ${}_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}$, for which

$$\mathscr{F}\left[{}_xD_0^\alpha f(x)\right](\kappa) = -|\kappa|^\alpha \mathscr{F}\left[f(x)\right](\kappa). \tag{4}$$

This special case has been used in the theory of Lévy flights [31, 32].

In this paper we also use the quantum fractional Riesz-Feller derivative ${}_xD_\theta^{*,\alpha}$ of order α and skewness θ, which is defined as a pseudo-differential operator with a symbol $\psi_\alpha^\theta(\kappa)$ given by [22, 40]

$$\mathscr{F}\left[{}_xD_\theta^{*,\alpha}f(x)\right](\kappa) = \psi_\alpha^\theta(\kappa)\mathscr{F}\left[f(x)\right](\kappa). \tag{5}$$

Note that the quantum fractional Riesz-Feller derivative is the Riesz-Feller fractional derivative (1) multiplied by -1. Thus, the obtained solutions which correspond to the case of fractional Riesz-Feller space derivative (5) can be easily transformed to those obtained in a case where the quantum fractional Riesz-Feller space derivative (1) is applied.

The R-L fractional integral is defined by [11, 18, 36]

$$\left({}_yI_{a+}^\mu f\right)(y) = \frac{1}{\Gamma(\mu)}\int_a^y \frac{f(y')}{(y-y')^{1-\mu}}dy', \quad y > a, \quad \Re(\mu) > 0. \tag{6}$$

For $\mu = 0$, this is the identity operator, $\left({}_yI_{a+}^0 f\right)(y) = f(y)$. Similarly, R-L fractional derivative is defined by [11, 18, 36]

$$\left({}_yD_{a+}^\mu f\right)(y) = \left(\frac{d}{dy}\right)^n \left({}_yI_{a+}^{n-\mu} f\right)(y), \quad \Re(\mu) > 0, \quad n = \left[\Re(\mu)\right] + 1, \tag{7}$$

where $[\Re(\mu)]$ denotes the integer part of the real number $\Re(\mu)$. Hilfer generalized the fractional derivative (7) by the following fractional derivative of order $0 < \mu \leq 1$ and type $0 \leq \nu \leq 1$ [11]:

$$\left({}_y D_{a+}^{\mu,\nu} f \right)(y) = \left({}_y I_{a+}^{\nu(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}y} \left({}_y I_{a+}^{(1-\nu)(1-\mu)} f \right) \right)(y). \tag{8}$$

Note that when $0 < \mu \leq 1$, $\nu = 0$, $a = 0$, the generalized R-L fractional derivative (8) would correspond to the classical R-L fractional derivative [11, 18, 36]

$$\left({}^{RL}{}_y D_{0+}^{\mu} f \right)(y) = \frac{\mathrm{d}}{\mathrm{d}y} \left({}_y I_{0+}^{(1-\mu)} f \right)(y). \tag{9}$$

Conversely, when $0 < \mu \leq 1$, $\nu = 1$, $a = 0$, it corresponds to the Caputo fractional derivative [2]

$$\left({}^{C}{}_y D_{0+}^{\mu} f \right)(y) = \left({}_y I_{0+}^{(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}y} f \right)(y). \tag{10}$$

The difference between fractional derivatives of different types becomes apparent when we consider their Laplace transform. In Ref. [11] it is found for $0 < \mu < 1$ that

$$\mathscr{L}\left[{}_y D_{0+}^{\mu,\nu} f(y) \right](s) = s^{\mu} \mathscr{L}\left[f(y) \right](s) - s^{\nu(\mu-1)} \left({}_y I_{0+}^{(1-\nu)(1-\mu)} f \right)(0+), \tag{11}$$

where the initial-value term $\left({}_y I_{0+}^{(1-\nu)(1-\mu)} f \right)(0+)$ is evaluated in the limit $y \to 0+$, in the space of summable Lebesgue integrable functions

$$L(0, \infty) = \left\{ f : \|f\|_1 = \int_0^{\infty} |f(y)| \mathrm{d}y < \infty \right\}. \tag{12}$$

Hilfer, Luchko and Tomovski generalized Hilfer-composite derivative (8) to order $n-1 < \mu \leq n$ $(n \in N^+)$ and type $0 \leq \nu \leq 1$ in the following way [14]:

$$\left({}_y D_{a+}^{\mu,\nu} f \right)(y) = \left({}_y I_{a+}^{\nu(n-\mu)} \frac{\mathrm{d}^n}{\mathrm{d}y^n} \left({}_y I_{a+}^{(1-\nu)(n-\mu)} f \right) \right)(y). \tag{13}$$

Its Laplace transform is recently given by Tomovski [49]

$$\mathscr{L}\left[{}_y D_{0+}^{\mu,\nu} f(y) \right](s) = s^{\mu} \mathscr{L}\left[f(y) \right](s) - \sum_{k=0}^{n-1} s^{n-k-\nu(n-\mu)-1} \left[\frac{\mathrm{d}^k}{\mathrm{d}y^k} \left({}_y I_{0+}^{(1-\nu)(n-\mu)} f \right) \right](0+), \tag{14}$$

where initial-value terms $\left[\frac{\mathrm{d}^k}{\mathrm{d}y^k} \left({}_y I_{0+}^{(1-\nu)(n-\mu)} f \right) \right](0+)$ are evaluated in the limit $y \to 0+$.

Various operators for fractional integration were investigated by Srivastava and Saxena [46]. Srivastava and Tomovski [47] introduced an integral operator $(\mathscr{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa} \varphi)(y)$ of form

$$\left({}_y \mathscr{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa} \varphi \right)(y) = \int_a^y (y - \xi)^{\beta-1} E_{\alpha,\beta}^{\gamma,\kappa}(\omega(y - \xi)^{\alpha}) \varphi(\xi) \mathrm{d}\xi, \tag{15}$$

where $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ is the four parameter M-L function (A5). In case when $\omega = 0$ the integral operator (15) would correspond to the classical R-L integral operator. For $\kappa = 1$ integral operator (15) becomes the Prabhakar integral operator $\left({}_{y}\mathscr{E}_{a+;\alpha,\beta}^{\omega;\gamma}\varphi\right)(y)$ [35], which was extensively investigated by Kilbas, Saigo and Saxena [17], and will be used here with $\gamma = 1$ for representation of solutions. These generalized integral operators was shown to appear in the expression of solutions of fractional diffusion/wave equations with source terms [38, 39, 51–53].

3. Fractional Laplace and Fractional Poisson Equations

In this section we investigate generalized form of the Laplace equation for the field variable $N(x,y)$ in two dimensions

$$\frac{\partial^2}{\partial x^2}N(x,y) + \frac{\partial^2}{\partial y^2}N(x,y) = 0, \tag{16}$$

on the upper half plane $y \geq 0$ and $-\infty < x < \infty$, with boundary conditions

$$N(x,0+) = f(x), \quad \frac{\mathrm{d}}{\mathrm{d}y}N(x,0+) = g(x), \tag{17a}$$

$$\lim_{x\to\pm\infty} N(x,y) = 0. \tag{17b}$$

Since there is no dependence on time variable, Laplace equation gives the steady-state solution of, for example, diffusion/heat conduction and wave equations. Thus, initial conditions are not required, only we use boundary conditions, which may be defined in a different ways. Therefore, Laplace equation in two dimensions (16) may arise in analysis of two dimensional steady-state diffusion/heat conduction, static deflection of a membrane, electrostatic potential, etc.

If in the Laplace equation in two dimensions (16) we add a source term $\Phi(x,y)$, then it becomes Poisson equation

$$\frac{\partial^2}{\partial x^2}N(x,y) + \frac{\partial^2}{\partial y^2}N(x,y) = \Phi(x,y). \tag{18}$$

This equation has applications in different field of science, such as gravitation theory, electromagnetism, elasticity, etc. For example, $N(x,y)$ may be interpreted as a temperature field variable subject to external force (source) $\Phi(x,y)$.

Before to formulate the corresponding fractional form of the Laplace equation (16) and Poisson equation (18) we prove the following Lemmas.

Lemma 1. *Let $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, $\varsigma \geq 0$ and $\hat{r}(\kappa)$ is a given function. Then the following relation holds true*

$$\mathscr{L}^{-1}\left[\frac{s^{\varsigma-\nu(2-\mu)}}{s^{\mu} \pm \hat{r}(\kappa)}\right](y) = y^{1-(1-\nu)(2-\mu)-\varsigma}E_{\mu,2-(1-\nu)(2-\mu)-\varsigma}\left(\mp\hat{r}(\kappa)y^{\mu}\right), \tag{19}$$

where $E_{\alpha,\beta}(z)$ is the two parameter M-L function (A2).

Proof. From relation (A3), we directly prove Lemma 1. □

Lemma 2. *Let $1 < \mu \leq 2$ and $\hat{r}(\kappa)$ and $\hat{\Phi}(\kappa, y)$ are given functions. Then the following relation holds true*

$$\mathscr{L}^{-1}\left[\frac{1}{s^{\mu} \pm \hat{r}(\kappa)}\mathscr{L}\left[\hat{\Phi}(\kappa, y)\right](\kappa, s)\right](\kappa, y) = \left(\mathscr{E}_{0+;\mu,\mu}^{\mp\hat{r}(\kappa);1}\hat{\Phi}\right)(\kappa, y), \tag{20}$$

where $\mathscr{E}_{0+;\mu,\mu}^{\mp\hat{r}(\kappa);1}\tilde{\Phi}$ is the Prabhakar integral operator (see definition (15)) and $\hat{\Phi}(\kappa, y)$ is a given function.

Proof. From relation (A3) it follows that

$$\frac{1}{s^{\mu} \pm \hat{r}(\kappa)} = \mathscr{L}\left[y^{\mu-1}E_{\mu,\mu}\left(\mp\hat{r}(\kappa)y^{\mu}\right)\right](\kappa, s). \tag{21}$$

Thus by applying the convolution theorem of the Laplace transform one obtains

$$\mathscr{L}^{-1}\left[\frac{1}{s^{\mu} \pm \hat{r}(\kappa)}\mathscr{L}\left[\hat{\Phi}(\kappa, t)\right](\kappa, s)\right](\kappa, t) = \int_0^y (y-\xi)^{\mu-1}E_{\mu,\mu}^1\left(\mp\hat{r}(\kappa)(y-\xi)^{\mu}\right)\hat{\Phi}(\kappa, \xi)d\xi, \tag{22}$$

from where we obtain the proof of Lemma 2. □

Theorem 1. *The solution of the following fractional Poisson equation*

$$_xD_{\theta}^{\alpha}N(x, y) + {}_yD_{0+}^{\mu,\nu}N(x, y) = \Phi(x, y), \tag{23}$$

where $x \in R$, $y \in R^+$, $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, with boundary conditions

$$\left({}_yI_{0+}^{(1-\nu)(2-\mu)}N\right)(x, 0+) = f(x), \quad \left(\frac{\mathrm{d}}{\mathrm{d}y}\left({}_yI_{0+}^{(1-\nu)(2-\mu)}N\right)\right)(x, 0+) = g(x), \tag{24a}$$

$$\lim_{x \to \pm\infty} N(x, y) = 0, \tag{24b}$$

is given by

$$\begin{aligned}
N(x, y) =& \frac{y^{-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{f}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa \\
&+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{g}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^y (y-\xi)^{\mu-1}E_{\mu,\mu}\left((y-\xi)^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{\Phi}(\kappa, \xi)e^{-\iota\kappa x}\mathrm{d}\xi\mathrm{d}\kappa \\
=& \frac{y^{-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{f}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa
\end{aligned}$$

$$+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{g}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left({}_{y}\mathscr{E}_{0+;\mu,\mu}^{\psi_{\alpha}^{\theta}(\kappa);1}\hat{\hat{\Phi}}\right)(\kappa,y)e^{-\iota\kappa x}\mathrm{d}\kappa, \tag{25}$$

where $\hat{\Phi}(\kappa,y) = \mathscr{F}\left[\Phi(x,y)\right](\kappa,y)$.

Proof. From the Laplace transform (14) to equation (23) and the boundary conditions (24a), it follows

$$_{x}D_{\alpha}^{\theta}\tilde{N}(x,s) + s^{\mu}\tilde{N}(x,s) - s^{1-\nu(2-\mu)}f(x) - s^{-\nu(2-\mu)}g(x) = \tilde{\Phi}(x,s), \tag{26}$$

where $\tilde{N}(x,s) = \mathscr{L}\left[N(x,y)\right](x,s)$. By applying Fourier transform (5) to relation (26) we find

$$\hat{\tilde{N}}(\kappa,s) = \frac{s^{1-\nu(2-\mu)}}{s^{\mu} - \psi_{\alpha}^{\theta}(\kappa)}\hat{f}(\kappa) + \frac{s^{-\nu(2-\mu)}}{s^{\mu} - \psi_{\alpha}^{\theta}(\kappa)}\hat{g}(\kappa) + \frac{1}{s^{\mu} - \psi_{\alpha}^{\theta}(\kappa)}\hat{\tilde{\Phi}}(\kappa,s), \tag{27}$$

where $\hat{\tilde{\Phi}}(\kappa,s) = \mathscr{F}\left[\tilde{\Phi}(x,s)\right](\kappa,s)$. Employing the results from Lemma 1 and Lemma 2, by inverse Fourier transform we obtain solution (25). Thus, we finish with the proof of Theorem 1. $\square$

Remark 1. *If in equation* (23) *instead of fractional Riesz-Feller derivative we use quantum fractional Riesz-Feller derivative we obtain the following equation*

$$_{x}D_{\theta}^{*,\alpha}N(x,y) + {}_{y}D_{0+}^{\mu;\nu}N(x,y) = \Phi(x,y). \tag{28}$$

For same boundary conditions as those used in Theorem 1, we obtain the solution in the following form

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(-y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{f}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa$$

$$+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(-y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{g}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left({}_{y}\mathscr{E}_{0+;\mu,\mu}^{-\psi_{\alpha}^{\theta}(\kappa);1}\hat{\hat{\Phi}}\right)(\kappa,y)e^{-\iota\kappa x}\mathrm{d}\kappa. \tag{29}$$

Corollary 1. *If we consider source term of form* $\Phi(x,y) = \delta(x)\frac{y^{-\beta}}{\Gamma(1-\beta)}$, *the solutions of fractional equations* (23) *and* (28) *are given by*

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(\pm y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{f}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa$$

$$+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(\pm y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{g}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa$$

$$+\frac{y^{\mu-\beta}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,\mu-\beta+1}\left(\pm y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)e^{-\iota\kappa x}d\kappa$$

$$=\frac{y^{-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,1-(1-\nu)(2-\mu)}\left(\pm y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{f}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{1-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,2-(1-\nu)(2-\mu)}\left(\pm y^{\mu}\psi_{\alpha}^{\theta}(\kappa)\right)\hat{g}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{\mu-\beta}}{|x|}H_{3,3}^{2,1}\left[\mp\frac{|x|^{\alpha}}{y^{\mu}e^{\iota\frac{\theta\pi}{2}}}\;\middle|\;\begin{array}{l}(1,1),(1+\mu-\beta,\mu),(1,\frac{\alpha}{2})\\(1,\alpha),(1,1),(1,\frac{\alpha}{2})\end{array}\right],\tag{30}$$

where the upper signs in the solution correspond to the case of fractional Riesz-Feller derivative and lower signs to quantum fractional Riesz-Feller derivative.

Example 1. *If we consider $\Phi(x,y)=\delta(x)\delta(y)$ and boundary conditions $f(x)=\delta(x)$, $g(x)=0$, for $\theta=0$, from relations (A8) and (A9), we obtain solutions (25) and (29) in terms of Fox H-functions*

$$N(x,y)=\frac{y^{-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,1-(1-\nu)(2-\mu)}\left(\pm y^{\mu}|\kappa|^{\alpha}\right)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{\mu-1}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,\mu}\left(\pm y^{\mu}|\kappa|^{\alpha}\right)e^{-\iota\kappa x}d\kappa$$

$$=\frac{y^{-(1-\nu)(2-\mu)}}{|x|}H_{3,3}^{2,1}\left[\mp\frac{|x|^{\alpha}}{y^{\mu}}\;\middle|\;\begin{array}{l}(1,1),(1-(1-\nu)(2-\mu),\mu),(1,\frac{\alpha}{2})\\(1,\alpha),(1,1),(1,\frac{\alpha}{2})\end{array}\right]$$

$$+\frac{y^{\mu-1}}{|x|}H_{3,3}^{2,1}\left[\mp\frac{|x|^{\alpha}}{y^{\mu}}\;\middle|\;\begin{array}{l}(1,1),(\mu,\mu),(1,\frac{\alpha}{2})\\(1,\alpha),(1,1),(1,\frac{\alpha}{2})\end{array}\right],\tag{31}$$

where the upper signs in the solution correspond to the case of fractional Riesz-Feller derivative and lower signs to quantum fractional Riesz-Feller derivative. Moreover, for $\alpha=2$, solution (31) in case of quantum fractional Riesz-Feller derivative becomes

$$N(x,y)=\frac{y^{-(1-\nu)(2-\mu)}}{2|x|}H_{2,2}^{2,0}\left[\frac{|x|}{y^{\mu/2}}\;\middle|\;\begin{array}{l}(1-(1-\nu)(2-\mu),\frac{\mu}{2}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{2})\end{array}\right]$$

$$+\frac{y^{\mu-1}}{2|x|}H_{2,2}^{2,0}\left[\frac{|x|}{y^{\mu/2}}\;\middle|\;\begin{array}{l}(\mu,\frac{\mu}{2}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{2})\end{array}\right]$$

$$=\frac{y^{-(1-\nu)(2-\mu)}}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{y^{\mu/2}}\;\middle|\;\begin{array}{l}(1-(1-\nu)(2-\mu),\frac{\mu}{2})\\(1,1)\end{array}\right]$$

$$+\frac{y^{\mu-1}}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{y^{\mu/2}}\;\middle|\;\begin{array}{l}(\mu,\frac{\mu}{2})\\(1,1)\end{array}\right].\tag{32}$$

Remark 2. *For the asymptotic behavior of solution (32) for $\frac{|x|}{y^{\mu/2}}\gg 1$, we obtain*

$$N(x,y)\simeq\frac{\left(\frac{\mu}{2}\right)^{1-2\nu+\frac{1}{2-\mu}}}{2\sqrt{(2-\mu)\pi}}\frac{y^{-(1-\nu)(2-\mu)}}{|x|}\left(\frac{|x|}{y^{\mu/2}}\right)^{\frac{1+2(1-\nu)(2-\mu)}{2-\mu}}$$

$$\times \exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}}\left(\frac{|x|}{y^{\mu/2}}\right)^{\frac{2}{2-\mu}}\right]$$

$$+\frac{\left(\frac{\mu}{2}\right)^{\frac{1-2\mu}{2-\mu}}}{2\sqrt{(2-\mu)\pi}}\frac{y^{\mu-1}}{|x|}\left(\frac{|x|}{y^{\mu/2}}\right)^{\frac{1-2\mu}{2-\mu}}\exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}}\left(\frac{|x|}{y^{\mu/2}}\right)^{\frac{2}{2-\mu}}\right], \qquad (33)$$

where we employ relations (A10), (A11), (A12), (A13) and (A14).

Remark 3. From the series representation (A7) of Fox H-function, we obtain the following series representation of solution (32)

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)-\frac{\mu}{2}}}{2}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma\left(1-(1-\nu)(2-\mu)-\frac{\mu}{2}(j+1)\right)}\left(\frac{|x|}{y^{\mu/2}}\right)^j$$

$$+\frac{y^{\frac{\mu}{2}-1}}{2}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma\left(-\mu j\right)}\left(\frac{|x|}{y^{\mu/2}}\right)^j$$

$$=\frac{y^{-(1-\nu)(2-\mu)-\frac{\mu}{2}}}{2}\phi\left(-\frac{\mu}{2},1-(1-\nu)(2-\mu)-\frac{\mu}{2};-\frac{|x|}{y^{\mu/2}}\right)$$

$$+\frac{y^{\frac{\mu}{2}-1}}{2}\phi\left(-\mu,0;-\frac{|x|}{y^{\mu/2}}\right), \qquad (34)$$

from where by using the first few terms of the series (34) we can obtain the asymptotic behavior for $\frac{|x|}{y^{\mu/2}}\ll 1$. Here, $\phi(a,b;z)$ is the Wright function (A16).

Example 2. For $\Phi(x,y)=\delta(x)\delta(y)$ and boundary conditions $f(x)=0$, $g(x)=\delta(x)$, for $\theta=0$, from (A8) and (A9), we obtain the solutions (25) and (29) in the following form

$$N(x,y) = \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,2-(1-\nu)(2-\mu)}\left(\pm y^{\mu}|\kappa|^{\alpha}\right)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{\mu-1}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,\mu}\left(\pm y^{\mu}|\kappa|^{\alpha}\right)e^{-\iota\kappa x}d\kappa$$

$$=\frac{y^{1-(1-\nu)(2-\mu)}}{|x|}H_{3,3}^{2,1}\left[\mp\frac{|x|^{\alpha}}{y^{\mu}}\left|\begin{array}{c}(1,1),(2-(1-\nu)(2-\mu),\mu),(1,\frac{\alpha}{2})\\(1,\alpha),(1,1),(1,\frac{\alpha}{2})\end{array}\right.\right]$$

$$+\frac{y^{\mu-1}}{|x|}H_{3,3}^{2,1}\left[\mp\frac{|x|^{\alpha}}{y^{\mu}}\left|\begin{array}{c}(1,1),(\mu,\mu),(1,\frac{\alpha}{2})\\(1,\alpha),(1,1),(1,\frac{\alpha}{2})\end{array}\right.\right], \qquad (35)$$

where the upper signs in the solution correspond to the case of fractional Riesz-Feller derivative and lower signs to quantum fractional Riesz-Feller derivative. For $\alpha=2$ solution (35) in case of quantum fractional Riesz-Feller derivative becomes

$$N(x,y) = \frac{y^{1-(1-\nu)(2-\mu)}}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{y^{\mu/2}}\left|\begin{array}{c}(2-(1-\nu)(2-\mu),\frac{\mu}{2})\\(1,1)\end{array}\right.\right]$$

$$+ \frac{y^{\mu-1}}{2|x|} H_{1,1}^{1,0} \left[\frac{|x|}{y^{\mu/2}} \, \middle| \, \begin{matrix} (\mu, \frac{\mu}{2}) \\ (1,1) \end{matrix} \right]. \tag{36}$$

The asymptotic behavior and series representation of this solution can be found in a same way as it was done in Remark 6 and Remark 7.

Corollary 2. *The solution of the following fractional form of the Laplace equation*

$$_x D_\theta^\alpha N(x,y) + _y D_{0+}^{\mu,\nu} N(x,y) = 0, \tag{37}$$

where $x \in R$, $y \in R^+$, $1 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2-\alpha\}$, $1 < \mu \le 2$, $0 \le \nu \le 1$, *with boundary conditions*

$$\left(_y I_{0+}^{(1-\nu)(2-\mu)} N \right)(x, 0+) = f(x), \quad \left(\frac{d}{dy} \left(_y I_{0+}^{(1-\nu)(2-\mu)} N \right) \right)(x, 0+) = g(x), \tag{38a}$$

$$\lim_{x \to \pm\infty} N(x,y) = 0, \tag{38b}$$

is given by

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu, 1-(1-\nu)(2-\mu)} \left(y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{f}(\kappa) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu, 2-(1-\nu)(2-\mu)} \left(y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{g}(\kappa) e^{-\iota\kappa x} d\kappa, \tag{39}$$

where $\hat{f}(\kappa) = \mathcal{F}\left[f(x) \right](\kappa)$, $\hat{g}(\kappa) = \mathcal{F}\left[g(x) \right](\kappa)$.

Proof. The proof of Corrolary 2 follows directly from Theorem 1 if we substitute $\Phi(x,y) = 0$. $\square$

Remark 4. *If in equation (37) instead of fractional Riesz-Feller derivative we use quantum fractional Riesz-Feller derivative we obtain the following equation*

$$_x D_\theta^{*,\alpha} N(x,y) + _y D_{0+}^{\mu,\nu} N(x,y) = \Phi(x,y). \tag{40}$$

For same boundary conditions as those in Theorem 2, the solution of equation (40) is given by

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu, 1-(1-\nu)(2-\mu)} \left(-y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{f}(\kappa) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu, 2-(1-\nu)(2-\mu)} \left(-y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{g}(\kappa) e^{-\iota\kappa x} d\kappa. \tag{41}$$

Corollary 3. *Solutions of equation (37) and (40) in case of Caputo fractional derivative ($\nu = 1$), become*

$$N(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_\mu \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{f}(\kappa) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2} \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{g}(\kappa) e^{-\iota\kappa x} d\kappa, \tag{42}$$

and for R-L fractional derivative $(\nu = 0)$

$$N(x,y) = \frac{y^{\mu-2}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu-1} \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{f}(\kappa) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y^{\mu-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu} \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) \hat{g}(\kappa) e^{-\iota\kappa x} d\kappa, \tag{43}$$

where the upper signs in the solution correspond to the case of fractional Riesz-Feller derivative and lower signs to quantum fractional Riesz-Feller derivative.

Example 3. *If we use the following boundary conditions $f(x) = \delta(x)$ and $g(x) = 0$, solutions (39) and (41) are given by*

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)} \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) e^{-\iota\kappa x} d\kappa, \tag{44}$$

which for Caputo fractional derivative become

$$N(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_\mu \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) e^{-\iota\kappa x} d\kappa, \tag{45}$$

and for R-L fractional derivative

$$N(x,y) = \frac{y^{\mu-2}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu-1} \left(\pm y^\mu \psi_\alpha^\theta(\kappa) \right) e^{-\iota\kappa x} d\kappa, \tag{46}$$

where it is used that $\mathscr{F}[\delta(x)] = 1$, and we use the upper signs in the solution in case of fractional Riesz-Feller derivative and lower signs in case of quantum fractional Riesz-Feller derivative.

Remark 5. *From relation between M-L and Fox H-function (A8), by using Mellin-cosine transform formula (A9), for the solution (44) we find*

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{\pi} \int_0^\infty \cos(\kappa x) H_{1,2}^{1,1} \left[\mp y^\mu e^{\iota\frac{\theta\pi}{2}} |\kappa|^\alpha \left| \begin{array}{c} (0,1) \\ (0,1), ((1-\nu)(2-\mu),\mu) \end{array} \right. \right] d\kappa$$

$$= \frac{y^{-(1-\nu)(2-\mu)}}{|x|} H_{3,3}^{2,1} \left[\mp \frac{|x|^\alpha}{y^\mu e^{\iota\frac{\theta\pi}{2}}} \left| \begin{array}{c} (1,1), (1-(1-\nu)(2-\mu),\mu), (1,\frac{\alpha}{2}) \\ (1,\alpha), (1,1), (1,\frac{\alpha}{2}) \end{array} \right. \right], \tag{47}$$

which for $\theta = 0$ becomes

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{|x|} H_{3,3}^{2,1} \left[\mp \frac{|x|^\alpha}{y^\mu} \left| \begin{array}{c} (1,1), (1-(1-\nu)(2-\mu),\mu), (1,\frac{\alpha}{2}) \\ (1,\alpha), (1,1), (1,\frac{\alpha}{2}) \end{array} \right. \right]. \tag{48}$$

Note that for $\mu = 2$, $\nu = 1$ solution (48) *in case of quantum fractional Riesz-Feller derivative is given by*

$$N(x, y) = \frac{1}{\alpha |x|} H^{2,1}_{3,3} \left[\frac{|x|}{y^{2/\alpha}} \,\middle|\, \begin{array}{c} (1, \frac{1}{\alpha}), (1, \frac{2}{\alpha}), (1, \frac{1}{2}) \\ (1, 1), (1, \frac{1}{\alpha}), (1, \frac{1}{2}) \end{array} \right]. \tag{49}$$

and for $\alpha = 2$ by

$$N(x, y) = \frac{y^{-(1-\nu)(2-\mu)}}{2|x|} H^{2,0}_{2,2} \left[\frac{|x|}{y^{\mu/2}} \,\middle|\, \begin{array}{c} (1 - (1-\nu)(2-\mu), \frac{\mu}{2}), (1, \frac{1}{2}) \\ (1, 1), (1, \frac{1}{2}) \end{array} \right]$$

$$= \frac{y^{-(1-\nu)(2-\mu)}}{2|x|} H^{1,0}_{1,1} \left[\frac{|x|}{y^{\mu/2}} \,\middle|\, \begin{array}{c} (1 - (1-\nu)(2-\mu), \frac{\mu}{2}) \\ (1, 1) \end{array} \right], \tag{50}$$

where we apply the definition (A6) *and the known properties of H-function* [29].

Remark 6. *From the results in Remark 2, for the asymptotic behavior of solution* (50) *for $\frac{|x|}{y^{\mu/2}} \gg 1$, we obtain*

$$N(x, y) \simeq \frac{\left(\frac{\mu}{2}\right)^{1 - 2\nu + \frac{1}{2-\mu}}}{2\sqrt{(2-\mu)\pi}} \frac{y^{-(1-\nu)(2-\mu)}}{|x|} \left(\frac{|x|}{y^{\mu/2}} \right)^{\frac{1 + 2(1-\nu)(2-\mu)}{2-\mu}}$$

$$\times \exp \left[-\frac{2-\mu}{2} \left(\frac{\mu}{2} \right)^{\frac{\mu}{2-\mu}} \left(\frac{|x|}{y^{\mu/2}} \right)^{\frac{2}{2-\mu}} \right]. \tag{51}$$

Remark 7. *The series representation of solution* (50) *is given by*

$$N(x, y) = \frac{y^{-(1-\nu)(2-\mu)-\frac{\mu}{2}}}{2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma\left(1 - (1-\nu)(2-\mu) - \frac{\mu}{2}(j+1)\right)} \left(\frac{|x|}{y^{\mu/2}} \right)^j$$

$$= \frac{y^{-(1-\nu)(2-\mu)-\frac{\mu}{2}}}{2} \phi \left(-\frac{\mu}{2}, 1 - (1-\nu)(2-\mu) - \frac{\mu}{2}; -\frac{|x|}{y^{\mu/2}} \right). \tag{52}$$

By using the first few terms of series (52), *we can obtain the asymptotic behavior of solution* (50) *for $\frac{|x|}{y^{\mu/2}} \ll 1$.*

Example 4. *For the following boundary conditions $f(x) = 0$ and $g(x) = \delta(x)$, solutions* (39) *and* (41) *are given by*

$$N(x, y) = \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu, 2-(1-\nu)(2-\mu)} \left(\pm y^{\mu} \psi^{\theta}_{\alpha}(\kappa) \right) e^{-\iota \kappa x} d\kappa$$

$$= \frac{y^{1-(1-\nu)(2-\mu)}}{|x|} H^{2,1}_{3,3} \left[\mp \frac{|x|^{\alpha}}{y^{\mu} e^{\iota \frac{\theta \pi}{2}}} \,\middle|\, \begin{array}{c} (1, 1), (2 - (1-\nu)(2-\mu), \mu), (1, \frac{\alpha}{2}) \\ (1, \alpha), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right]. \tag{53}$$

The case with $\theta = 0$, $\alpha = 2$, and quantum fractional Riesz-Feller derivative, yields

$$N(x, y) = \frac{y^{1-(1-\nu)(2-\mu)}}{2|x|} H^{2,0}_{2,2} \left[\frac{|x|}{y^{\mu/2}} \,\middle|\, \begin{array}{c} (2 - (1-\nu)(2-\mu), \frac{\mu}{2}), (1, \frac{1}{2}) \\ (1, 1), (1, \frac{1}{2}) \end{array} \right]$$

$$= \frac{y^{1-(1-\nu)(2-\mu)}}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{y^{\mu/2}} \middle| \begin{array}{c} (2-(1-\nu)(2-\mu), \frac{\mu}{2}) \\ (1,1) \end{array} \right]. \tag{54}$$

From solution (54) in case of Caputo fractional derivative ($\nu = 1$) one finds

$$N(x,y) = \frac{y}{2|x|} H_{2,2}^{2,0}\left[\frac{|x|}{y^{\mu/2}} \middle| \begin{array}{c} (2,\frac{\mu}{2}),(1,\frac{1}{2}) \\ (1,1),(1,\frac{1}{2}) \end{array} \right] = \frac{y}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{y^{\mu/2}} \middle| \begin{array}{c} (2,\frac{\mu}{2}) \\ (1,1) \end{array} \right], \tag{55}$$

and for R-L fractional derivative ($\nu = 0$) [37]

$$N(x,y) = \frac{y^{\mu-1}}{2|x|} H_{2,2}^{2,0}\left[\frac{|x|}{y^{\mu/2}} \middle| \begin{array}{c} (\mu,\frac{\mu}{2}),(1,\frac{1}{2}) \\ (1,1),(1,\frac{1}{2}) \end{array} \right] = \frac{y^{\mu-1}}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{y^{\mu/2}} \middle| \begin{array}{c} (\mu,\frac{\mu}{2}) \\ (1,1) \end{array} \right]. \tag{56}$$

Remark 8. *Here we note that the considered equation (28) can be transformed to the following general space-time fractional wave equation in presence of an external source $\Phi(x,t)$*

$$_tD_{0+}^{\mu,\nu}N(x,t) = {_xD_\theta^\alpha}N(x,t) + \Phi(x,t), \tag{57}$$

where we use fractional Riesz-Feller space derivative ${_xD_\theta^\alpha} = -{_xD_\theta^{*,\alpha}}$ given by (1), $x \in R$, $t \geq 0$, $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, with initial conditions

$$\left(_tI_{0+}^{(1-\nu)(2-\mu)}N\right)(x,0+) = f(x), \quad \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(_tI_{0+}^{(1-\nu)(2-\mu)}N\right)\right)(x,0+) = g(x), \tag{58a}$$

and boundary conditions

$$\lim_{x\to\pm\infty} N(x,t) = 0. \tag{58b}$$

So, its solution is given by

$$\begin{aligned}
N(x,t) = &\frac{t^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(-t^\mu \psi_\alpha^\theta(\kappa)\right)\hat{f}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa \\
&+ \frac{t^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(-t^\mu \psi_\alpha^\theta(\kappa)\right)\hat{g}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa \\
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(_t\mathscr{E}_{0+;\mu,\mu}^{-\psi_\alpha^\theta(\kappa);1}\hat{\Phi}\right)(\kappa,t)e^{-\iota\kappa x}\mathrm{d}\kappa, \tag{59}
\end{aligned}$$

where $\hat{\Phi}(\kappa,t) = \mathscr{F}[\Phi(x,t)](\kappa,t)$. Thus, all the previously obtained results, series representations and asymptotic behaviors in case of quantum fractional Riesz-Feller derivative can be used for this fractional wave equation with fractional Riesz-Feller space derivative and Hilfer-composite fractional time derivative. From this solution many obtained results for fractional wave equations with Caputo or R-L time fractional derivative can be recovered. For example, if $\Phi(x,t) = 0$ we obtain the general space-time fractional wave equation $_tD_{0+}^{\mu,\nu}N(x,t) = {_xD_\theta^\alpha}N(x,t)$ which contains a number of limiting cases.

4. Fractional Helmholtz Equation

The inhomogeneous Helmholtz equation in two variables is given by

$$\frac{\partial^2}{\partial x^2}N(x,y) + \frac{\partial^2}{\partial y^2}N(x,y) + k^2 N(x,y) = \Phi(x,y), \tag{60}$$

where $N(x,y)$ is the field variable, k is the wave number, and $\Phi(x,y)$ is a given function. For $\Phi(x,y) = 0$ it becomes homogeneous Helmholtz equation. Furthermore, if $k = 0$ it is related with the Poisson and Laplace equations considered in previous section. For a given form of $\Phi(x,y)$, equation (60) corresponds to the time-independent wave equation, which may be used for modeling vibrating membrane.

Here we analyze fractional generalization of the inhomogeneous Helmholtz equation (60).

Theorem 2. *The solution of the following inhomogeneous fractional Helmholtz equation*

$$_x D_\theta^\alpha N(x,y) + {}_y D_{0+}^{\mu,\nu} N(x,y) + k^2 N(x,y) = \Phi(x,y), \tag{61}$$

where $x \in R$, $y \geq 0$, $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, *with boundary conditions*

$$\left({}_y I_{0+}^{(1-\nu)(2-\mu)} N\right)(x,0+) = f(x), \quad \left(\frac{d}{dy}\left({}_y I_{0+}^{(1-\nu)(2-\mu)} N\right)\right)(x,0+) = g(x), \tag{62a}$$

$$\lim_{x \to \pm\infty} N(x,y) = 0, \tag{62b}$$

is given by

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(y^\mu\left(\psi_\alpha^\theta(\kappa) - k^2\right)\right) \hat{f}(\kappa) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(y^\mu\left(\psi_\alpha^\theta(\kappa) - k^2\right)\right) \hat{g}(\kappa) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left({}_y \mathscr{E}_{0+;\mu,\mu}^{\psi_\alpha^\theta(\kappa)-k^2;1} \hat{\Phi}\right)(\kappa,y) e^{-\iota\kappa x} d\kappa, \tag{63}$$

where $\hat{\Phi}(\kappa,y) = \mathscr{F}\left[\Phi(x,y)\right](\kappa,y)$.

Proof. In a same way as previously, by Laplace transform (14) to equation (61) and from the boundary conditions (62a), we obtain

$$_x D_\alpha^\theta \tilde{N}(x,s) + s^\mu \tilde{N}(x,s) - s^{1-\nu(2-\mu)} f(x) - s^{-\nu(2-\mu)} g(x) + k^2 \tilde{N}(x,s) = \tilde{\Phi}(x,s), \tag{64}$$

where $\tilde{N}(x,s) = \mathscr{L}\left[N(x,t)\right]$. The Fourier transform (5) to (64) yields

$$\hat{\tilde{N}}(\kappa,s) = \frac{s^{1-\nu(2-\mu)}}{s^\mu - \left(\psi_\alpha^\theta(\kappa) - k^2\right)} \hat{f}(\kappa) + \frac{s^{-\nu(2-\mu)}}{s^\mu - \left(\psi_\alpha^\theta(\kappa) - k^2\right)} \hat{g}(\kappa) + \frac{1}{s^\mu - \left(\psi_\alpha^\theta(\kappa) - k^2\right)} \hat{\tilde{\Phi}}(\kappa,s), \tag{65}$$

where $\hat{\tilde{\Phi}}(\kappa,s) = \mathscr{F}\left[\tilde{\Phi}(x,s)\right]$. By application of Lemma (1) and Lemma 2, by inverse Fourier transform we obtain solution (63). $\qquad\square$

Remark 9. *If in equation* (61) *instead of fractional Riesz-Feller derivative we use quantum fractional Riesz-Feller derivative we obtain the following equation*

$$_xD_\theta^{*,\alpha}N(x,y) + _yD_{0+}^{\mu,\nu}N(x,y) + k^2N(x,t) = \Phi(x,y). \tag{66}$$

If we use same boundary conditions as those used in Theorem (2), *we obtain the following solution*

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,1-(1-\nu)(2-\mu)}\left(-y^\mu\left(\psi_\alpha^\theta(\kappa)+k^2\right)\right)\hat{f}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{1-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,2-(1-\nu)(2-\mu)}\left(-y^\mu\left(\psi_\alpha^\theta(\kappa)+k^2\right)\right)\hat{g}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(_y\mathscr{E}_{0+;\mu,\mu}^{-(\psi_\alpha^\theta(\kappa)+k^2);1}\hat{\Phi}\right)(\kappa,y)e^{-\iota\kappa x}d\kappa. \tag{67}$$

Corollary 4. *For $\nu = 0$ (R-L fractional derivative) one finds the following solutions* [37]

$$N(x,y) = \frac{y^{\mu-2}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,\mu-1}\left(y^\mu\left(\pm\psi_\alpha^\theta(\kappa)-k^2\right)\right)\hat{f}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{\mu-1}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,\mu}\left(y^\mu\left(\pm\psi_\alpha^\theta(\kappa)-k^2\right)\right)\hat{g}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(_y\mathscr{E}_{0+;\mu,\mu}^{\pm\psi_\alpha^\theta(\kappa)-k^2;1}\hat{\Phi}\right)(\kappa,y)e^{-\iota\kappa x}d\kappa, \tag{68}$$

and for $\nu = 1$ - solutions

$$N(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}E_\mu\left(y^\mu\left(\pm\psi_\alpha^\theta(\kappa)-k^2\right)\right)\hat{f}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y}{2\pi}\int_{-\infty}^{\infty}E_{\mu,2}\left(y^\mu\left(\pm\psi_\alpha^\theta(\kappa)-k^2\right)\right)\hat{g}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(_y\mathscr{E}_{0+;\mu,\mu}^{\pm\psi_\alpha^\theta(\kappa)-k^2;1}\hat{\Phi}\right)(\kappa,y)e^{-\iota\kappa x}d\kappa, \tag{69}$$

where the upper signs in the solution correspond to the case of fractional Riesz-Feller derivative and lower signs to quantum fractional Riesz-Feller derivative.

Corollary 5. *The solutions of equations* (61) *and* (66) *for a source term of form*
$\Phi(x,y) = \delta(x)\frac{y^{-\beta}}{\Gamma(1-\beta)}$ *are given by*

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,1-(1-\nu)(2-\mu)}\left(y^\mu\left(\pm\psi_\alpha^\theta(\kappa)-k^2\right)\right)\hat{f}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+\frac{y^{1-(1-\nu)(2-\mu)}}{2\pi}\int_{-\infty}^{\infty}E_{\mu,2-(1-\nu)(2-\mu)}\left(y^\mu\left(\pm\psi_\alpha^\theta(\kappa)-k^2\right)\right)\hat{g}(\kappa)e^{-\iota\kappa x}d\kappa$$

$$+ \frac{y^{\mu-\beta}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu-\beta+1} \left(y^{\mu} \left(\pm \psi_{\alpha}^{\theta}(\kappa) - k^2\right)\right) e^{-\iota\kappa x} d\kappa, \tag{70}$$

where the upper signs in the solution correspond to the case of fractional Riesz-Feller derivative and lower signs to quantum fractional Riesz-Feller derivative.

Example 5. *The solution of the following fractional Helmholtz equation*

$$_x D_{\theta}^{\alpha} N(x,y) + _y D_{0+}^{\mu,\nu} N(x,y) + k^2 N(x,y) = \delta(x)\delta(y), \tag{71}$$

where $x \in R$, $x \geq 0$, $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, with boundary conditions

$$\left(_y I_{0+}^{(1-\nu)(2-\mu)} N\right)(x,0+) = \delta(x), \quad \left(\frac{d}{dy}\left(_y I_{0+}^{(1-\nu)(2-\mu)} N\right)\right)(x,0+) = 0, \tag{72a}$$

$$\lim_{x \to \pm\infty} N(x,y) = 0, \tag{72b}$$

is given by

$$N(x,y) = \frac{y^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)} \left(y^{\mu} \left(\psi_{\alpha}^{\theta}(\kappa) - k^2\right)\right) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y^{\mu-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu} \left(y^{\mu} \left(\psi_{\alpha}^{\theta}(\kappa) - k^2\right)\right) e^{-\iota\kappa x} d\kappa. \tag{73}$$

Example 6. *If the boundary conditions are given by $f(x) = 0$ and $g(x) = \delta(x)$, equation from Example (5) has a solution of form*

$$N(x,y) = \frac{y^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)} \left(y^{\mu} \left(\psi_{\alpha}^{\theta}(\kappa) - k^2\right)\right) e^{-\iota\kappa x} d\kappa$$

$$+ \frac{y^{\mu-1}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,\mu} \left(y^{\mu} \left(\psi_{\alpha}^{\theta}(\kappa) - k^2\right)\right) e^{-\iota\kappa x} d\kappa. \tag{74}$$

Remark 10. *Note that equation (66) can be transformed to the following general space-time fractional wave equation in presence of an external source $\Phi(x,t)$*

$$_t D_{0+}^{\mu,\nu} N(x,t) = _x D_{\theta}^{\alpha} N(x,t) - k^2 N(x,t) + \Phi(x,t), \tag{75}$$

where we use fractional Riesz-Feller space derivative $_x D_{\theta}^{\alpha} = -_x D_{\theta}^{*,\alpha}$ given by (1), $x \in R$, $t \geq 0$, $1 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$, $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, with initial conditions

$$\left(_t I_{0+}^{(1-\nu)(2-\mu)} N\right)(x,0+) = f(x), \quad \left(\frac{d}{dt}\left(_t I_{0+}^{(1-\nu)(2-\mu)} N\right)\right)(x,0+) = g(x), \tag{76a}$$

and boundary conditions

$$\lim_{x \to \pm\infty} N(x,t) = 0. \tag{76b}$$

Thus, its solution is given by

$$
\begin{aligned}
N(x,t) = {} & \frac{t^{-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,1-(1-\nu)(2-\mu)}\left(-t^{\mu}\left(\psi_{\alpha}^{\theta}(\kappa)+k^2\right)\right)\hat{f}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa \\
& + \frac{t^{1-(1-\nu)(2-\mu)}}{2\pi} \int_{-\infty}^{\infty} E_{\mu,2-(1-\nu)(2-\mu)}\left(-t^{\mu}\left(\psi_{\alpha}^{\theta}(\kappa)+k^2\right)\right)\hat{g}(\kappa)e^{-\iota\kappa x}\mathrm{d}\kappa \\
& + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left({}_{t}\mathscr{E}_{0+;\mu,\mu}^{-\left(\psi_{\alpha}^{\theta}(\kappa)+k^2\right);1}\hat{\Phi}\right)(\kappa,t)e^{-\iota\kappa x}\mathrm{d}\kappa,
\end{aligned}
\tag{77}
$$

where $\hat{\Phi}(\kappa,t) = \mathscr{F}[\Phi(x,t)](\kappa,t)$. This solution contains a number of limiting cases.

5. Conclusions

We consider fractional generalization of the Laplace equation, Poisson equation and Helmholtz equations in two variables. Since there is no dependence on the time variable, the solutions of these equations can be considered as a steady-state solutions. The fractional derivatives used in this paper are of Riesz-Feller and Hilfer-composite form. M-L type functions, Fox *H*-functions, and the Prabhakar integral operator containing two parameter M-L function in the kernel are used to express solutions analytically. Several special cases of these equations are investigated. Asymptotic behavior of solutions is analyzed, and series expression of solutions are provided. The general space-time fractional wave equation in presence of an external source is considered as well.

Appendix: Mittag-Leffler and Fox H-functions

The standard (one parameter) M-L function, introduced by Mittag-Leffler, is defined by [4, 5, 11, 18, 29, 34, 36]

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{A1}$$

where $(z \in \mathbb{C}; \Re(\alpha) > 0)$. Later, two parameter M-L function which was introduced by Wiman, and further analyzed by Agarwal and Humbert, is given by [4, 5, 11, 18, 29, 34, 36]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{A2}$$

where $(z, \beta \in \mathbb{C}; \Re(\alpha) > 0)$. The M-L functions (A1) and (A2) are entire functions of order $\rho = 1/\Re(\alpha)$ and type 1. Note that $E_{\alpha,1}(z) = E_\alpha(z)$. These functions are generalization of the exponential, hyperbolic and trigonometric functions since $E_{1,1}(z) = e^z$, $E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$ and $E_{2,2}(-z^2) = \sin(z)/z$.

The Laplace transform of the M-L function is given by [11, 18, 29, 34, 36]

$$\mathscr{L}[t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha)](s) = \int_0^\infty e^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha) dt = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}, \tag{A3}$$

where $\Re(s) > |a|^{1/\alpha}$.

Prabhakar [35] introduced the following three parameter M-L function

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \tag{A4}$$

where $\beta, \gamma, z \in \mathbb{C}$, $\Re(\alpha) > 0$, $(\gamma)_k$ is the Pochhammer symbol. It is an entire function of order $\rho = 1/\Re(\alpha)$ and type 1. Note that $E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z)$. Later, in [47] it is used the following four parameter generalized M-L function

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \sum_{n=0}^\infty \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!}, \tag{A5}$$

where $(z, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > \max\{0, \Re(\kappa) - 1\}; \Re(\kappa) > 0)$ and $(\gamma)_{\kappa n}$ is a notation of the Pochhammer symbol, as a kernel of a generalized integral operator. It is an entire function of order $\rho = \frac{1}{\Re(\alpha-\kappa)+1}$ and type $\sigma = \frac{1}{\rho}\left(\frac{\{\Re(\alpha)\}^{\Re(\kappa)}}{\{\Re(\alpha)\}^{\Re(\alpha)}}\right)^\rho$ [47]. Note that $E_{\alpha,\beta}^{\gamma,1}(z) = E_{\alpha,\beta}^\gamma(z)$.

The Fox H-function (or simply H-function) is defined by the following Mellin-Barnes integral [29]

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q) \end{matrix}\right] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix}\right] = \frac{1}{2\pi\iota}\int_\Omega \theta(s) z^s ds, \tag{A6}$$

where $\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+B_j s) \prod_{j=n+1}^{p} \Gamma(a_j-A_j s)}$, $0 \leq n \leq p$, $1 \leq m \leq q$, $a_i, b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}^+$, $i = 1, \ldots, p$, $j = 1, \ldots, q$. The contour Ω starting at $c - \iota\infty$ and ending at $c + \iota\infty$ separates the poles of the function $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$ from those of the function $\Gamma(1 - a_i - A_i s)$, $i = 1, \ldots, n$. The expansion for the H-function (A6) is given by [29]

$$
\begin{aligned}
&H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right. \right] \\
&= \sum_{h=1}^{m} \sum_{k=0}^{\infty} \frac{\prod_{j=1, j \neq h}^{m} \Gamma\left(b_j - B_j \frac{b_h+k}{B_h}\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j + A_j \frac{b_h+k}{B_h}\right)}{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j + B_j \frac{b_h+k}{B_h}\right) \prod_{j=n+1}^{p} \Gamma\left(a_j - A_j \frac{b_h+k}{B_h}\right)} \cdot \frac{(-1)^k z^{(b_h+k)/B_h}}{k! B_h}.
\end{aligned}
\tag{A7}
$$

From the Mellin-Barnes integral representation of two parameter M-L function, one can find the following relation with the Fox H-function [29]

$$
E_{\alpha,\beta}(z) = \frac{1}{2\pi\iota} \int_{\Omega} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} z^s \mathrm{d}s = H_{1,2}^{1,1}\left[-z \left| \begin{array}{l} (0,1) \\ (0,1), (1-\beta, \alpha) \end{array} \right. \right],
\tag{A8}
$$

where the contour Ω starts at $c - \iota\infty$, ends at $c + \iota\infty$, and separates the poles of function $\Gamma(s)$ from those of the function $\Gamma(1-s)$. It is shown by Mathai, Saxena and Haubold that the integral converges for all z [29].

The Mellin-cosine transform of the H-function is given by [29]

$$
\begin{aligned}
&\int_0^{\infty} k^{\rho-1} \cos(kx) H_{p,q}^{m,n}\left[ak^{\delta} \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right. \right] \mathrm{d}k \\
&= \frac{2^{\rho-1}\sqrt{\pi}}{x^{\rho}} H_{p+2,q}^{m,n+1}\left[a\left(\frac{2}{x}\right)^{\delta} \left| \begin{array}{l} (2-\frac{\rho}{2}, \frac{\mu}{2}), (a_p, A_p), (\frac{1-\rho}{2}, \frac{\delta}{2}) \\ (b_q, B_q) \end{array} \right. \right],
\end{aligned}
\tag{A9}
$$

where $\Re(\rho) + \delta \min_{1 \leq j \leq m} \Re\left(\frac{b_j}{B_j}\right) > 0$, $\rho = \delta \max_{1 \leq j \leq n} \Re\left(\frac{a_j-1}{A_j}\right) < 0$, $|\arg(a)| < \pi\theta/2$, $\theta = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0$.

The asymptotic expansion of the Fox H-function $H_{p,q}^{m,0}(z)$ where $q = m$ for large z is [1, 29]

$$
H_{p,q}^{m,0}(z) \sim B z^{(1-\alpha)/m^*} \exp\left(-m^* C^{1/m^*} z^{1/m^*}\right),
\tag{A10}
$$

where

$$
\alpha = \sum_{k=1}^{p} a_k - \sum_{k=1}^{q} b_k + \frac{1}{2}(q-p+1),
\tag{A11}
$$

$$
m^* = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0,
\tag{A12}
$$

$$
C = \prod_{k=1}^{p} A_k^{A_k} \prod_{k=1}^{q} B_k^{-B_k},
\tag{A13}
$$

$$B = (2\pi)^{(m-p-1)/2} C^{(1-\alpha)/m^*} m^{*-1/2} \prod_{k=1}^{p} A_k^{1/2-a_k} \prod_{k=1}^{m} B_k^{b_k-1/2}. \tag{A14}$$

Closely related to the Fox H-function is the Fox-Wright function defined by [29]

$$_p\Psi_q(z) = {}_p\Psi_q\left[z \left| \begin{array}{c} (a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q) \end{array} \right.\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + nA_j\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + nB_j\right)} \cdot \frac{z^n}{n!}, \tag{A15}$$

which as a special case gives the Wright function [29]

$$\phi(a,b;z) = {}_0\Psi_1(z) = {}_0\Psi_1\left[z \left| \begin{array}{c} \\ (b,a) \end{array} \right.\right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(b+na)} \cdot \frac{z^n}{n!}. \tag{A16}$$

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