EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 4, 2014, 442-461
ISSN 1307-5543 – www.ejpam.com

Analyzing Periodic Solutions of an ODE Suspension Bridge Model using Difference Equations and Polynomial Methods

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Abstract. In [13], McKenna and Moore studied oscillations in a suspension bridge by investigating periodic solutions to a differential equations model for the bridge and its linearized version numerically. In this paper, the author seeks to build a rigorous mathematical foundation for the numerical experiments of McKenna and Moore in [13] by studying an associated discrete difference equations model using an interplay of ideas from engineering, discrete dynamical systems, algebraic geometry and the theory of polynomials.

2010 Mathematics Subject Classifications: 39A05, 39A11

Key Words and Phrases: suspension bridge, torsional angle, discrete model, periodic solution, equilibrium, bifurcation, elliptic curve, basin of attraction, global attractivity, eigenvalues

1. Introduction

Suspension bridge dynamics such as factors leading to the 1940 Tacoma Narrows Bridge Collapse in Washington, USA, have been widely studied by civil engineers, architects and applied mathematicians all across the world (see [1, 6–8, 13, 14]). Some researchers prefer to use suspension bridge models involving ordinary differential equations (see [1, 13, 14]) while others prefer to use partial differential equations models (see [6–8]) to study the dynamics of suspension bridges. Often both groups of researchers turn to numerical algorithms to gain further insight into their model dynamics. More specifically, they run their numerical algorithms on discretized versions of their original model for a finite number of initial conditions to get approximate solutions. Some exceptions to this rule are paper [14] by A. Pascoletti and F. Zanolin and paper [8] by Z. Ding in which they present rigorous mathematical proofs to back their results.

A danger in relying too heavily on numerical algorithms to study the dynamics of actual suspension bridges is that one can very easily miss those few useful initial points which lead to extraordinary or unusual bridge dynamics from the point of view of applications (see [13]) since computers do not possess the mathematical intuition needed to look for special initial

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conditions. It is also extremely difficult to predict exact bifurcation values for parameters by relying solely on computer-aided numerical simulations (see [13]). In this paper, the author studies the dynamics of a discrete difference equations model for a suspension bridge using a rigorous mathematical approach involving the theory of difference equations (see [2–4, 9–12, 15]) to get useful global attractivity results and parameter bifurcation values backed by mathematical theorems and proofs.

In [13], P. J. McKenna and K. S. Moore studied oscillations in a suspension bridge by analyzing periodic solutions to the system of nonlinear ordinary differential equations

$$\begin{aligned} \theta'' &= -\frac{6K}{m}\cos\theta\sin\theta - \delta_1\theta' + \lambda\sin\mu t \\ y'' &= -\frac{2K}{m}y - \delta_2 y' + g \end{aligned} \tag{1}$$

where δ_1, δ_2 are damping constants, K, m, λ and μ are positive parameters, g is the force due to gravity and $\lambda\sin\mu t$ is an external force at time t. They used a numerical continuation algorithm to demonstrate the existence of three periodic solutions. The paper relied heavily on numerical experiments and did not focus as much on developing the mathematical theory to back their numerical observations. It was also missing three key aspects, namely, (a) bounds on the number of real equilibria and real periodic solutions, (b) existence conditions for the real equilibria and periodic solutions, and (c) global attractivity results for these solutions including basins of attraction and precise bifurcation values for the parameters λ and μ.

In this paper, the author will set up a rigorous mathematical foundation for the McKenna-Moore suspension bridge model (1) by first discretizing it and then employing analytical and geometrical methods from the theory of difference equations (see [2–4, 9–12, 15]) to analyze equilibria and periodic solutions of the resulting difference equation. For the rest of this paper, the author will use the term 'periodic solutions' to mean periodic solutions of minimal period two. Using this approach, the author will successfully come up with missing mathematical explanations for numerical phenomena observed by McKenna and Moore in [13]. She will also successfully resolve the three key aspects missing from [13], namely, bounds, existence conditions and global attractivity of the real equilibria and periodic solutions of equation (1) in this paper.

This paper is organized as follows. In Section 2, we introduce a discretization of the the McKenna-Moore suspension bridge model (1) involving a second-order nonlinear difference equation. In Section 3, we look at the linearization of the discrete nonlinear model introduced in Section 2. In Section 4, we introduce a modification to our discrete model to make it more realistic. In Section 5, we establish the number of real equilibria and real periodic solutions one can expect to see in our modified discrete model. In Section 6, we establish local and global attractivity results for our model, including basins of attraction and bifurcation values for the parameters λ and μ. In Section 7, we give a physical interpretation of our mathematical results for our modified discrete suspension bridge model from Section 4.

2. A Discrete Nonlinear Suspension Bridge Model

We start with the McKenna-Moore suspension bridge model (1) from the previous section. Replacing the parameters in (1) by actual numerical values from engineers' reports of the 1940 Tacoma Narrows Bridge Collapse which were used by McKenna and Moore in their paper [13] and discretizing the first equation in (1) by setting $\theta' := \theta_{n+1} - \theta_n$, we get the nonlinear nonautonomous difference equation

$$\theta_{n+1} = 1.99\theta_n - 0.99\theta_{n-1} - 2.4\cos\theta_{n-1}\sin\theta_{n-1} + \lambda\sin\mu(n-1) \tag{2}$$

where λ, μ are positive parameters and $-\pi/2 < \theta_n < \pi/2$ for all $n \in \mathbb{N}$. Note that the periodic external forcing term $\lambda\sin\mu(n-1)$ depends on the step number $n-1$. More precisely, it depends on the position of the suspension bridge and hence on its torsional angle θ_{n-1} at step number $n-1$. For example, if the periodic external forcing term is due to blowing wind, then the effect of the wind on the bridge would depend on the position of the bridge with respect to the direction of the wind gusts. Incorporating this in (2), we get the updated autonomous difference equation

$$\begin{aligned}\theta_{n+1} &= 1.99\theta_n - 0.99\theta_{n-1} - 2.4\cos\theta_{n-1}\sin\theta_{n-1} + \lambda\sin\mu\theta_{n-1} \\ &=: f(\theta_{n-1}, \theta_n)\end{aligned} \tag{3}$$

Applying the transformation $\theta_{n+1} := u_n$ and $u_{n+1} := f(\theta_n, u_n)$ to (3) changes it to the nonlinear system of two difference equations

$$\begin{cases}\theta_{n+1} = & u_n \\ u_{n+1} = & 1.99u_n - 0.99\theta_n - 2.4\cos\theta_n\sin\theta_n + \lambda\sin\mu\theta_n\end{cases} \tag{4}$$

whose associated map $T(\theta, u)$ is defined as

$$T\begin{bmatrix}\theta \\ u\end{bmatrix} = \begin{bmatrix} u \\ 1.99u - 0.99\theta - 2.4\cos\theta\sin\theta + \lambda\sin\mu\theta\end{bmatrix} \tag{5}$$

Note that the periodic solutions of (3) are precisely the intersection points of the *equilibrium curves* of the map $T^2(\theta, u) := (F(\theta, u), G(\theta, u))$ which are defined by the equations

$$\begin{cases}F(\theta, u) = & \theta \\ G(\theta, u) = & u\end{cases} \tag{6}$$

In particular, the map $T^2(\theta, u)$ for equation (3) has the form

$$T^2\begin{bmatrix}\theta \\ u\end{bmatrix} = \begin{bmatrix} -0.99(-1.0101\lambda\sin\mu\theta + \theta + 2.42424\sin\theta\cos\theta - 2.0101u) \\ \\ 2.9701(0.670011\lambda\sin\mu\theta + 0.336689\lambda\sin\mu u - 0.663311\theta - \\ 1.60803\sin\theta\cos\theta + u - 0.808054\sin u\cos u)\end{bmatrix} \tag{7}$$

with its associated equilibrium curves given by

$$\begin{cases} E_1: & -0.99(-1.0101\lambda\sin\mu\theta+\theta+2.42424\sin\theta\cos\theta-2.0101u)-\theta=0 \\ E_2: & 2.9701(0.670011\lambda\sin\mu\theta+0.336689\lambda\sin\mu u-0.663311\theta \\ & -1.60803\sin\theta\cos\theta-0.808054\sin u\cos u)=0 \end{cases} \tag{8}$$

Replacing $\sin\theta$, $\cos\theta$, $\sin u$ and $\cos u$ by their Taylor series expansions in (8), one can think of the curves E_1 and E_2 as polynomials of infinite degrees, that is, of degrees n_1 and n_2 where $n_1 \to \infty$ and $n_2 \to \infty$. It follows from Bézout's Theorem [16] that the number of intersections of E_1 and E_2 is bounded by the product $n_1 n_2 \to \infty$. In other words, one can expect to see infinitely many periodic solutions for equation (3). However this is not a realistic scenario from the applications point of view. In Section 4, we will consider a modification of equation (3) with a finite number of periodic solutions. In the next section, we consider a linearization of (3) which will be used to give mathematical explanations for certain numerical observations of McKenna and Moore in [13].

3. A Linearization of the Discrete Nonlinear Model

Recall that for small θ values, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Putting these changes in equation (3) gives the following linearization of (3)

$$\theta_{n+1} = 1.99\theta_n - 0.99\theta_{n-1} - 2.4\theta_{n-1} + \lambda\mu\theta_{n-1} \tag{9}$$

where λ, μ are positive parameters and $-\pi/2 < \theta_n < \pi/2$ for all $n \in \mathbb{N}$. Its associated linearized system has the form

$$\begin{cases} \theta_{n+1} = & u_n \\ u_{n+1} = & 1.99u_n - 0.99\theta_n - 2.4\theta_n + \lambda\mu\theta_n \end{cases} \tag{10}$$

It is straightforward to see that the equilibrium curves (6) of the map $T^2(\theta, u)$ of (10) belong to straight lines passing through the origin with explicit formulas

$$\begin{cases} E_1: & u = 0.502513(4.39-\lambda\mu)t \\ E_2: & u = \frac{1.99(3.39-\lambda\mu)}{\lambda\mu-0.4299}t \end{cases}$$

Moreover it is easy to see that E_1 and E_2 intersect at infinitely many points exactly when they have the same slope, that is exactly when $\lambda\mu = 2.4, 6.38$. For all other $\lambda\mu$-values, E_1 and E_2 intersect only at the origin which also happens to be the equilibrium solution of the map $T^2(\theta, u)$. It follows that the linearization (9) of equation (3) has infinitely many periodic solutions for $\lambda\mu = 2.4$ and 6.38, and no nontrivial periodic solutions for all other $\lambda\mu$-values.

This suggests that linearization is not a very practical way to study oscillations in a suspension bridge since the dynamics in this case are too extreme to occur in real life. But it does help to explain mathematically the numerical observations of McKenna and Moore in [13], an explanation that was missing from their paper. More specifically, McKenna and Moore noted in their paper that for their linearized differential equation

$$\theta'' = -2.4\theta - 0.01\theta' + \lambda \sin \mu t \tag{11}$$

periodic solutions occur for $\mu \approx 1.55$ and for *small* λ whose exact range was unspecified in the paper. Moreover for $\mu > 1.55$, they observed no bifurcation from one to many periodic solutions. They also noted that for $\mu < 1.55$ but close to this value, their numerical algorithm *did not converge* to a periodic solution. They did not address the case where $\mu < 1.55$ and far away from this value at all in their paper [13]. In short, the case $\mu < 1.55$ was pretty much left unanalyzed in their paper.

For their numerical experiments, McKenna and Moore used the four sets of $\lambda\mu$-values given below to study the occurrence of periodic solutions in (11).

$$\{\lambda, \mu\} = \{0.0126, 1\}, \{0.0117, 1.2\}, \{0.0088, 1.4\}, \{0.0197, 1.5\}$$

In the first three cases, they observed bifurcations from single to multiple periodic solutions while in the fourth case, they simply noted the existence of multiple periodic solutions. A quick check shows that in the first three cases, the product $\lambda\mu$ is roughly constant with $\lambda\mu \approx 0.013$ or 0.014 with 3-digit rounding and in the fourth case, $\lambda\mu \approx 0.03$.

Drawing analogy to the bifurcation analysis of the discrete linearized difference equations model (9) presented earlier in this section, one can conclude that the McKenna-Moore linearized equation (11) has infinitely many periodic solutions for $\lambda\mu \approx 0.013$ (or 0.014) and 0.03, and no nontrivial periodic solutions for all other $\lambda\mu$-values. This would explain why their numerical algorithm did not converge to a periodic solution in some cases (no nontrivial periodic solutions in theses cases!) while in other cases they observed no bifurcation from one to many periodic solutions (infinitely many periodic solutions in these cases!). To summarize, the authors in [13] missed the fact that the observed bifurcations from one to multiple periodic solutions in equation (11) were being caused not separately by the parameters λ or μ but rather by their joint product $\lambda\mu$. Analyzing its associated discrete linearized difference equation (9) as shown above helped to figure this out. In the next section, we look at a modification of equation (3).

4. A Modified Discrete Nonlinear Suspension Bridge Model

In this section, we will consider a modified nonlinear discrete difference equations model for a suspension bridge with a finite number of periodic solutions. The goal is to come up with a more realistic model from an applications point of view than the original discrete nonlinear model (3) with the infinite number of periodic solutions which was introduced in Section 2. The strategy will be to replace $\cos\theta$ and $\sin\theta$ by their second-order Taylor polynomial approximations in (3) and analyze the resulting model mathematically. The proofs for higher-order Taylor polynomial approximations are essentially the same and will be omitted.

One way to decide which order Taylor Polynomial to use is to see which order best approximates an actual data set of torsional angle values for a given suspension bridge taken at different time intervals. Replacing $\cos\theta$ and $\sin\theta$ by their respective second-order Taylor polynomial approximations $1-\theta^2$ and θ in (3), we get the new equation

$$\theta_{n+1} = 1.99\theta_n - 0.99\theta_{n-1} - 2.4(1-\theta_{n-1}^2)\theta_{n-1} + \lambda\mu\theta_{n-1} \tag{12}$$

where λ, μ are positive parameters and $-\pi/2 < \theta_n < \pi/2$ for all $n \in \mathbb{N}$. Its associated 2D-system is

$$\begin{cases} \theta_{n+1} = & u_n \\ u_{n+1} = & 1.99u_n - 0.99\theta_n - 2.4(1-\theta_n^2)\theta_n + \lambda\mu\theta_n \end{cases} \tag{13}$$

with associated map $T(\theta,u)$ defined as

$$T\begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} u \\ 1.99u - 0.99\theta - 2.4(1-\theta^2)\theta + \lambda\mu\theta \end{bmatrix} \tag{14}$$

Setting $T(\theta,u) = (\theta,u)$ and solving for θ in (14) gives the formulas for the three equilibria of (12) as

$$\begin{aligned} &E_1 := (0,0) \\ &E_2 := (0.645497\sqrt{2.4-\lambda\mu}, 0.645497\sqrt{2.4-\lambda\mu}) \\ &E_3 := (-0.645497\sqrt{2.4-\lambda\mu}, -0.645497\sqrt{2.4-\lambda\mu}) \end{aligned}$$

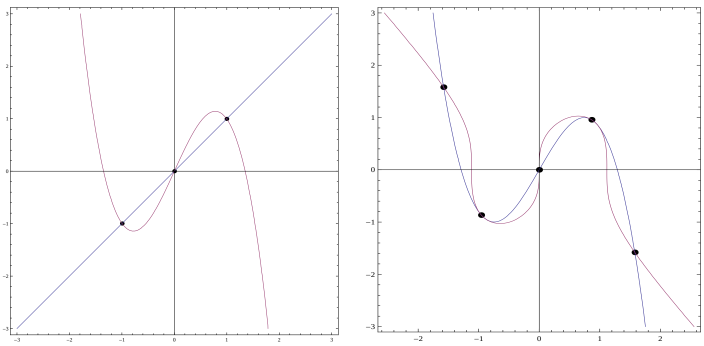
To find the number of real periodic solutions of (12), note that these are precisely the intersection points of the equilibrium curves of the map $T^2(\theta,u)$ whose explicit formulas can be found by solving the equation $T^2(\theta,u) = (\theta,u)$ where

$$T^2\begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} 1.99u - 0.99\theta - 2.4(1-\theta^2)\theta + \lambda\mu\theta \\ 1.99(1.99u - 0.99\theta - 2.4(1-\theta^2)\theta + \lambda\mu\theta) + \lambda\mu u - 2.4\left(1-u^2\right)u - 0.99u \end{bmatrix} \tag{15}$$

A simple calculation shows that the equilibrium curves of $T^2(\theta,u)$ are elliptic curves with formulas

$$\begin{cases} E_1: & -0.99(-1.0101\lambda\mu\theta + \theta + 2.424224\theta(1-\theta^2) - 2.0101u) - \theta = 0 \\ E_2: & 2.9701(0.670011\lambda\mu\theta + 0.336689\lambda\mu u - 0.663311\theta - 1.60803\theta(1-\theta^2) \\ & -0.808054u(1-u^2) = 0 \end{cases} \tag{16}$$

The equilibrium curves for the maps $T(\theta,u)$ and $T^2(\theta,u)$ are shown in Figure 1. According to Bézout's Theorem, the maximum number of intersections of the third-degree elliptic curves E_1 and E_2 counting complex intersections and multiplicities is $3 \times 3 = 9$. Thus Bézout's Theorem gives an upper bound for the number of real periodic solutions to (12), which is what we are interested in for studying real oscillations in a suspension bridge. In the next section, we will apply Descartes' Rule of Signs for real roots of polynomials (see [16]) to predict the number of real periodic solutions of (12) for different $\lambda\mu$-parameter regions.



(a) (b)

Figure 1: Equilibrium curves for the maps $T(\theta, u)$ (on the left) and $T^2(\theta, u)$ (on the right). The black dots represent equilibria (left) and periodic solutions (right) of equation (12).

5. Number of Real Equilibria and Periodic Solutions

In [13], McKenna and Moore used a numerical continuation algorithm to demonstrate the existence of three periodic solutions to their continuous suspension bridge model (1) through a finite number of numerical experiments (five in their paper). However, they did not give a rigorous mathematical proof for the existence of (or lack of) three periodic solutions in all cases not covered by their finite number of simulations. More specifically, they did not address the question of whether to expect more or less than three periodic solutions for some $\lambda\mu$-parameter regions. They also did not distinguish between trivial periodic solutions, namely, equilibrium solutions, and nontrivial periodic solutions for their model (1).

In this section, we will rigorously prove that our suspension bridge model (12) can have at most three real equilibria and at most four real nontrivial periodic solutions for $0 < \lambda\mu \leq 6.38$. Moreover, we will show that it is possible to have $\lambda\mu$-parameter regions where at most one or even no real nontrivial periodic solutions exist. The main theorem of this section is as follows.

Theorem 1. *The number of real equilibria and real nontrivial periodic solutions of equation* (12) *for various $\lambda\mu$-parameter regions is as shown in Table* 1.

Table 1: Table showing the number of real equilibria and real nontrivial periodic solutions of equation (12).

$\lambda\mu$ - region	No. of real equilibria	No. of real periodic solns.
$0 < \lambda\mu < 2.4$	three	$\leq$ four pairs
$\lambda\mu = 2.4$	$(0,0)$ unique equilibrium	$\leq$ three pairs
$2.4 < \lambda\mu < 4.39$	$(0,0)$ unique equilibrium	$\leq$ three pairs
$\lambda\mu = 4.39$	$(0,0)$ unique equilibrium	$\leq$ one pair
$4.39 < \lambda\mu < 6.38$	$(0,0)$ unique equilibrium	$\leq$ one pair
$\lambda\mu = 6.38$	$(0,0)$ unique equilibrium	none
$\lambda\mu > 6.38$	$(0,0)$ unique equilibrium	none

Proof. The number of real equilibria in Table 1 follows directly from their formulas given in the previous section. Recall that the periodic solutions of equation (12) are precisely the intersection points of the elliptic curves E_1 and E_2 which were defined in (16). Solving the equations for E_1 and E_2 simultaneously for the variable θ gives rise to the ninth-degree polynomial equation

$$\begin{aligned} g(\theta) = & -4.210004\theta[\theta^8 + (-5.4875 + 1.25\lambda\mu)\theta^6 \\ & + (10.0376 - 4.57292\lambda\mu + 0.520833\lambda^2\mu^2)\theta^4 \\ & + (-7.3777 + 4.46878\lambda\mu - 0.952691\lambda^2\mu^2 + 0.072338\lambda^3\mu^3)\theta^2 \\ & + (1.82765 - 1.04799\lambda\mu + 0.119361\lambda^2\mu^2)] \\ =: & -4.210004\theta h(\theta) \end{aligned} \tag{17}$$

where

$$\begin{aligned} h(\theta) = & \theta^8 + (-5.4875 + 1.25\lambda\mu)\theta^6 + (10.0376 - 4.57292\lambda\mu + 0.520833\lambda^2\mu^2)\theta^4 \\ & + (-7.3777 + 4.46878\lambda\mu - 0.952691\lambda^2\mu^2 + 0.072338\lambda^3\mu^3)\theta^2 \\ & + (1.82765 - 1.04799\lambda\mu + 0.119361\lambda^2\mu^2) \end{aligned} \tag{18}$$

Observe that the coefficients of all terms except the leading θ^8-term in $h(\theta)$ are polynomials in the 'new' variable $\lambda\mu$. Renaming $\lambda\mu \to s$, one can rewrite $h(\theta)$ in (18) as follows.

$$h(\theta) = \theta^8 + P_1(s)\theta^6 + P_2(s)\theta^4 + P_3(s)\theta^2 + P_4(s) \tag{19}$$

where

$$\begin{aligned} P_1(s) &:= -5.4875 + 1.25s \\ P_2(s) &:= 10.0376 - 4.57292s + 0.520833s^2 \\ P_3(s) &:= -7.3777 + 4.46878s - 0.952691s^2 + 0.072338s^3 \\ P_4(s) &:= 1.82765 - 1.04799s + 0.119361s^2 \end{aligned}$$

The graphs of $P_1(s)$, $P_2(s)$, $P_3(s)$ and $P_4(s)$ are shown in Figure 2. Solving the equations $P_1(s) = 0$, $P_2(s) = 0$ and $P_3(s) = 0$ simultaneously, we see that the common s-intercept of the three polynomial curves $P_1(s)$, $P_2(s)$ and $P_3(s)$ is at $s = 4.39$, that is, at $\lambda\mu = 4.39$. Moreover, the parabola $P_2(s)$ has a real, repeated root at $s = 4.39$ and is positive everywhere else. The line $P_1(s)$ and the cubic $P_3(s)$ are clearly positive for $s > 4.39$ and negative elsewhere. Moreover, the two s-intercepts of the parabola $P_4(s)$ are given by $s = 2.4$ and $s = 6.38$. In particular, $P_4(s)$ is negative between these two s-values and positive everywhere else. These observations are summarized in Table 2 in terms of the coefficients of the polynomial $h(\theta)$ given in (19).

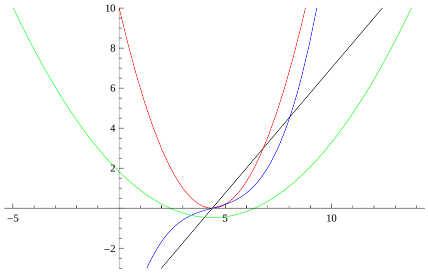


Figure 2: Diagram showing the polynomials $P_1(s)$ in black, $P_2(s)$ in red, $P_3(s)$ in blue and $P_4(s)$ in green.

Table 2: Table showing the signs of the coefficients of $h(\theta)$ for various $\lambda\mu$-parameter regions.

$\lambda\mu$ - region	θ^8 Coeff.	θ^6 Coeff.: $P_1(s)$	θ^4 Coeff.: $P_2(s)$	θ^2 Coeff.: $P_3(s)$	Constant: $P_4(s)$
$0 < \lambda\mu < 2.4$	+	-	+	-	+
$\lambda\mu = 2.4$	+	-	+	-	0
$2.4 < \lambda\mu < 4.39$	+	-	+	-	-
$\lambda\mu = 4.39$	+	0	0	0	-
$4.39 < \lambda\mu < 6.38$	+	+	+	+	-
$\lambda\mu = 6.38$	+	+	+	+	0
$\lambda\mu > 6.38$	+	+	+	+	+

It follows from Table 2 that for $0 < \lambda\mu < 2.4$, there are four sign changes between the coefficients of $h(\theta)$. Hence by Descartes' Rule of Signs [16], there exist at most four positive periodic solutions of equation (12). We also saw in Section 4 that (12) has a positive equilibrium $\theta = 0.645497\sqrt{2.4-\lambda\mu}$ for $\lambda\mu < 2.4$. Since equilibria are trivially periodic solutions, equation (12) has at most $4-1=3$ nontrivial positive periodic solutions for $\lambda\mu < 2.4$. Also note that $h(-\theta)$ has the same number of sign changes between its coefficients as $h(\theta)$. Hence one has from Descartes' Rule of Signs that (12) has at most 3 nontrivial negative periodic solutions for $\lambda\mu < 2.4$. Since periodic solutions of discrete 2D-systems occur in pairs, one can conclude that there exist at most three pairs of real nontrivial periodic solutions of equation (12) for $0 < \lambda\mu < 2.4$. Similar arguments can be given for the remaining $\lambda\mu$-parameter cases in Table 2. $\square$

To summarize, in this section we looked at the exact number of equilibria for equation (12) and computed strict upper bounds for the number of real nontrivial periodic solutions of (12) in the parameter region $0 < \lambda\mu < 2.4$. In the next section, we will address global attractivity and basins of attraction of these real equilibria and periodic solutions.

6. Global Attractivity and Basins of Attraction of the Real Equilibria and Periodic Solutions

An important question that McKenna and Moore failed to address in [13] was global attractivity properties of the real equilibria and periodic solutions of their continuous suspension bridge model (1), including basins of attraction of these solutions. One can expect global attractivity properties of real periodic solutions to play a very important role in predicting the stability of a swaying bridge, with an attracting periodic solution implying stable bridge oscillations in the long run and a repelling or saddle point periodic solution signifying the onset of unstable or even chaotic bridge oscillations in the long run possibly leading to disastrous consequences such as the 1940 Tacoma Narrows Bridge Collapse in Washington, USA.

In this section, we will compute for our discrete suspension bridge model (12) precise $\lambda\mu$-parameter regions where there exist a unique equilibrium, multiple equilibria, a unique real periodic solution and multiple real periodic solutions. We will also compute basins of attraction for these. We start by stating a local stability result for the equilibria E_1, E_2, and E_3 of (12) whose formulas were first introduced in Section 4 and are given below for easy reference.

$$\begin{aligned} E_1 &:= (0,0) \\ E_2 &:= (0.645497\sqrt{2.4-\lambda\mu}, 0.645497\sqrt{2.4-\lambda\mu}) \\ E_3 &:= (-0.645497\sqrt{2.4-\lambda\mu}, -0.645497\sqrt{2.4-\lambda\mu}) \end{aligned} \tag{20}$$

Lemma 1. *The equilibria E_1, E_2, and E_3 are unstable for all admissible $\lambda\mu$-parameter regions.*

Proof. It is easy to check that the eigenvalues λ_1 and λ_2 (with $\lambda_1 \leq \lambda_2$) of the jacobian of the map $T(\theta, u)$ evaluated at the three equilibria are as follows:

$$\begin{aligned} E_1 &: \lambda_1 = 1 - \sqrt{\lambda\mu - 2.4}, \lambda_2 = 1 + \sqrt{\lambda\mu - 2.4} \\ E_2 &: \lambda_1 = 1 - 1.414\sqrt{2.4-\lambda\mu}, \lambda_2 = 1 + 1.414\sqrt{2.4-\lambda\mu} \\ E_3 &: \lambda_1 = 1 - 1.414\sqrt{2.4-\lambda\mu}, \lambda_2 = 1 + 1.414\sqrt{2.4-\lambda\mu} \end{aligned}$$

Note that E_2 and E_3 have the same set of eigenvalues. Moreover, an easy calculation shows that the eigenvalues of E_1 satisfy the following inequalities for the given $\lambda\mu$-parameter regions:

$$\begin{cases} 0 < \lambda\mu < 2.4: & |\lambda_1| > 1, |\lambda_2| > 1 \text{ (a)} \\ \lambda\mu = 2.4: & |\lambda_1| = |\lambda_2| = 1 \quad \text{(b)} \\ 2.4 < \lambda\mu < 6.38: & |\lambda_1| < 1, |\lambda_2| > 1 \text{ (c)} \\ \lambda\mu = 6.38: & |\lambda_1| = 1, |\lambda_2| > 1 \text{ (d)} \\ \lambda\mu > 6.38: & |\lambda_1| > 1, |\lambda_2| > 1 \text{ (e)} \end{cases} \tag{21}$$

Hence E_1 is a repeller in cases (a) and (e), a saddle point equilibrium in case (c) and a nonhyperbolic equilibrium in cases (b) and (d). Similarly, one can show that the eigenvalues of E_2

and E_3 satisfy

$$\begin{cases} 0 < \lambda\mu < 0.41: & |\lambda_1| > 1, |\lambda_2| > 1 \\ \lambda\mu = 0.41: & |\lambda_1| = 1, |\lambda_2| > 1 \\ 0.41 < \lambda\mu < 2.4: & |\lambda_1| < 1, |\lambda_2| > 1 \end{cases} \tag{22}$$

Hence E_2 and E_3 are both repellers, saddle point equilibria or nonhyperbolic equilibria. □

The next theorem follows directly from (20), (21) and (22).

Theorem 2. *The following are true for the equilibria E_1, E_2 and E_3 of equation* (12)*:*

1. *If $0 < \lambda\mu < 2.4$, then all three equilibria E_1, E_2 and E_3 exist. The zero equilibrium E_1 is a repelling equilibrium. The nonzero equilibria E_2 and E_3 satisfy:*

   i. *If $0 < \lambda\mu < 0.41$, then they are both repelling equilibria.*

   ii. *If $0.41 < \lambda\mu < 2.4$, then they are both saddle point equilibria.*

   iii. *If $\lambda\mu = 0.41$, then they are both nonhyperbolic equilibria.*

2. *If $2.4 \le \lambda\mu < 6.38$, then E_1 is a unique saddle point equilibrium.*

3. *If $\lambda\mu \ge 6.38$, then E_1 is a repelling equilibrium.*

Note that the south-east partial ordering "$\preceq_{se}$" and the north-east partial ordering "$\preceq_{ne}$" from the theory of cooperative and competitive maps (see see [5, 11, 12, 15]) are defined as follows:

$$\begin{array}{lll} (x_1, y_1) \preceq_{se} (x_2, y_2) & \text{if and only if} & x_1 \le x_2 \text{ and } y_1 \ge y_2 \\ (x_1, y_1) \preceq_{ne} (x_2, y_2) & \text{if and only if} & x_1 \le x_2 \text{ and } y_1 \le y_2 \end{array} \tag{23}$$

It is a well-known fact that cooperative maps preserve the "$\preceq_{ne}$" ordering (see [5, 11, 12, 15]). Our next lemma gives precise parameter and initial value conditions for $T^2(\theta, u)$ to be cooperative. It will play a key role in the proof of our main theorem for this section.

Lemma 2. *The map $T^2(\theta, u)$ is cooperative exactly when one of the following holds:*

i. $3.39 \le \lambda\mu \le 6.38$

ii. $0 \le \lambda\mu < 3.39$ *and $\theta \notin (\theta_1, \theta_2)$, where θ_1 and θ_2 are given by the formulas*

$$\theta_1 := -0.372678\sqrt{3.39 - \lambda\mu} \quad \textit{and} \quad \theta_2 := 0.372678\sqrt{3.39 - \lambda\mu}$$

Proof. Note that the Jacobian of the map $T^2(\theta, u)$ is given by

$$Jac_{T^2} = \begin{pmatrix} 7.2\theta^2 + \lambda\mu - 3.39 & 1.99 \\ 1.99(7.2\theta^2 + \lambda\mu - 3.39) & \lambda\mu + 7.2u^2 + 0.5701 \end{pmatrix} \tag{24}$$

If $3.39 \le \lambda\mu \le 6.38$, then all entries of the jacobian are positive which implies that the associated map $T^2(\theta, u)$ is cooperative (see [11, 12, 15]). If $0 < \lambda\mu < 3.39$, then the $1,1$-entry of the jacobian in (24),

$$f(\theta) := 7.2\theta^2 + \lambda\mu - 3.39$$

is a parabola with positive leading coefficient whose θ-axis intercepts are given by the formulas

$$\theta_1, \theta_2 = \pm 0.372678\sqrt{3.39 - \lambda\mu}, \quad \theta_1 < \theta_2 \tag{25}$$

Clearly $f(\theta) \geq 0$ for $\theta \in (-\infty, \theta_1] \bigcup [\theta_2, \infty)$. Hence the map $T^2(\theta, u)$ is cooperative in this θ-range. □

The main theorem of this paper is as follows:

Theorem 3. *Under the hypotheses of Lemma 2, equation* (12) *has exactly two real nontrivial periodic solutions* $\{\{p_1, q_1\}, \{p_2, q_2\}\}$ *and* $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ *where*

$$\{p_1, q_1\} \preceq_{se} (0,0) \preceq_{se} \{p_2, q_2\} \quad \textit{and} \quad \{s_1, t_1\} \preceq_{ne} (0,0) \preceq_{ne} \{s_2, t_2\}$$

$\{\{p_1, q_1\}, \{p_2, q_2\}\}$ *is a repelling periodic solution and* $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ *is locally asymptotically stable. The global dynamics of equation* (12) *is as follows:*

1. *If* $0 < \lambda\mu < 2.4$*, then both pairs of periodic solutions are present. Moreover,*
   - i. *If* $0 < \lambda\mu < 0.41$*, then every solution converges to the stable periodic solution* $\{\{s_1, t_1\}, \{s_2, t_2\}\}$.
   - ii. *If* $0.41 \leq \lambda\mu < 2.4$*, then every solution converges to the stable periodic solution* $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ *or to the saddle point equilibria* E_2 *or* E_3.
2. *If* $2.4 \leq \lambda\mu < 6.38$*, then the repelling periodic solution* $\{\{p_1, q_1\}, \{p_2, q_2\}\}$ *is the only one present. Every solution converges to the zero saddle point equilibrium* E_1.
3. *If* $\lambda\mu \geq 6.38$*, then equation* (12) *does not possess any periodic solutions. The zero equilibrium* E_1 *is a repelling equilibrium. Every solution escapes to infinity.*

The proof of Theorem 3 follows from the statements of Theorem 2, Lemmas 3-5, Propositions 1-2, Corollaries 1-2 and the fact that all orbits of a bounded cooperative map must converge to a locally asymptotically stable equilibrium or to a saddle point equilibrium (see for example [5, 11, 12, 15]).

The next lemma gives bounds for the coordinates of the map $T^2(\theta, u)$. It will be crucial for the rest of the proofs in this section.

Lemma 3. *Under the assumptions* $-\pi/2 \leq \theta \leq \pi/2$ *and* $0 < \lambda\mu \leq 6.38$*, there exist two real numbers* m_1 *and* m_2 *for which the map* $T^2(\theta, u) = (F(\theta, u), G(\theta, u))$ *satisfies*

$$-m_1 \leq F(\theta, u) \leq m_1 \quad \textit{and} \quad -m_2 \leq G(\theta, u) \leq m_2$$

Proof. Note that under the given assumptions, the first coordinate of the map

$$T^2 \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} 1.99u - 0.99\theta - 2.4(1 - \theta^2)\theta + \lambda\mu\theta \\ 1.99(1.99u - 0.99\theta - 2.4(1 - \theta^2)\theta + \lambda\mu\theta) + \lambda\mu u - 2.4\left(1 - u^2\right)u - 0.99u \end{bmatrix}$$

which was defined in (15) has bounds given by

$$-17.7528 \le 1.99u - 0.99\theta - 2.4(1-\theta^2)\theta + \lambda\mu\theta \le 17.7528 \tag{26}$$

and its second coordinate has bounds given by

$$\begin{aligned} -9.09491 \le & 1.99(1.99u - 0.99\theta - 2.4(1-\theta^2)\theta + \lambda\mu\theta) + \lambda\mu u - 2.4\left(1-u^2\right)u \\ & -0.99u \le 9.09491 \end{aligned} \tag{27}$$

Choose $m_1 = 17.7528$ and $m_2 = 9.09491$ in the statement of the lemma. □

Define the four regions $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ and $\mathscr{R}_4$ as follows.

$$\begin{aligned} \mathscr{R}_1 &:= [0, m_1] \times [0, m_2] \\ \mathscr{R}_2 &:= [-m_1, 0] \times [0, m_2] \\ \mathscr{R}_3 &:= [-m_1, 0] \times [-m_2, 0] \\ \mathscr{R}_4 &:= [0, m_1] \times [-m_2, 0] \end{aligned} \tag{28}$$

The following lemma establishes an invariant attracting set for the map $T^2(\theta, u)$ when the latter is cooperative. It will play a key role in establishing global attractivity results for equation (12) later on in this section.

Lemma 4. *Under the hypotheses of Lemma 2, the regions $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ and $\mathscr{R}_4$ satisfy:*

$$T^2(\mathscr{R}_1) \subseteq \mathscr{R}_1, \quad T(\mathscr{R}_2) \subseteq \mathscr{R}_1, \quad T^2(\mathscr{R}_3) \subseteq \mathscr{R}_3, \quad T(\mathscr{R}_4) \subseteq \mathscr{R}_3$$

In particular, $\mathscr{R}_1 \cup \mathscr{R}_3$ is an invariant attracting set for the map $T^2(\theta, u)$.

Proof. To see that $T^2(\mathscr{R}_1) \subseteq \mathscr{R}_1$, observe that all $(\theta, u) \in \mathscr{R}_1$ satisfy

$$(0,0) \preceq_{ne} (\theta, u) \preceq_{ne} (m_1, m_2)$$

Under the hypotheses of Lemma 2, the map $T^2(\theta, u)$ is cooperative. Since it also has $(0,0)$ as a fixed point, it follows from the "$\preceq_{ne}$" order-preserving property of cooperative maps (see [12, 15]) that

$$(0,0) = T^2(0,0) \preceq_{ne} T^2(\theta, u) \preceq_{ne} T^2(m_1, m_2) \preceq_{ne} (m_1, m_2)$$

which is what we wanted to prove. The proof of $T^2(\mathscr{R}_3) \subseteq \mathscr{R}_3$ is similar and we skip it.

To see that $T(\mathscr{R}_2) \subseteq \mathscr{R}_1$, note that the equilibrium curves $\theta = f(\theta, u)$ and $u = g(\theta, u)$ of the map $T(\theta, u) = (f(\theta, u), g(\theta, u))$ defined in (14) are both increasing curves passing through the origin. In particular, the region $\mathscr{R}_2$ lies entirely above them. Moreover, all $(\theta, u) \in \mathscr{R}_2$ satisfy $(\theta, u) \preceq_{ne} T(\theta, u)$. For example, one can check that $(0, u) \preceq_{ne} T(0, u)$ for $u > 0$. It is also easy to check that $T(\theta, u)$ is a cooperative map by looking at its jacobian. Hence for all $(\theta, u) \in \mathscr{R}_2$, one has

$$(\theta, u) \preceq_{ne} T(\theta, u) \preceq_{ne} T^2(\theta, u) \preceq_{ne} \dots$$

The monotone increasing sequence given above must enter the region $\mathscr{R}_1$ since otherwise it would have to converge to a fixed point of $T(\theta,u)$ in $\mathscr{R}_2$, contradicting the fact that the equilibrium curves of $T(\theta,u)$ do not lie in this region by our previous discussion. The proof of $T(\mathscr{R}_4)\subseteq\mathscr{R}_3$ is similar and we skip it. □

Our next lemma establishes the fact that any region bounded by two fixed points of a cooperative map in the "$\preceq_{ne}$" ordering must contain a locally asymptotically stable fixed point of this map.

Lemma 5. *Suppose $\widetilde{T}:\mathbb{R}\to\mathbb{R}$ is a cooperative map. In addition, suppose there exist two points (x_1^*,y_1^*) and (x_2^*,y_2^*) with $(x_1^*,y_1^*)\preceq_{ne}(x_2^*,y_2^*)$ in $\mathbb{R}\times\mathbb{R}$ such that*

$$\widetilde{T}(x_1^*,y_1^*)=(x_1^*,y_1^*) \quad \textit{and} \quad \widetilde{T}(x_2^*,y_2^*)=(x_2^*,y_2^*)$$

Then there exists a point $(p^*,q^*)\in[x_1^*,x_2^*]\times[y_1^*,y_2^*]$ such that $\widetilde{T}(p^*,q^*)=(p^*,q^*)$. Moreover, (p^*,q^*) must be a locally asymptotically stable fixed point of the map $\widetilde{T}$.

Proof. It is easy to see that $[x_1^*,x_2^*]\times[y_1^*,y_2^*]$ is an invariant region for $\widetilde{T}$ since

$$(x_1^*,y_1^*)=\widetilde{T}(x_1^*,y_1^*)\preceq_{ne}(x,y)\preceq_{ne}\widetilde{T}(x,y)\preceq_{ne}\widetilde{T}(x_2^*,y_2^*)=(x_2^*,y_2^*)$$

for all $(x,y)\in[x_1^*,x_2^*]\times[y_1^*,[y_2^*]$. The proof follows from this and the theory of cooperative maps. □

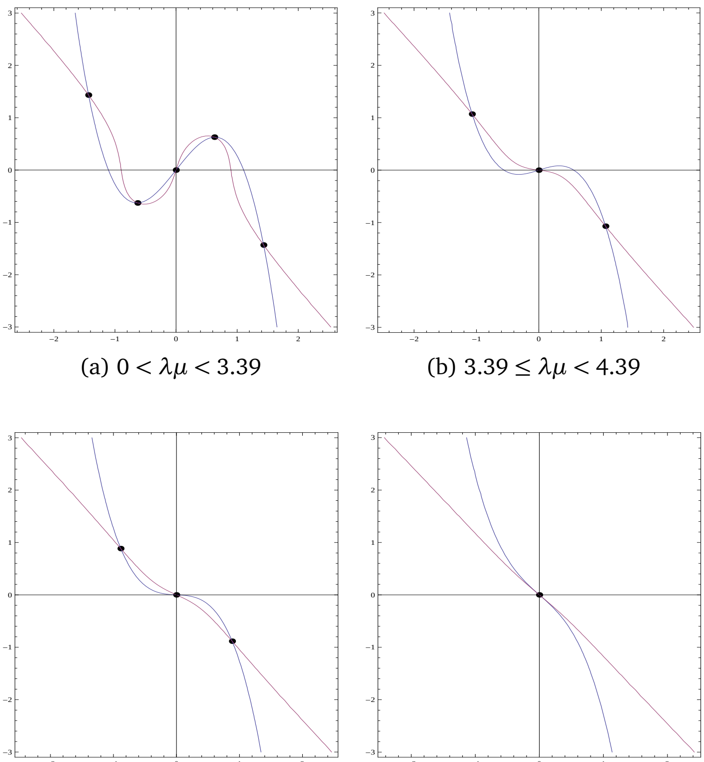
The next two propositions give existence and uniqueness conditions for the real periodic solutions of equation (12). They complete the proof of Theorem 3 and are as follows.

Proposition 1. *Suppose the hypotheses of Lemma 2 hold. If $0<\lambda\mu<6.38$, then there always exists a unique periodic solution $\{\{p_1,q_1\},\{p_2,q_2\}\}$ of equation (12) in the bounded region $\mathscr{R}_2\cup\mathscr{R}_4$ defined by (28). This periodic solution is repelling in nature and disappears for $\lambda\mu\geq 6.38$.*

Proof. Observe that the θ-intercepts of the equilibrium curves E_1 and E_2 of the map $T^2(\theta,u)$ defined in (16) are respectively given by

$$\begin{aligned} E_1:&\quad \theta=0,\quad \pm0.645497\sqrt{4.39-\lambda\mu}\\ E_2:&\quad \theta=0,\quad \pm0.645497\sqrt{3.39-\lambda\mu}\end{aligned} \tag{29}$$

It follows that if $0<\lambda\mu<3.39$, then both E_1 and E_2 have a positive θ-axis intercept and a negative θ-axis intercept in addition to the $(0,0)$-intercept. If $3.39\leq\lambda\mu<4.39$, then only E_1 has the two nonzero θ-axis intercepts. And if $4.39\leq\lambda\mu<6.38$, then both E_1 and E_2 have only the $(0,0)$-intercept. Moreover, it is straightforward to check that both E_1 and E_2 have negative slopes in $\mathscr{R}_2\cup\mathscr{R}_4$ which satisfy Slope$(E_1)<$ Slope (E_2) at all three θ-intercepts. Hence E_1 and E_2 must intersect transversally in $\mathscr{R}_2\cup\mathscr{R}_4$ whenever $0<\lambda\mu<6.38$. If $\lambda\mu\geq6.38$, then this is no longer true since the slopes of E_1 and E_2 now satisfy Slope$(E_1)=$ Slope (E_2) at the origin. The various scenarios are shown in Figure 3. The repelling nature of $\{\{p_1,q_1\},\{p_2,q_2\}\}$ is a direct consequence of the fact that $T(\mathscr{R}_2)\subseteq\mathscr{R}_1$ and $T(\mathscr{R}_4)\subseteq\mathscr{R}_3$ by Lemma 4. □



(a) $0 < \lambda\mu < 3.39$ (b) $3.39 \leq \lambda\mu < 4.39$

(c) $4.39 \leq \lambda\mu < 6.38$ (d) $\lambda\mu \geq 6.38$

Figure 3: Diagram showing the nontrivial periodic solution in $\mathscr{R}_2 \cup \mathscr{R}_4$ along with the $(0,0)$ equilibrium.

Proposition 2. *Suppose the hypotheses of Lemma 2 hold. If* $0 < \lambda\mu < 2.4$*, then there always exists a unique periodic solution* $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ *of equation* (12) *in the bounded region* $\mathscr{R}_1 \cup \mathscr{R}_3$ *defined by* (28)*. This periodic solution is locally asymptotically stable and disappears for* $\lambda\mu \geq 2.4$*.*

Proof. We prove uniqueness of the locally asymptotically stable periodic solution $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ by contradiction. Suppose there exists a second periodic solution, say $\{\{v_1, w_1\}, \{v_2, w_2\}\}$, of equation (2) in $\mathscr{R}_1 \cup \mathscr{R}_3$. Then it must satisfy

$$\{v_1, w_1\} \preceq_{ne} \{s_1, t_1\} \preceq_{ne} (0,0) \preceq_{ne} \{s_2, t_2\} \preceq_{ne} \{v_2, w_2\} \tag{30}$$

Otherwise if $\{v_1, w_1\} \preceq_{se} \{s_1, t_1\}$, then the fact that $\{v_1, w_1\} \preceq_{ne} (0,0)$, an unstable fixed point of $T^2(\theta, u)$ by Lemma 1, along with Lemma 5 would imply that $\{v_1, w_1\}$ is locally asymptotically stable. This would make $\{v_1, w_1\}$ and $\{s_1, t_1\}$ locally asymptotically stable neighbors in the "$\preceq_{se}$" ordering, contradicting a theorem by Dancer and Hess in [5] which says that stable and unstable fixed points of an order-preserving map must alternate. Hence $\{\{v_1, w_1\}, \{v_2, w_2\}\}$ must be an unstable periodic solution of (28) satisfying the "$\preceq_{ne}$" ordering in (30). However in this case, the invariant region defined by $[-m_1, v_1] \times [-m_2, w_1]$ cannot contain a locally asymptotically stable fixed point of the cooperative map T^2, contradicting Lemma 5.

There also cannot exist an additional stable periodic solution to get around this problem because if it did, then we would have two unstable periodic solutions including the repelling equilibrium $(0,0)$ alternating with two stable periodic solutions including $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ in the $\preceq_{ne}$ ordering of periodic solutions. However, this would leave no room to fit the two nonzero unstable equilibria E_2 and E_3, which also exist in this case by Theorem 2 part 1, in this ordering without having two unstable equilibria as neighbors, thus violating Dancer and Hess' result in [5] once again.

The fact that the unique periodic solution $\{\{s_1, t_1\}, \{s_2, t_2\}\}$ must be locally asymptotically stable follows directly from the theory of cooperative maps which guarantees the existence of a stable fixed point of the cooperative map $T^2(\theta, u)$ in the invariant attracting region $\mathscr{R}_1 \cup \mathscr{R}_3$. To see that the periodic solution disappears for $\lambda\mu \geq 2.4$, one just needs to observe that in this case, the critical points and θ-intercepts of the equilibrium curve E_2 are smaller in magnitude than those of the equilibrium curve E_1. Since the critical points of both curves lie in $\mathscr{R}_1 \cup \mathscr{R}_3$, it follows that they cannot intersect here. □

The next corollary is a direct consequence of Theorem 3.

Corollary 1. *If the hypotheses of Lemma 2 are not satisfied, i.e., if the map $T^2(\theta, u)$ is not cooperative or is unbounded, then the solutions of equation* (12) *may exhibit unpredictable or even chaotic behavior.*

The next corollary addresses bifurcation values of the parameters λ and μ.

Corollary 2. *At $\lambda\mu = 2.4$ and 6.38, the point $(0,0)$ is a unique nonhyperbolic equilibrium and undergoes a Neimark-Sacker bifurcation. Moreover if $\lambda\mu > 6.38$, then every nonzero solution escapes to infinity.*

Proof. The proof of the first statement is a direct consequence of Table 1 and (21)-(22) in the proof of Lemma 1. In particular, note that at $\lambda\mu = 6.38$, the $(0,0)$ equilibrium changes from a repeller to a saddle point and a new pair of periodic solutions is created. Similarly at $\lambda\mu = 2.4$, $(0,0)$ changes from a saddle point to a repeller and a new pair of of periodic solutions is created. Thus the nonhyperbolic equilibrium $(0,0)$ undergoes a Neimark-Sacker bifurcation at both these $\lambda\mu$-values. It is easy to see that the map $T^2(\theta, u)$ in (15) is unbounded for $\lambda\mu > 6.38$. This and the fact that $T^2(\theta, u)$ is cooperative guarantee unbounded growth for the orbits of $T^2(\theta, u)$. As a result, every nonzero solution escapes to infinity. □

In the next section, we give a summary of our mathematical results in this paper and their physical interpretation for our discrete suspension bridge model (12).

7. Conclusion and Model Interpretation

In this paper, we mathematically analyzed a discrete difference equations version of McKenna and Moore's continuous nonlinear differential equations model for a suspension bridge in [13].

More specifically, we computed the exact number of real equilibria and real nontrivial periodic solutions for our discrete suspension bridge model under some very general hypotheses. We also gave precise parameter regions for our model to have (a) unique and multiple equilibria, and (b) unique and multiple real periodic solutions. In addition, we established the different types of local and global attractivity behaviors that these equilibria and real periodic solutions exhibit along with the precise parameter regions where they exhibit these behaviors. Finally, we used a linearization of our discrete model to gain valuable insights that helped us offer missing mathematical explanations for phenomena that McKenna and Moore observed for their linearized continuous model in [13] via numerical simulations. Our results are summarized in Table 3.

The physical interpretation of our mathematical results for the discrete suspension bridge model (12) is that the suspension bridge will eventually stop oscillating and come to rest at its original position for all initial conditions $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $2.4 < \lambda\mu \leq 6.38$. It will either come to a complete rest at one of two nonzero equilibrium positions or its oscillations will eventually settle down to one of two stable periodic motions for all initial conditions $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $0.41 < \lambda\mu \leq 2.4$. Its oscillations will eventually settle down to one of two stable periodic motions for all initial conditions $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $0 \leq \lambda\mu \leq 0.41$. The model breaks down for $\lambda\mu > 6.38$.

ACKNOWLEDGEMENTS The author is deeply indebted to Dr. Edward F. Aboufadel from the Math Department of Grand Valley State University in Allendale, MI, USA, for his constant encouragement, support and mentoring over the years. She would also like to thank Dr. Orlando Merino from the Math Department of the University of Rhode Island in Kingston, RI, USA, for making her the mathematician that she is today.

Table 3: Table summarizing our results in this paper.

$\lambda\mu$ - parameter region	No. of equilibria	Local attractivity	No. of periodic solns	Local attractivity	Global dynamics
$0 < \lambda\mu < 0.41$	3	3 repelling equilibria	2	At least one is locally asymptotically stable	Every Solution $\downarrow$
$\lambda\mu = 0.414$	3	$(0,0)$ a repeller, 2 nonhyperbolic equilibria	2	At least one is locally asymptotically stable	converges to $\downarrow$
$0.41 < \lambda\mu < 2.4$	3	$(0,0)$ a repeller, 2 saddle point equilibria	2	At least one is locally asymptotically stable	an equilibrium or a periodic soln. $\downarrow$
$\lambda\mu = 2.4$	1	$(0,0)$ nonhyperbolic, Neimark-Sacker bifurcation occurs	1	It is a repelling periodic solution	for θ-values not in $\downarrow$
$2.4 < \lambda\mu \leq 3.39$	1	The $(0,0)$ equil. is a saddle point	1	It is a repelling periodic solution	(θ_1, θ_2)
$3.39 < \lambda\mu < 4.39$	1	The $(0,0)$ equil. is a saddle point	1	It is a repelling periodic solution	Every soln. converges to $(0,0)$
$\lambda\mu = 4.39$	1	The $(0,0)$ equil. is a saddle point	1	It is a repelling periodic solution	Every soln. converges to $(0,0)$
$4.39 \leq \lambda\mu < 6.38$	1	The $(0,0)$ equil. is a saddle point	1	It is a repelling periodic solution	Every soln. converges to $(0,0)$
$\lambda\mu = 6.38$	1	$(0,0)$ nonhyperbolic, Neimark-Sacker bifurcation occurs	0	$---$	Every soln. converges to $(0,0)$
$\lambda\mu > 6.38$	1	$(0,0)$ is a repelling equilibrium	0	$---$	Every soln. escapes to ∞

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