On Generalized Ideals of Left Almost Semigroups

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Abstract. In this paper, we study \((m, n)\)-ideals of an \(\mathcal{L}\mathcal{A}\)-semigroup in detail. We characterize \((0, 2)\)-ideals of an \(\mathcal{L}\mathcal{A}\)-semigroup \(S\) and prove that \(A\) is a \((0, 2)\)-ideal of \(S\) if and only if \(A\) is a left ideal of some left ideal of \(S\). We also show that an \(\mathcal{L}\mathcal{A}\)-semigroup \(S\) is \(0\)-\((0, 2)\)-bisimple if and only if \(S\) is right \(0\)-simple. Furthermore we study 0-minimal \((m, n)\)-ideals in an \(\mathcal{L}\mathcal{A}\)-semigroup \(S\) and prove that \(S\) is \(0\)-minimal \((m, n)\)-ideal if and only if \(S\) is \(0\)-simple. Finally we discuss \((m, n)\)-ideals in an \(\mathcal{L}\mathcal{A}\)-regular \(\mathcal{L}\mathcal{A}\)-semigroup \(S\) and show that \(S\) is \((0, 1)\)-regular if and only if \(L = SL\) where \(L\) is a \((0, 1)\)-ideal of \(S\).

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1. Introduction

A left almost semigroup (\(\mathcal{L}\mathcal{A}\)-semigroup) is a groupoid \(S\) satisfying the left invertive law \((ab)c = (cb)a\) for all \(a, b, c \in S\). This left invertive law has been obtained by introducing braces on the left of ternary commutative law \(abc = cba\). The concept of an \(\mathcal{L}\mathcal{A}\)-semigroup was first given by Kazim and Naseeruddin in 1972 [3]. An \(\mathcal{L}\mathcal{A}\)-semigroup satisfies the medial law \((ab)(cd) = (ac)(bd)\) for all \(a, b, c, d \in S\). Since \(\mathcal{L}\mathcal{A}\)-semigroups satisfy medial law, they belong to the class of entropic groupoids which are also called abelian quasigroups [12]. If an \(\mathcal{L}\mathcal{A}\)-semigroup \(S\) contains a left identity (unitary \(\mathcal{L}\mathcal{A}\)-semigroup), then it satisfies the paramedial law \((ab)(cd) = (dc)(ba)\) and the identity \(a(bc) = b(ac)\) for all \(a, b, c, d \in S\) [7].

An \(\mathcal{L}\mathcal{A}\)-semigroup is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An \(\mathcal{L}\mathcal{A}\)-semigroup is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures. It has been investigated in [7] that if an \(\mathcal{L}\mathcal{A}\)-semigroup contains a right identity, then it becomes a commutative semigroup. The connection of a commutative inverse semigroup

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with an \( \mathcal{L} \mathcal{A} \)-semigroup has been given by Yousafzai et al. in [16] as, a commutative inverse semigroup \((S, \cdot)\) becomes an \( \mathcal{L} \mathcal{A} \)-semigroup \((S, \ast)\) under \( a \ast b = ba^{-1}r^{-1} \), \( \forall a, b, r \in S \). An \( \mathcal{L} \mathcal{A} \)-semigroup \( S \) with left identity becomes a semigroup under the binary operation "\( \circ \)" defined as \( x \circ y = (xe)y \) for all \( x, y \in S \) [17]. An \( \mathcal{L} \mathcal{A} \)-semigroup is the generalization of a semigroup theory [7] and has vast applications in collaboration with semigroups like other branches of mathematics. Khan et al. studied an intra-regular class of an \( \mathcal{L} \mathcal{A} \)-semigroup in [4] and proved some interesting problems by using different ideals. They proved that the set of all two-sided ideals of intra-regular \( \mathcal{L} \mathcal{A} \)-semigroup forms a semilattice structure. They characterized an intra-regular \( \mathcal{L} \mathcal{A} \)-semigroup by using left, right, two-sided and bi-ideals. An \( \mathcal{L} \mathcal{A} \)-semigroup is the generalization of a semigroup theory [7]. Many interesting results on \( \mathcal{L} \mathcal{A} \)-semigroups have been investigated in [5, 9–11, 15].

Yaqoob, Corsini and Yousafzai [13] extended the concept of LA-semigroups and introduced a new structure called left almost semihypergroup. Further Yaqoob and Gulistan [14] defined partial ordering on left almost semihypergroups. Gulistan et al. [2] defined \( H \)-\( \mathcal{L} \mathcal{A} \)-semigroups which is a new generalization of \( \mathcal{L} \mathcal{A} \)-semigroups and \( \mathcal{L} \mathcal{A} \)-semihypergroups.

### 2. Preliminaries and Examples

If \( S \) is an \( \mathcal{L} \mathcal{A} \)-semigroup with product \( \cdot : S \times S \rightarrow S \), then \( ab \cdot c \) and \( (ab)c \) both denote the product \((a \cdot b) \cdot c\).

If there is an element 0 of an \( \mathcal{L} \mathcal{A} \)-semigroup \((S, \cdot)\) such that \( x \cdot 0 = 0 \cdot x = x \), \( \forall x \in S \), we call 0 a zero element of \( S \).

**Example 1.** Let \( S = \{a, b, c, d, e\} \) with a left identity \( d \). Then the following multiplication table shows that \((S, \cdot)\) is a unitary \( \mathcal{L} \mathcal{A} \)-semigroup with a zero element \( a \).

\[
\begin{array}{c|ccccc}
\cdot & a & b & c & d & e \\
\hline
a & a & a & a & a & a \\
b & a & e & e & c & e \\
c & a & e & e & b & e \\
d & a & b & c & d & e \\
e & a & e & e & e & e \\
\end{array}
\]

**Example 2.** Let \( S = \{a, b, c, d\} \). Then the following multiplication table shows that \((S, \cdot)\) is an \( \mathcal{L} \mathcal{A} \)-semigroup with a zero element \( a \).

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & d & d & c \\
c & a & c & c & c \\
d & a & c & c & c \\
\end{array}
\]

The above \( \mathcal{L} \mathcal{A} \)-semigroup \( S \) has commutative powers, that is \( aa \cdot a = a \cdot aa \) for all \( a \in S \) which is called a locally associative \( \mathcal{L} \mathcal{A} \)-semigroup [8]. Note that \( S \) has no associative powers for all \( a \in S \) because \((bb \cdot b)b \neq b(bb \cdot b)\) for \( b \in S \).
Assume that \( S \) is an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup. Let us define \( a^1 = a \), \( a^{m+1} = a^m a \) and \( a^m = (((aa)a)a) \ldots a) = a^{m-1}a \) for all \( a \in S \) where \( m \geq 1 \). It is easy to see that \( a^m = a^{m-1}a = aa^{m-1} \) for all \( a \in S \) and \( m \geq 3 \) if \( S \) has a left identity. Also, we can show by induction, \( (ab)^m = a^m b^m \) and \( a^m a^n = a^{m+n} \) hold for all \( a, b \in S \) and \( m, n \geq 3 \).

A subset \( A \) of an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) is called a right (left) ideal of \( S \) if \( AS \subseteq A \) (\( SA \subseteq A \)), and is called an ideal of \( S \) if it is both left and right ideal of \( S \).

A subset \( A \) of an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) is called an \( \mathcal{L} \cdot \mathcal{A} \)-subsemigroup of \( S \) if \( A^2 \subseteq A \).

The concept of \((m, n)\)-ideals of a semigroup and an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup was given in [6] and [1] respectively.

An \( \mathcal{L} \cdot \mathcal{A} \)-subsemigroup \( A \) of an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) is said to be an \((m, n)\)-ideal of \( S \) if \( A^m S \cdot A^n \subseteq A \) where \( m, n \) are non-negative integers such that \( m = n \neq 0 \). Here \( A^m \) or \( A^n \) are suppressed if \( m = 0 \) or \( n = 0 \), that is \( A^0 S = S \) or \( SA^0 = S \). Note that if \( m = n = 1 \), then an \((m, n)\)-ideal \( A \) of an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) is called a bi-ideal of \( S \). If we take \( m = 0 \) or \( n = 0 \), then an \((m, n)\)-ideal \( A \) of an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) becomes a left or a right ideal of \( S \).

An \((m, n)\)-ideal \( A \) of an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) with zero is said to be 0-minimal if \( A \neq \{0\} \) and \( \{0\} \) is the only \((m, n)\)-ideal of \( S \) properly contained in \( A \).

An \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) with zero is said to be 0-(0, 2)-bisimple if \( S^2 \neq \{0\} \) and \( \{0\} \) is the only proper \((0, 2)\)-bi-ideal of \( S \).

An \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) with zero is said to be nilpotent if \( S^l = \{0\} \) for some positive integer \( l \).

Let \( m, n \) be non-negative integers and \( S \) be an \( \mathcal{L} \cdot \mathcal{A} \)-semigroup. We say that \( S \) is \((m, n)\)-regular if for every element \( a \in S \) there exists some \( x \in S \) such that \( a = (a^m) x a^n \). Note that \( a^0 \) is defined as an operator element such that \( a^0 y = y \) and \( z a^0 = z \) for any \( y, z \in S \).

3. 0-Minimal \((0, 2)\)-Bi-Ideals in Unitary \( \mathcal{L} \cdot \mathcal{A} \)-Semigroups

If \( S \) is a unitary \( \mathcal{L} \cdot \mathcal{A} \)-semigroup, then it is easy to see that \( S^2 = S \), \( SA^2 = A^2 S \) and \( A \subseteq SA \forall A \subseteq S \). Note that every right ideal of a unitary \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \) is a left ideal of \( S \) but the converse is not true in general. Example 1 shows that there exists a subset \( \{a, b, c\} \) of \( S \) which is a left ideal of \( S \) but not a right ideal of \( S \). It is easy to see that \( SA \) and \( SA^2 \) are the left and right ideals of a unitary \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \). Thus \( SA^2 \) is an ideal of a unitary \( \mathcal{L} \cdot \mathcal{A} \)-semigroup \( S \).

Lemma 1. Let \( S \) be a unitary \( \mathcal{L} \cdot \mathcal{A} \)-semigroup. Then \( A \) is a \((0, 2)\)-ideal of \( S \) if and only if \( A \) is an ideal of some left ideal of \( S \).

Proof. Let \( A \) be a \((0, 2)\)-ideal of \( S \), then \( SA \cdot A = AA \cdot S = SA^2 \subseteq A \) and \( A \cdot SA = S \cdot AA = SS \cdot AA = SA^2 \subseteq A \). Hence \( A \) is an ideal of a left ideal \( SA \) of \( S \).

Conversely, assume that \( A \) is a left ideal of a left ideal \( L \) of \( S \), then
\[
SA^2 = AA \cdot S = SA \cdot A \subseteq SL \cdot A \subseteq LA \subseteq A,
\]
and clearly \( A \) is an \( \mathcal{L} \cdot \mathcal{A} \)-subsemigroup of \( S \), therefore \( A \) is a \((0, 2)\)-ideal of \( S \). \( \square \)
Corollary 1. Let $S$ be a unitary $\mathcal{L} \mathcal{A}$-semigroup. Then $A$ is a $(0,2)$-ideal of $S$ if and only if $A$ is a left ideal of some left ideal of $S$.

Lemma 2. Let $S$ be a unitary $\mathcal{L} \mathcal{A}$-semigroup. Then $A$ is a $(0,2)$-bi-ideal of $S$ if and only if $A$ is an ideal of some right ideal of $S$.

Proof. Let $A$ be a $(0,2)$-bi-ideal of $S$, then $SA^2 \cdot A = A^2 S \cdot A = AS \cdot A^2 \subseteq SA^2 \subseteq A$ and $A \cdot SA^2 = SS \cdot AA^2 = A^2 A \cdot SS = SA \cdot A^2 \subseteq SA^2 \subseteq A$. Hence $A$ is an ideal of some right ideal $SA^2$ of $S$.

Conversely, assume that $A$ is an ideal of a right ideal $R$ of $S$, then $SA^2 = A \cdot SA = A \cdot (SS)A = A \cdot (AS)S \subseteq A \cdot (RS)R \subseteq AR \subseteq A$, and $(AS)A \subseteq (RS)A \subseteq RA \subseteq A$, which shows that $A$ is a $(0,2)$-ideal of $S$. 

Theorem 1. Let $S$ be a unitary $\mathcal{L} \mathcal{A}$-semigroup. Then the following statements are equivalent.

(i) $A$ is a $(1,2)$-ideal of $S$;

(ii) $A$ is a left ideal of some bi-ideal of $S$;

(iii) $A$ is a bi-ideal of some ideal of $S$;

(iv) $A$ is a $(0,2)$-ideal of some right ideal of $S$;

(v) $A$ is a left ideal of some $(0,2)$-ideal of $S$.

Proof. (i) $\Rightarrow$ (ii). It is easy to see that $SA^2 \cdot S$ is a bi-ideal of $S$. Let $A$ be a $(1,2)$-ideal of $S$, then

$$(SA^2 \cdot S)A = (SA^2 \cdot SS)A = (SS \cdot A^2 S)A = (S \cdot A^2 S)A = A^2 S \cdot A$$

$$= AS \cdot A^2 \subseteq A,$$

which shows that $A$ is a left ideal of a bi-ideal $SA^2 \cdot S$ of $S$.

(ii) $\Rightarrow$ (iii). Let $A$ be a left ideal of a bi-ideal $B$ of $S$, then

$$A \cdot SA^2 = (S \cdot AA^2)A \subseteq [S(SA \cdot AA)]A = [S(AA \cdot AS)]A$$

$$= [AA \cdot S(AS)]A = [(S(AS) \cdot A)A]A = [(AS \cdot A)A]A$$

$$\subseteq [(BS \cdot B)A]A \subseteq BA \cdot A \subseteq A,$$

which shows that $A$ is a bi-ideal of an ideal $SA^2$ of $S$.

(iii) $\Rightarrow$ (iv). Let $A$ be a bi-ideal of an ideal $I$ of $S$, then

$$SA^2 \cdot A^2 = (A^2 \cdot AA)S = (A \cdot A^2 A)S \subseteq [A \cdot (AI)A]S = AA \cdot S$$

$$= SA \cdot S \subseteq SI \cdot S \subseteq I,$$

which shows that $A$ is a $(0,2)$-ideal of a right ideal $SA^2$ of $S$. 

(iv) $\implies$ (v). It is easy to see that $SA^3$ is a $(0, 2)$-ideal of $S$. Let $A$ be a $(0, 2)$-ideal of a right ideal $R$ of $S$, then

$$A \cdot SA^3 = A(SS \cdot A) = A(AA^2 \cdot S) \subseteq A[(SA \cdot AA)S] = A[(AA \cdot AS)S] = (AA)[(A \cdot AS)S] = [S \cdot A(AS)]A^2 = [A \cdot S(AS)]A^2 \subseteq RS \cdot A^2 \subseteq RA^2 \subseteq A,$$

which shows that $A$ is a left ideal of a $(0, 2)$-ideal $SA^3$ of $S$.

(v) $\implies$ (i). Let $A$ be a left ideal of a $(0, 2)$-ideal $O$ of $S$, then

$$AS \cdot A^2 = (AA \cdot SS)A = SA^2 \cdot A \subseteq SO^2 \cdot A \subseteq OA \subseteq A,$$

which shows that $A$ is a $(1, 2)$-ideal of $S$.

Lemma 3. Let $S$ be a unitary $\mathcal{L}_\mathcal{A}$-semigroup and $A$ be an idempotent subset of $S$. Then $A$ is a $(1, 2)$-ideal of $S$ if and only if there exist a left ideal $L$ and a right ideal $R$ of $S$ such that $RL \subseteq A \subseteq R \cap L$.

Proof. Assume that $A$ is a $(1, 2)$-ideal of $S$ such that $A$ is idempotent. Setting $L = SA$ and $R = SA^2$, then

$$RL = SA^2 \cdot SA = A^2 \cdot SA = (SA \cdot SS)A^2 = (SS \cdot AS)A^2 = [S(AS \cdot A)]A^2 = [A(AS \cdot A)]A^2 \subseteq AS \cdot A^2 \subseteq A.$$

It is clear that $A \subseteq R \cap L$.

Conversely, let $R$ be a right ideal and $L$ be a left ideal of $S$ such that $RL \subseteq A \subseteq R \cap L$, then

$$AS \cdot A^2 = AS \cdot AA \subseteq RS \cdot SL \subseteq RL \subseteq A.$$

Assume that $S$ is a unitary $\mathcal{L}_\mathcal{A}$-semigroup with zero. Then it is easy to see that every left (right) ideal of $S$ is a $(0, 2)$-ideal of $S$. Hence if $O$ is a $0$-minimal $(0, 2)$-ideal of $S$ and $A$ is a left (right) ideal of $S$ contained in $O$, then either $A = \{0\}$ or $A = O$.

Lemma 4. Let $S$ be a unitary $\mathcal{L}_\mathcal{A}$-semigroup with zero. Assume that $A$ is a $0$-minimal ideal of $S$ and $O$ is an $\mathcal{L}_\mathcal{A}$-subsemigroup of $A$. Then $O$ is a $(0, 2)$-ideal of $S$ contained in $A$ if and only if $O^2 = \{0\}$ or $O = A$.

Proof. Let $O$ be a $(0, 2)$-ideal of $S$ contained in a $0$-minimal ideal $A$ of $S$. Since $SO^2 \subseteq O \subseteq A$. Therefore $O$ is an ideal of $S$, therefore by minimality of $A$, $SO^2 = \{0\}$ or $SO^2 = A$. If $SO^2 = A$, then $A = SO^2 \subseteq O$ and therefore $O = A$. Let $SO^2 = \{0\}$, then $O^2S = SO^2 = \{0\} \subseteq O^2$, which shows that $O^2$ is a right ideal of $S$, and hence an ideal of $S$ contained in $A$, therefore by minimality of $A$, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$, then $O = A$.

Conversely, let $O^2 = \{0\}$, then $SO^2 = O^2S = \{0\}S = \{0\} = O^2$. Now if $O = A$, then $SO^2 = SS \cdot OO = SA \cdot SA \subseteq A = O$, which shows that $O$ is a $(0, 2)$-ideal of $S$ contained in $A$. □
Corollary 2. Let $S$ be a unitary $\mathcal{L}_{A}$-semigroup with zero. Assume that $A$ is a 0-minimal left ideal of $S$ and $O$ is an $\mathcal{L}_{A}$-subsemigroup of $A$. Then $O$ is a $(0, 2)$-ideal of $S$ contained in $A$ if and only if $O^2 = \{0\}$ or $O = A$.

Lemma 5. Let $S$ be a unitary $\mathcal{L}_{A}$-semigroup with zero and $O$ be a 0-minimal $(0, 2)$-ideal of $S$. Then $O^2 = \{0\}$ or $O$ is a 0-minimal right (left) ideal of $S$.

Proof. Let $O$ be a 0-minimal $(0, 2)$-ideal of $S$, then

$$S(O^2)^2 = SS \cdot O^2 O^2 = O^2 O^2 \cdot S = SO^2 \cdot O^2 \subseteq OO^2 \subseteq O^2,$$

which shows that $O^2$ is a $(0, 2)$-ideal of $S$ contained in $O$, therefore by minimality of $O$, $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then $OS = OO \cdot SS = SO^2 \subseteq O$, which shows that $O$ is a right ideal of $S$. Let $R$ be a right ideal of $S$ contained in $O$, then $R^2 S = RR \cdot S \subseteq RS \cdot S \subseteq R$. Thus $R$ is a $(0, 2)$-ideal of $S$ contained in $O$, and again by minimality of $O$, $O = \{0\}$ or $R = O$. \qed

The following Corollary follows from Lemma 4 and Corollary 2.

Corollary 3. Let $S$ be a unitary $\mathcal{L}_{A}$-semigroup. Then $O$ is a minimal $(0, 2)$-ideal of $S$ if and only if $O$ is a minimal left ideal of $S$.

Theorem 2. Let $S$ be a unitary $\mathcal{L}_{A}$-semigroup. Then $A$ is a minimal $(2, 1)$-ideal of $S$ if and only if $A$ is a minimal bi-ideal of $S$.

Proof. Let $A$ be a minimal $(2, 1)$-ideal of $S$. Then

$$[(A^2 S \cdot A)^2 S](A^2 S \cdot A) = [(A^2 S \cdot A)(A^2 S \cdot A)]S(A^2 S \cdot A) \subseteq [(AS \cdot A)(AS \cdot A)]S(AS \cdot A) = [(AS \cdot A)(AA)]S(AS \cdot A) = (A^2 S \cdot S)(AS \cdot A) \subseteq (AS \cdot S)(AS \cdot A) = (AS \cdot AS)(SA) = A^2 S \cdot SA = AS \cdot SA^2 = (SA^2 \cdot S)A = (A^2 S \cdot S)A = (SS \cdot AA)A = A^2 S \cdot A,$$

and similarly we can show that $(A^2 S \cdot A)^2 \subseteq A^2 S \cdot A$. Thus $A^2 S \cdot A$ is a $(2, 1)$-ideal of $S$ contained in $A$, therefore by minimality of $A$, $A^2 S \cdot A = A$. Now

$$AS \cdot A = (AS)(A^2 S \cdot A) = [(A^2 S \cdot A)S]A = (SA \cdot A^2 S)A = [A^2(SA \cdot S)]A \subseteq A^2 S \cdot A = A,$$

It follows that $A$ is a bi-ideal of $S$. Suppose that there exists a bi-ideal $B$ of $S$ contained in $A$, then $B^2 S \cdot B \subseteq BS \cdot B \subseteq B$, so $B$ is a $(2, 1)$-ideal of $S$ contained in $A$, therefore $B = A$. 

Conversely, assume that $A$ is a minimal bi-ideal of $S$, then it is easy to see that $A$ is a $(2, 1)$-ideal of $S$. Let $C$ be a $(2, 1)$-ideal of $S$ contained in $A$, then

$$[(C^{2}S \cdot C)S](C^{2}S \cdot C) = (SC \cdot C^{2}S)(C^{2}S \cdot C) = (SC \cdot CS)(C^{2}S \cdot C) = [C(SC \cdot S)](C^{2}S \cdot C) = [(C^{2}S \cdot C)(S \cdot C^{2}S)]C = [(C^{2}S \cdot C)(C^{2}S)]C = [C^{2}((C^{2}S \cdot C)S)]C \subseteq C^{2}S \cdot C.$$  

This shows that $C^{2}S \cdot C$ is a bi-ideal of $S$, and by minimality of $A$, $C^{2}S \cdot C = A$. Thus $A = C^{2}S \cdot C \subseteq C$, and therefore $A$ is a minimal $(2, 1)$-ideal of $S$. \hfill \Box

**Theorem 3.** Let $A$ be a $0$-minimal $(0, 2)$-bi-ideal of a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup $S$ with zero. Then exactly one of the following cases occurs:

(i) $A = \{0, a\}, a^2 = 0$;

(ii) $\forall a \in A \setminus \{0\}, Sa^2 = A$.

**Proof.** Assume that $A$ is a $0$-minimal $(0, 2)$-bi-ideal of $S$. Let $a \in A \setminus \{0\}$, then $Sa^2 \subseteq A$. Also $Sa^2$ is a $(0, 2)$-bi-ideal of $S$, therefore $Sa^2 = \{0\}$ or $Sa^2 = A$.

Let $Sa^2 = \{0\}$. Since $a^2 \in A$, we have either $a^2 = a$ or $a^2 = 0$ or $a^2 \in A \setminus \{0, a\}$. If $a^2 = a$, then $a^3 = a^2a = a$, which is impossible because $a^3 \in a^2S = Sa^2 = \{0\}$. Let $a^2 \in A \setminus \{0, a\}$, we have

$$S \cdot \{0, a^2\} \{0, a^2\} = SS \cdot a^2a^2 = Sa^2 \cdot Sa^2 = \{0\} \subseteq \{0, a^2\},$$

and

$$\{0, a^2\}S \{0, a^2\} = \{0, a^2\} \{0, a^2\} = a^2S \cdot a^2 \subseteq Sa^2 = \{0\} \subseteq \{0, a^2\}.$$  

Therefore $\{0, a^2\}$ is a $(0, 2)$-bi-ideal of $S$ contained in $A$. We observe that $\{0, a^2\} \neq \{0\}$ and $\{0, a^2\} \neq A$. This is a contradiction to the fact that $A$ is a $0$-minimal $(0, 2)$-bi-ideal of $S$. Therefore $a^2 = 0$ and $A = \{0, a\}$.

If $Sa^2 \neq \{0\}$, then $Sa^2 = A$. \hfill \Box

**Corollary 4.** Let $A$ be a $0$-minimal $(0, 2)$-bi-ideal of a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup $S$ with zero such that $A^2 \neq 0$. Then $A = Sa^2$ for every $a \in A \setminus \{0\}$.

**Lemma 6.** Let $S$ be a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup. Then every right ideal of $S$ is a $(0, 2)$-bi-ideal of $S$.

**Proof.** Assume that $A$ is a right ideal of $S$, then

$$SA^2 = AA \cdot SS = AS \cdot AS \subseteq AA \subseteq AS \subseteq A, \quad AS \cdot A \subseteq A,$$

and clearly $A^2 \subseteq A$, therefore $A$ is a $(0, 2)$-bi-ideal of $S$. \hfill \Box

The converse of Lemma 6 is not true in general. Example 1 showed that there exists a $(0, 2)$-bi-ideal $A = \{a, c, e\}$ of $S$ which is not a right ideal of $S$. 

Theorem 4. Let $S$ be a unitary $\mathcal{L}\mathcal{A}$-semigroup with zero. Then $Sa^2 = S$ for every $a \in S \setminus \{0\}$ if and only if $S$ is 0-(0, 2)-bisimple if and only if $S$ is right 0-simple.

Proof. Assume that $Sa^2 = S$ for every $a \in S \setminus \{0\}$. Let $A$ be a (0, 2)-bi-ideal of $S$ such that $A \neq \{0\}$. Let $a \in A \setminus \{0\}$, then $S = Sa^2 \subseteq SA^2 \subseteq A$. Therefore $S = A$. Since $S = Sa^2 \subseteq SS = S^2$, we have $S^2 = S \neq \{0\}$. Thus $S$ is $0-(0, 2)$-bisimple. The converse statement follows from Corollary 4.

Let $R$ be a right ideal of $0-(0, 2)$-bisimple $S$. Then by Lemma 6, $R$ is a (0, 2)-bi-ideal of $S$ and so $R = \{0\}$ or $R = S$.

Conversely, assume that $S$ is right 0-simple. Let $a \in S \setminus \{0\}$, then $Sa^2 = S$. Hence $S$ is 0-(0, 2)-bisimple.

Theorem 5. Let $A$ be a 0-minimal (0, 2)-bi-ideal of a unitary $\mathcal{L}\mathcal{A}$-semigroup $S$ with zero. Then either $A^2 = \{0\}$ or $A$ is right 0-simple.

Proof. Assume that $A$ is 0-minimal (0, 2)-bi-ideal of $S$ such that $A^2 \neq \{0\}$. Then by using Corollary 4, $Sa^2 = A$ for every $a \in A \setminus \{0\}$. Since $a^2 \in A \setminus \{0\}$ for every $a \in A \setminus \{0\}$, we have $a^4 = (a^2)^2 \in A \setminus \{0\}$ for every $a \in A \setminus \{0\}$. Let $a \in A \setminus \{0\}$, then

$$\begin{align*}
(Aa^2)S \cdot Aa^2 &= a^2A \cdot S(Aa^2) = [(S \cdot Aa^2)A]a^2 \subseteq [(S \cdot A)A]a^2 \\
&= (AA \cdot SS)a^2 = SA^2 \cdot A \subseteq Aa^2,
\end{align*}$$

and

$$\begin{align*}
S(Aa^2)^2 &= S(Aa^2 \cdot Aa^2) = S(a^2A \cdot a^2A) = S[a^2(a^2A \cdot A)] \\
&= (aa)[S(a^2A \cdot A)] = [(a^2A \cdot A)S]a^2 \\
&\subseteq (AA \cdot SS)a^2 = SA^2 \cdot A \subseteq Aa^2,
\end{align*}$$

which shows that $Aa^2$ is a (0, 2)-bi-ideal of $S$ contained in $A$. Hence $Aa^2 = \{0\}$ or $Aa^2 = A$. Since $a^4 \in Aa^2$ and $a^4 \in A \setminus \{0\}$, we get $Aa^2 = A$. Thus by using Theorem 4, $A$ is right 0-simple.}

4. $(m, n)$-Ideals in Unitary $\mathcal{L}\mathcal{A}$-Semigroups

In this section, we characterize a unitary $\mathcal{L}\mathcal{A}$-semigroup in terms of $(m, n)$-ideals with the assumption that $m, n \geq 3$. If we take $m, n \geq 2$, then all the results of this section can be trivially followed for a locally associative unitary $\mathcal{L}\mathcal{A}$-semigroup. If $S$ is a unitary $\mathcal{L}\mathcal{A}$-semigroup, then it is easy to see that $SA^m = A^mS$ and $A^mA^n = A^nA^m$ for $m, n \geq 3$ such that $A^0 = e$ if occurs, where $e$ is a left identity of $S$.

Lemma 7. Let $S$ be a unitary $\mathcal{L}\mathcal{A}$-semigroup. If $R$ and $L$ are the right and left ideals of $S$ respectively, then $RL$ is an $(m, n)$-ideal of $S$. 

\textbf{Proof.} Let \( R \) and \( L \) be the right and left ideals of \( S \) respectively, then

\[
(RL)^n S \cdot (RL)^n = (R^m L^m \cdot S)(R^n L^n) = (R^m L^m \cdot R^n)(SL^n) \\
= (L^m R^m \cdot R^n)(SL^n) = (R^n R^m \cdot L^m)(SL^n) \\
= (R^m R^n \cdot L^m)(SL^n) = (R^{m+n} L^m)(SL^n) \\
= SS \cdot L^m R^{m+n} = SL^{m+n} \cdot SR^{m+n} \\
= R^{m+n} S \cdot L^{m+n} S = SR^{m+n} \cdot SL^{m+n},
\]

and

\[
SR^{m+n} \cdot SL^{m+n} = (S \cdot R^{m+n-1} R)(S \cdot L^{m+n-1} L) \\
= [S(R^{m+n-2} R \cdot R)][S(L^{m+n-2} L \cdot L)] \\
= [S(R \cdot R^{m+n-2})][S(L \cdot L^{m+n-2})] \\
\subseteq (SS \cdot RR^{m+n-2})(SS \cdot LL^{m+n-2}) \\
\subseteq (SR \cdot SR^{m+n-2})(SL \cdot LL^{m+n-2}) \\
\subseteq (R^{m+n-2} S \cdot RS)(L \cdot SL^{m+n-2}) \\
\subseteq (R^{m+n-2} S \cdot SL^{m+n-2}) \\
= (RS \cdot R^{m+n-2})(SL^{m+n-1}) \\
\subseteq RR^{m+n-2} \cdot SL^{m+n-1} \\
\subseteq SR^{m+n-1} \cdot SL^{m+n-1},
\]

therefore

\[
(RL)^n S \cdot (RL)^n \subseteq SR^{m+n} \cdot SL^{m+n} \subseteq SR^{m+n-1} \cdot SL^{m+n-1} \subseteq \ldots \subseteq SR \cdot SL \\
\subseteq (SS \cdot R)L = (RS \cdot S)L \subseteq RL,
\]

and also

\[
RL \cdot RL = LR \cdot LR = (LR \cdot R)L = (R \cdot L)L \subseteq (RS \cdot S)L \subseteq RL.
\]

This shows that \( RL \) is an \((m, n)\)-ideal of \( S \).

\[\square\]

\textbf{Theorem 6.} Let \( S \) be a unitary \( \mathcal{L} \cdot\text{-semigroup} \) with zero. If \( S \) has the property that it contains no non-zero nilpotent \((m, n)\)-ideals and \( R \) \((L)\) is a 0-minimal right \((left)\) ideal of \( S \), then either \( RL = \{0\} \) or \( RL \) is a 0-minimal \((m, n)\)-ideal of \( S \).

\textbf{Proof.} Assume that \( R(L) \) is a 0-minimal right \((\text{left})\) ideal of \( S \) such that \( RL \neq \{0\} \), then by Lemma 7, \( RL \) is an \((m, n)\)-ideal of \( S \). Now we show that \( RL \) is a 0-minimal \((m, n)\)-ideal of \( S \).

Let \( \{0\} \neq M \subseteq RL \) be an \((m, n)\)-ideal of \( S \). Note that since \( RL \subseteq R \cap L \), we have \( M \subseteq R \cap L \). Hence \( M \subseteq R \) and \( M \subseteq L \). By hypothesis, \( M^m \neq \{0\} \) and \( M^n \neq \{0\} \). Since \( \{0\} \neq SM^m = M^m S \), therefore

\[
\{0\} \neq M^m S \subseteq R^m S = R^{m-1} R \cdot S = SR \cdot R^{m-1} = SR \cdot R^{m-2} R
\]
therefore \( \{0\} \neq M^mS \subseteq R^n \subseteq R^{n-1} \subseteq \ldots \subseteq R \). It is easy to see that \( M^mS \) is a right ideal of \( S \). Thus \( M^mS = R \) since \( R \) is 0-minimal. Also

\[
\{0\} \neq SM^n \subsetneq \{0\} \neq SL^n = S \cdot L^{n-1}L = L^{n-1} \cdot SL \subseteq L^{n-1}L = L^n,
\]

and

\[
L^n \subseteq SL^n = SS \cdot LL^{n-1} = L^{n-1}L \cdot S = (L^{n-2}L \cdot L)S = SL \cdot L^{n-2}L \subseteq L \cdot L^{n-2}L = L^{n-2}L \subseteq \ldots \subseteq L,
\]

therefore \( \{0\} \neq SM^n \subseteq L^n \subseteq L^{n-1} \subseteq \ldots \subseteq L \). It is easy to see that \( SM^n \) is a left ideal of \( S \). Thus \( SM^n = L \) since \( L \) is 0-minimal. Therefore

\[
M \subseteq RL = M^mS \cdot SM^n = M^mS \cdot SM^m = (SM^m \cdot S)M^n
\]

\[
=(SM^m \cdot SS)M^n = (S \cdot M^mS)M^n = (M^m \cdot SS)M^n
\]

\[
= M^mS \cdot M^n \subseteq M.
\]

Thus \( M = RL \), which means that \( RL \) is a 0-minimal \((m,n)\)-ideal of \( S \).

**Theorem 7.** Let \( S \) be a unitary \( \mathcal{H} \)-semigroup. If \( R \) \((L)\) is a 0-minimal right \((left)\) ideal of \( S \), then either \( R^mL^n = \{0\} \) or \( R^mL^n \) is a 0-minimal \((m,n)\)-ideal of \( S \).

**Proof.** Assume that \( R \) \((L)\) is a 0-minimal right \((left)\) ideal of \( S \) such that \( R^mL^n \neq \{0\} \), then \( R^m \neq \{0\} \) and \( L^n \neq \{0\} \). Hence \( \{0\} \neq R^n \subseteq R \) and \( \{0\} \neq L^n \subseteq L \), which shows that \( R^m = R \) and \( L^n = L \) since \( R \) \((L)\) is a 0-minimal right \((left)\) ideal of \( S \). Thus by Lemma 7, \( R^mL^n = RL \) is an \((m,n)\)-ideal of \( S \). Now we show that \( R^mL^n \) is a 0-minimal \((m,n)\)-ideal of \( S \). Let \( \{0\} \neq M \subseteq R^mL^n = RL \subseteq R \cap L \) be an \((m,n)\)-ideal of \( S \). Hence

\[
\{0\} \neq SM^2 = MM \cdot SS = MS \cdot MS \subseteq RS \cdot RS \subseteq R
\]

and \( \{0\} \neq SM \subseteq SL \subseteq L \). Thus \( R = SM^2 = MM \cdot SS = SM \cdot M \subseteq SM \) and \( SM = L \) since \( R \) \((L)\) is a 0-minimal right \((left)\) ideal of \( S \). Therefore

\[
M \subseteq R^mL^n \subseteq (SM)^m(SM)^n = SM^m \cdot SM^m \subseteq SM \cdot M^m \subseteq SM \cdot M^m \subseteq M
\]

Thus \( M = R^mL^n \), which shows that \( R^mL^n \) is a 0-minimal \((m,n)\)-ideal of \( S \).
Theorem 8. Let $S$ be a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup with zero. Assume that $A$ is an $(m, n)$-ideal of $S$ and $B$ is an $(m, n)$-ideal of $A$ such that $B$ is idempotent. Then $B$ is an $(m, n)$-ideal of $S$.

Proof. It is trivial that $B$ is an $\mathcal{L} \cdot \mathcal{A}$-subsemigroup $S$. Secondly, since $A^m S \cdot A^n \subseteq A$ and $B^m A \cdot B^n \subseteq B$, then

$$B^m S \cdot B^n = (B^m B^m \cdot S)(B^n B^n) = (B^m B^n)(S \cdot B^m B^m)$$
$$= [(S \cdot B^m B^n)B^n = [(B^n \cdot B^m B^n)(S)B^n$$
$$= [(B^m \cdot B^m B^n)(S)B^n = [S(B^n B^m \cdot B^m)]B^n$$
$$= [S(B^m \cdot B^m B^n)]B^n = [B^m(S S \cdot B^m B^n)]B^n$$
$$= [B^m(B^n B^m \cdot S)]B^n = [B^m(S B^n \cdot B^n)]B^n$$
$$= [B^m(S S \cdot B^{-1} B)]B^n = [B^m(B^m S \cdot B^n)]B^n$$
$$\subseteq [B^m(A^m S \cdot A^n)]B^n \subseteq B^m A \cdot B^n \subseteq B,$$

which shows that $B$ is an $(m, n)$-ideal of $S$. □

Lemma 8. Let $\langle a \rangle_{(m, n)} = a^m S \cdot a^n$, then $\langle a \rangle_{(m, n)}$ is an $(m, n)$-ideal of a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup $S$.

Proof. Assume that $S$ is a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup and $m, n$ are non-negative integers, then

$$\left(\langle a \rangle_{(m, n)}^m H\right) \{\langle a \rangle_{(m, n)}^n \} = \left(\langle a \rangle_{(m, n)}^m H(a^n)\right) \{\langle a \rangle_{(m, n)}^n \}$$
$$= \left\{\langle a \rangle_{(m, n)}^m H(a^n) \langle a \rangle_{(m, n)}^n \right\}$$
$$= \left\{\langle a \rangle_{(m, n)}^m H(a^n) \langle a \rangle_{(m, n)}^n \right\}$$
$$= \left\{\langle a \rangle_{(m, n)}^m H(a^n) \langle a \rangle_{(m, n)}^n \right\}$$
$$= \left\{\langle a \rangle_{(m, n)}^m H(a^n) \langle a \rangle_{(m, n)}^n \right\}$$
$$\subseteq \langle a \rangle_{(m, n)}^m H(a^n) \langle a \rangle_{(m, n)}^n$$
$$= \langle a \rangle_{(m, n)}^m H(a^n) \langle a \rangle_{(m, n)}^n$$

and similarly we can show that $\left(\langle a \rangle_{(m, n)}^m \right) \subseteq \langle a \rangle_{(m, n)}$. □

Theorem 9. Let $S$ be a unitary $\mathcal{L} \cdot \mathcal{A}$-semigroup and $\langle a \rangle_{(m, n)}$ be an $(m, n)$-ideal of $S$. Then the following statements hold:

(i) $\langle a \rangle_{(1, 0)}^m S = a^m S$;
(ii) $S \langle a \rangle_{(0, 1)}^n = S a^n$;
(iii) $\langle a \rangle_{(1, 0)}^m S \cdot \langle a \rangle_{(0, 1)}^n = (a^m S)a^n$. 

Proof. (i). As \((a)_{(1,0)} = aS\), we have
\[
\left( (a)_{(1,0)} \right)^m S = (aS)^m S = (aS)^{m-1} (aS) S = S(aS) \cdot (aS)^{m-1}
\]
\begin{align*}
&= (aS)(aS)^{m-1} = (aS) [(aS)^{m-2}(aS)] \\
&= (aS)^{m-2}(aS \cdot aS) = (aS)^{m-2}(a^2S) \\
&= \ldots = \begin{cases} 
(aS)^{m-(m-1)}(a^{m-1}S) & \text{if } m \text{ is odd} \\
(a^{m-1}S)(aS)^{m-(m-1)} & \text{if } m \text{ is even}
\end{cases}
\end{align*}
\[= a^m S.\]

Analogously, we can prove (ii) and (iii) is simple. \(\square\)

**Corollary 5.** Let \(S\) be a unitary \(\mathcal{L} - \mathcal{A}\)-semigroup and let \((a)_{(m,n)}\) be an \((m,n)\)-ideal of \(S\). Then the following statements hold:

(i) \(\left( (a)_{(1,0)} \right)^m S = Sa^m;\)

(ii) \(S \left( (a)_{(0,1)} \right)^n = a^n S;\)

(iii) \(\left( (a)_{(1,0)} \right)^m S \cdot \left( (a)_{(0,1)} \right)^n = (Sa^m)(a^n S).\)

Let \(\mathcal{L}_{(0,n)}, \mathcal{R}_{(m,0)}\) and \(\mathcal{A}_{(m,n)}\) denote the sets of \((0,n)\)-ideals, \((m,0)\)-ideals and \((m,n)\)-ideals of an \(\mathcal{L} - \mathcal{A}\)-semigroup \(S\) respectively.

**Theorem 10.** If \(S\) is a unitary \(\mathcal{L} - \mathcal{A}\)-semigroup, then the following statements hold:

(i) \(S\) is \((0,1)\)-regular if and only if \(\forall L \in \mathcal{L}_{(0,1)}, L = SL;\)

(ii) \(S\) is \((2,0)\)-regular if and only if \(\forall R \in \mathcal{R}_{(2,0)}, R = R^2 S\) such that every \(R\) is semiprime;

(iii) \(S\) is \((0,2)\)-regular if and only if \(\forall U \in \mathcal{A}_{(0,2)}, U = U^2 S\) such that every \(U\) is semiprime.

Proof. (i). Let \(S\) be \((0,1)\)-regular, then for \(a \in S\) there exists \(x \in S\) such that \(a = xa\). Since \(L\) is \((0,1)\)-ideal, therefore \(SL \subseteq L\). Let \(a \in L\), then \(a = xa \in SL \subseteq L\). Hence \(L = SL\). Converse is simple.

(ii). Let \(S\) be \((2,0)\)-regular and \(R\) be \((2,0)\)-ideal of \(S\), then it is easy to see that \(R = R^2 S\). Now for \(a \in S\) there exists \(x \in S\) such that \(a = a^2 x\). Let \(a^2 \in R\), then
\[
a = a^2 x \in RS = R^2 S \cdot S = SS \cdot R^2 = R^2 S = R,
\]
which shows that every \((2,0)\)-ideal is semiprime.

Conversely, let \(R = R^2 S\) for every \(R \in \mathcal{R}_{(2,0)}\). Since \(Sa^2\) is a \((2,0)\)-ideal of \(S\) such that \(a^2 \in Sa^2\), therefore \(a \in Sa^2\). Thus
\[
a \in Sa^2 = (Sa^2)^2 S = (Sa^2 \cdot Sa^2)S = (a^2 S \cdot a^2 S)S = [a^2(a^2S \cdot S)]S
\]
\[
= (a^2 \cdot Sa^2)S = (S \cdot Sa^2)a^2 \subseteq Sa^2 = a^2 S,
\]
which implies that \(S\) is \((2,0)\)-regular.

Analogously, we can prove (iii). \(\square\)
Lemma 9. If S is a unitary $\mathcal{L}\mathcal{A}$-semigroup, then the following statements hold:

(i) If S is $(0,n)$-regular, then $\forall L \in \mathcal{L}_{(0,n)}$, $L = SL^n$;

(ii) If S is $(m,0)$-regular, then $\forall R \in \mathcal{A}_{(m,0)}$, $R = R^m S$;

(iii) If S is $(m,n)$-regular, then $\forall U \in \mathcal{A}_{(m,n)}$, $U = (U^m S)U^n$.

Proof. It is simple. \hfill \Box

Corollary 6. If S is a unitary $\mathcal{L}\mathcal{A}$-semigroup, then the following statements hold:

(i) If S is $(0,n)$-regular, then $\forall L \in \mathcal{L}_{(0,n)}$, $L = L^n S$;

(ii) If S is $(m,0)$-regular, then $\forall R \in \mathcal{A}_{(m,0)}$, $R = SR^m$;

(iii) If S is $(m,n)$-regular, then $\forall U \in \mathcal{A}_{(m,n)}$, $U = U^{m+n} S = SM^{m+n}$.

Theorem 11. Let S be a unitary $(m,n)$-regular $\mathcal{L}\mathcal{A}$-semigroup such that $m = n$. Then for every $R \in \mathcal{A}_{(m,0)}$ and $L \in \mathcal{L}_{(0,n)}$, $R \cap L = R^m L \cap RL^n$.

Proof. It is simple. \hfill \Box

Theorem 12. Let S be a unitary $(m,n)$-regular $\mathcal{L}\mathcal{A}$-semigroup. If M (N) is a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S such that $MN \subseteq M \cap N$, then either $MN = \{0\}$ or MN is a 0-minimal $( m, n)$-ideal of S.

Proof. Let M (N) be a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S. Let $O = MN$, then clearly $O^2 \subseteq O$. Moreover

$$O^m S \cdot O^n = (MN)^m S \cdot (MN)^n = (M^n M^n) S \cdot M^n N^n \subseteq (M^m S) S \cdot SN^n$$

$$= SM^m \cdot SN^n = M^m S \cdot SN^n \subseteq MN = O,$$

which shows that O is an $(m,n)$-ideal of S. Let $\{0\} \neq P \subseteq O$ be a non-zero $(m,n)$-ideal of S. Since S is $(m,n)$-regular, therefore by using Lemma 9, we have

$$\{0\} \neq P = P^m S \cdot P^n = (P^m S^m S) P^n = (S \cdot P^m S) P^n = (P^n \cdot P^m S) (SS)$$

$$= (P^n S) (P^m S \cdot S) = P^n S \cdot SP^m = P^m S \cdot SP^n.$$

Hence $P^m S \neq \{0\}$ and $P^m S \neq \{0\}$. Further $P \subseteq O = MN \subseteq M \cap N$ implies that $P \subseteq M$ and $P \subseteq N$. Therefore $\{0\} \neq P^m S \subseteq M^m S \subseteq M$ which shows that $P^m S = M$ since M is 0-minimal. Likewise, we can show that $SP^n = N$. Thus we have

$$P \subseteq O = MN = P^m S \cdot SP^n = P^m S \cdot SP^m = (SP^m \cdot SS) P^n$$

$$= (S \cdot P^m S) P^n = P^m S \cdot P^n \subseteq P,$$

This means that $P = MN$ and hence $MN$ is 0-minimal. \hfill \Box
Theorem 13. Let $S$ be a unitary $(m,n)$-regular $\mathcal{L}.\mathcal{A}$-semigroup. If $M$ ($N$) is a $0$-minimal $(m,0)$-ideal (($0,n)$-ideal) of $S$, then either $M \cap N = \{0\}$ or $M \cap N$ is a $0$-minimal $(m,n)$-ideal of $S$.

Proof. Once we prove that $M \cap N$ is an $(m,n)$-ideal of $S$, the rest of the proof is same as in Theorem 11. Let $O = M \cap N$, then it is easy to see that $O^2 \subseteq O$. Moreover

$O^m S \cdot O^n \subseteq M^m S \cdot N^n \subseteq M N^n \subseteq S N^n \subseteq N$. But, we also have

$$O^m S \cdot O^n \subseteq M^m S \cdot N^n = (M^m \cdot S S)N^n = (S \cdot M^m S)N^n = (N^n \cdot M^m S)S$$

$$= (M^m \cdot N^n S)(S S) = (M^m S)(N^n S \cdot S) = M^m S \cdot S N^n$$

$$= M^m S \cdot N^n S = N^n (M^m S \cdot S) = N^n \cdot S M^m = N^n \cdot M^m S$$

$$= M^m \cdot N^n S = M^m \cdot S N^n \subseteq M^m N \subseteq M^m S \subseteq M.$$

Thus $O^m S \cdot O^n \subseteq M \cap N = O$ and therefore $O$ is an $(m,n)$-ideal of $S$. 

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