Honorary Invited Paper

On the structure of regular $\tilde{H}$-cryptogroups

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Abstract. We introduce the concepts of Green $\sim$-relations on $\tilde{H}$-abundant semigroups. By using the generalized strong semilattice of semigroups, we show that an $\tilde{H}$-cryptogroup is a regular $\tilde{H}$-cryptogroup if and only if it is an $\tilde{H}G$-strong semilattice of completely $\tilde{J}$-simple semigroups. This result not only extends a known result of Petrich from the class of completely regular semigroups to the class of semiabundant semigroups but also generalizes a well known result of Fountain on superabundant semigroups from the class of abundant semigroups to the class of semiabundant semigroups.

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1. Introduction

It was proved by Clifford [1] that a regular semigroup is a union of groups if and only if it is a semilattice of completely simple semigroups. It is also known that if the set of all idempotents of a completely regular semigroup $S$ is the center of $S$, then $S$ can be expressed by a strong semilattice of groups (see [1]). Thus, we usually regard the completely regular semigroups as generalized groups. Moreover, by Petrich and Reilly, we call a completely regular semigroup $S$ a normal cryptogroup if the Green relation $H$ on $S$ is a normal band congruence on $S$. In particular, a completely regular semigroup $S$ is a normal cryptogroup if and only if $S$ can be expressed by a strong semilattice of completely simple semigroups (see [12] and [13]). This result was further generalized by Fountain by proving that an abundant semigroup $S$ is a superabundant semigroup if and only if $S$ is a semilattice of completely $J^*$-simple semigroups [4]. The structure of superabundant semigroups whose set of idempotents forms a subsemigroup have been recently extensively investigated by Ren and Shum in [15] and [16].

The Green $\ast$-relations on a semigroup $S$ were first defined by Pastijn [11] which can be regarded as the Green relations in some oversemigroups of $S$. These relations were formulated by

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Later on, El-Qallali further generalized the Green ∗-relations to Green ~-relations [3] as follows:

$$\mathcal{L}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)xa = ya \Leftrightarrow xb = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*, \mathcal{D}^* = \mathcal{L}^* \lor \mathcal{R}^*.$$

We can easily see that $\mathcal{\tilde{L}}$ and $\mathcal{\tilde{R}}$ are equivalent relations on $S$, however, the $\mathcal{\tilde{L}}$ relation is not necessary to be right compatible with the semigroup multiplication and the $\mathcal{\tilde{R}}$ relation is not necessary to be left compatible with the semigroup multiplication. We now denote the $\mathcal{\tilde{L}}$-class containing the element $a$ of the semigroup $S$ by $\mathcal{\tilde{L}}_a$ and we observe that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{\tilde{L}}$. Among the usual Green relations or the above relations, $\mathcal{L}$- or the generalized $\mathcal{L}$-relations are duals of the corresponding $\mathcal{R}$-relations or generalized $\mathcal{R}$-relations. In what follows, we only discuss the properties which are related to the $\mathcal{L}$-relation and the generalized $\mathcal{L}$-relation, respectively. One can easily see that there is at most one idempotent of the semigroup $S$ in each $\mathcal{H}$-class. If $e \in \mathcal{H}_a \cap E(S)$, for some $a \in S$, then we simply denote the idempotent $e$ by $x^0$, for any $x \in \mathcal{H}_a$. Clearly, for any $x \in \mathcal{H}_a$ with $a \in S$, we have $x = xx^0 = x^0x$.

If a semigroup $S$ is regular, then every $\mathcal{L}$-class of $S$ contains at least one idempotent, and so does every $\mathcal{R}$-class of $S$. Thus, regular semigroups are obviously special abundant semigroups. Thus, Fountain called such semigroup superabundant [4] if its every $\mathcal{H}$-classes contains an idempotent. Obviously, completely regular semigroups are special superabundant semigroups. Following El-Qallali [3], we call a semigroup $S$ a semiabundant semigroup if every $\mathcal{L}$-class and every $\mathcal{R}$-class of $S$ contain at least one idempotent. A semigroup $S$ is called $\mathcal{H}$-abundant if every $\mathcal{H}$-class contains an idempotent of $S$. Clearly, the $\mathcal{H}$-abundant semigroups are generalizations of superabundant semigroups in the class of semiabundant semigroups. One can easily see that $\mathcal{\tilde{L}} = \mathcal{L}$ on the set of regular elements in any $\mathcal{H}$-abundant semigroup.

Throughout this paper, we call a band $B$ a regular band (right quasi normal band) if $B$ satisfies the identity $axya = axaya(xy = xay)$. According to Petrich and Reilly [12], a completely regular semigroup $S$ was called a regular cryptogroup if the Green relation $\mathcal{H}$ on $S$ is a regular band congruence on $S$. The structure of regular cryptogroup was investigated by Kong-Shum in [8] and [9]. In the class of abundant semigroups, Guo and Shum [5] called an abundant semigroup whose set of idempotents forms a regular band a cyber group. The semilattice structure of regular cyber groups have been recently investigated in [9].

Naturally, one would ask : can we establish an analogous result of superabundant semigroups [4] in the class of semiabundant semigroups or an analogous result of cryptogroups [12] in the
class of \( \tilde{\mathcal{H}} \)-abundant semigroups? In this paper, we will establish a theorem for \( \tilde{\mathcal{H}} \)-cryptogroups by using the Green \( \sim \) relations and the \( KG \)-strong semilattice of semigroups, as described in [10]. We will show that an \( \tilde{\mathcal{H}} \)-cryptogroup is a regular \( \tilde{\mathcal{H}} \)-cryptogroup if and only if it is an \( \tilde{\mathcal{H}}G \)-strong semilattice of completely \( \tilde{J} \)-simple semigroups. Our results in this paper also generalize and enrich the corresponding results given in [1], [4], [7], [8] and [13].

2. \( KG \)-strong semilattices

We now restate the concept of \( G \)-strong semilattice decomposition of semigroup \( S \) given by Kong and Shum in [8] and [9].

Let \( S = (Y; S_\alpha) \) be a semilattice of the semigroups \( S_\alpha \), where each \( S_\alpha \) is a subsemigroup of the semigroup \( S \) and \( Y \) is a semilattice. We define the \( G \)-strong semilattice of semigroups by generalizing the well known strong semilattice of semigroups (see [9]).

**Definition 2.1** Let \( S = (Y; S_\alpha) \) be a semigroup. Suppose that the following conditions \( S \) are satisfied:

(i) \( (\forall \alpha, \beta \in Y, \alpha \geq \beta) \), there exists a family of homomorphisms \( \varphi_{d(\alpha, \beta)} : S_\alpha \rightarrow S_\beta \), where \( d(\alpha, \beta) \in D(\alpha, \beta) \) and \( D(\alpha, \beta) \) is a non-empty index set.

(ii) \( (\forall \alpha \in Y) \), \( D(\alpha, \alpha) \) is a singleton. Denote the element in \( D(\alpha, \alpha) \) by \( d(\alpha, \alpha) \). In this case, the homomorphism \( \varphi_{d(\alpha, \alpha)} : S_\alpha \rightarrow S_\alpha \) is the identity automorphism of the semigroup \( S_\alpha \).

(iii) \( (\forall \alpha, \beta, \gamma \in Y, \alpha \geq \beta \geq \gamma) \), if we write \( \varphi_{\alpha, \beta} = \{ \varphi_{d(\alpha, \beta)} : d(\alpha, \beta) \in D(\alpha, \beta) \} \) then \( \varphi_{\alpha, \beta} \varphi_{\beta, \gamma} = \varphi_{\alpha, \gamma} \), where

\[
\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} = \{ \varphi_{d(\alpha, \beta)} \varphi_{d(\beta, \gamma)} : \forall d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma) \}.
\]

(iv) for each \( \alpha, \beta \in Y \), there is a mapping from \( S_\alpha \) into the set \( \varphi_{\beta, \alpha} \) whose value at any given element \( a \in S_\alpha \) is denoted by \( \varphi_{d(\beta, \alpha)}^a \) such that for all \( b \in S_\beta \),

\[
ab = (a \varphi_{d(\alpha, \beta)}^b)(b \varphi_{d(\beta, \alpha)}^a).
\]

Then the above semilattice of semigroups is called the generalized strong semilattice of semigroups \( S_\alpha \) and in brevity, the “\( G \)-strong semilattice” of semigroups \( S_\alpha \) and denoted it by \( S = G[Y; S_\alpha, \varphi_{\alpha, \beta}] \).

The following definition is a more general version of \( G \)-strong semilattices.

**Definition 2.2** Let \( K \) be any equivalent relation on a \( G \)-strong semilattice of semigroups \( S = G[Y; S_\alpha, \varphi_{\alpha, \beta}] \). Then, we call \( S \) a “\( KG \)-strong semilattice of semigroups \( S_\alpha \)” if for every \( \alpha, \beta \in Y \), the mapping \( a \mapsto \varphi_{\alpha, \beta}^{a,b} \) has the property that \( \varphi_{d(\beta, \alpha)}^a = \varphi_{d(\beta, \alpha)}^b \) whenever the elements \( a, b \in S_\alpha \) are in the same \( K \)-class of \( S \).

Thus, it is clear that the \( G \)-strong semilattice of semigroups \( S \) can be determined by an equivalent
of semigroups since $\rho G$ is weaker than the usual strong semilattice. In fact, if

it is clear that the Green $L$-relation $\sim$ is both a left (right) ideal.

Remark 2.3 It is clear that the $KG$-strong semilattice is stronger than the $G$-strong semilattice but it is weaker than the usual strong semilattice. In fact, if $\rho$ and $\delta$ are equivalent relations on the semigroup $S = (Y; S_\alpha)$ with $\rho \subseteq \delta$, then one can observe that $\delta G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is “stronger” than $\rho G[Y; S_\alpha, \varphi_{\alpha,\beta}]$. As special cases, $1_S G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is the “weakest” $KG$-strong semilattice of semigroups since $1_S$ is the “smallest” equivalent relation on $S$ and also $\eta G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is the strongest $KG$-strong semilattice of semigroups since $\eta$ is the “greatest” equivalent relation on $S$, where $1_S$ is the identity relation on $S$ and $\eta$ is the semilattice congruence on $S$ which partitions the semigroup $S$ into disjoint subsemigroups $S_\alpha(\alpha \in Y)$ of $S$. Hence, we can easily see that $\eta G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is the usual strong semilattice of semigroups since in this case, every index set $D(\alpha, \beta)$ is a singleton for $\alpha \geq \beta$ on $Y$ and hence there exists one and only one structure homomorphism in the set of structure homomorphisms $\varphi_{\alpha,\beta}$.

We have already defined the Green $\sim$-relations $\tilde{L}$, $\tilde{R}$, $\tilde{H}$ and $\tilde{D}$ on a semigroup $S$. In order to define the Green $\sim$-relation $\tilde{J}$ on $S$, we consider the left $\sim$-ideal $L$ of a semigroup $S$.

Definition 2.4 A left (right) ideal $L$ $(R)$ of a semigroup $S$ is called a left $\sim$-ideal of $S$ if $\tilde{L}_a \subseteq L(a \subseteq R)$ holds, for all $a \in L(a \in R)$. We call a subset $I$ of a semigroup $S$ a $\sim$-ideal of $S$ if it is both a left $\sim$-ideal and a right $\sim$-ideal.

It is noteworthy that if $S$ is a regular semigroup, then every left (right, two-sided) ideal of $S$ is a left (right, two-sided) $\sim$-ideal. We also observe that for any idempotent $e$ in a semigroup $S$, the left (right) ideal $S(eS)$ is a left(right) $\sim$-ideal. For if $a \in Se$, then $a = ae$, and hence for any element $b$ in $\tilde{L}_a$, we have $b = be \in Se$.

By Definition 2.4, we see that the semigroup $S$ is always a $\sim$-ideal of itself, and we denote the smallest $\sim$-ideal containing the element $a$ of $S$ by $\tilde{J}(a)$. Now, we define $\tilde{J} = \{(a, b) \in S \times S : \tilde{J}(a) = \tilde{J}(b)\}$.

Definition 2.5 An $\tilde{H}$-abundant semigroup $S$ is called completely $\tilde{J}$-simple if $S$ does not contain any non-trivial proper $\sim$-ideal of $S$.

We now give some properties of the $\tilde{H}$-abundant semigroups. Some of the properties may have already been known or can be easily derived, however, for the sake of completeness, we provide here the proofs.

Lemma 2.6 Let $S$ be an $\tilde{H}$-abundant semigroup. Then the following properties hold:

(i) The Green $\sim$-relation $\tilde{H}$ is a congruence on $S$ if and only if for any $a, b \in S$, $(ab)^0 = (a^0b^0)^0$.

(ii) If $e, f$ are $\tilde{D}$-related idempotents of $S$, then $e \tilde{D} f$.

(iii) $\tilde{D} = \tilde{L} \circ \tilde{R} = \tilde{R} \circ \tilde{L}$. 

(iv) If \(e, f\) are idempotents in \(S\) such that \(eJf\), then \(eDf\).

**Proof.**

(i) *(Necessity).* For any \(a, b \in S\), we have \(a\tilde{H}a^0\) and \(b\tilde{H}b^0\). Since \(\tilde{H}\) is a congruence on \(S\), \(ab\tilde{H}(ab)^0\). But \(ab\tilde{H}(ab)^0\), and so \((ab)^0 = (a^0b^0)^0\) since every \(\tilde{H}\)-class contains a unique idempotent.

*(Sufficiency)*. We only need to show that \(\tilde{H}\) is compatible with the semigroup multiplication of \(S\) since \(\tilde{H}\) is an equivalent relation on \(S\). Let \((a, b) \in \tilde{H}\) and \(c \in S\). Then \((ca)^0 = (c^0a^0)^0 = (c^0b^0)^0 = (cb)^0\) and hence, \(\tilde{H}\) is left compatible to the semigroup multiplication. Dually, \(\tilde{H}\) is right compatible with the semigroup multiplication and thus \(\tilde{H}\) is a congruence on \(S\).

(ii) Since \(e\tilde{D}f\), there exist elements \(a_1, \ldots, a_k\) of \(S\) such that \(eL\alpha_1R\alpha_2 \cdots a_k L f\). Since \(S\) is an \(\tilde{H}\)-abundant semigroup, \(eL\alpha_1R\alpha_2 \cdots a_k L f\). Thus \(e\tilde{D}f\).

(iii) If \(a, b \in S\) and \(a\tilde{D}b\), then by (ii), \(a^0\tilde{D}b^0\). Hence there exist elements \(c, d\) in \(S\) with \(a^0LcRh^0\) and \(a^0RdLb^0\), and consequently, \(aLc\tilde{R}b\) and \(a\tilde{R}db\). Thus the result is proved.

(iv) Since \(SeS = SfS\), there exist elements \(x, y, s, t\) in \(S\) such that \(f = set\) and \(e = xy\). Let \(h = (fy)^0\) and \(k = (se)^0\). Then \(hfy = fyy = ffy\) and so \(h = h^2 = fh\) and \(sek = se = see\), and thereby, \(k = k^2 = ke\). Hence, \(h, f, e\) are the idempotents satisfying the relations \(hL Rh\) and \(ekLk\). These imply that \(ehf\tilde{R}eh\) and \(ekf\tilde{L}kf\). Now by \(eh = xfyh = xfy = e\) and \(kf = kset = set = f\), we have \(eh\tilde{R}ef\tilde{L}f\). This shows that \(e\tilde{D}f\).

Similar to the definition of cyber group given by Guo and Shum [5], we formulate the following definition.

**Definition 2.7** An \(\tilde{H}\)-abundant semigroup \(S\) is called an \(\tilde{H}\)-cryptogroup if the Green \(\sim\)-relation \(\tilde{H}\) is a congruence on \(S\). Also, we call an \(\tilde{H}\)-abundant semigroup \(S\) a regular \(\tilde{H}\)-cryptogroup if \(\tilde{H}\) is a congruence on \(S\) such that \(S/\tilde{H}\) is a regular band. Thus, \(\tilde{H}\)-cryptogroups are analogy of cryptogroups in the class of \(\tilde{H}\)-abundant semigroups. Also, we see in [5] that an \(\tilde{H}\)-cryptogroup is a generalized cyber groups.

The \(\tilde{H}\)-cryptogroup \(S\) has the following properties:

**Lemma 2.8**

(i) For any element \(a\) of the \(\tilde{H}\)-cryptogroup \(S\), \(\tilde{J}(a) = Sa^0S\).

(ii) For the \(\tilde{H}\)-cryptogroup \(S\), \(\tilde{J} = \tilde{D}\).

(iii) If the \(\tilde{H}\)-cryptogroup \(S\) is completely \(\tilde{J}\)-simple, then the idempotents of \(S\) are primitive.
(iv) If the $\tilde{\mathcal{H}}$-cryptogroup $S$ is completely $\tilde{\mathcal{J}}$-simple, then the regular elements of $S$ generate a regular subsemigroup of $S$.

Proof.

(i) Obviously, we have $a^0 \in \tilde{\mathcal{J}}(a)$ and so $Sa^0S \subseteq \tilde{\mathcal{J}}(a)$. We need to show that the ideal $Sa^0S$ is in fact a $\sim$-ideal and since $a = a \alpha a^0 \in Sa^0S$, $\tilde{\mathcal{J}}(a) \subseteq Sa^0S$. Let $b = xa y \in Sa^0S(x, y \in S)$ and $k = (a^0y)^0$. Then $a^0a^0y = a^0y = ka^0y$ so that $a^0(a^0y)^0 = k^2 = k$. Also since $\mathcal{H}$ is a congruence, $xa^0y \mathcal{H} xk$. Now let $h = (xk)^0 = (xa^0y)^0$. Then $xkh = xk = xkk$ so that $h = h^2 = hk = ha^0k \in Sa^0S$. Hence if $c \in Lb, d \in \bar{R}a$, then $c = ch, d = hd \in Sa^0S$ and hence, $Sa^0S$ is a $\sim$-ideal, as required.

(ii) Let $(a, b) \in S$ with $a \mathcal{J} b$. Then by (i), we have $Sa^0S = Sb^0S$. Now, by Lemma 2.6 (iv), $a^0Db^0$ and so $a \mathcal{H}a^0Db^0 \mathcal{H} b$. This implies that $a \mathcal{D} b$ and hence $c \mathcal{J} \mathcal{H} b$. Conversely, let $a, b \in S$ with $a \mathcal{D} b$. Then by Lemma 2.6 (iii), there exists an element $c \in S$ such that $a \mathcal{L} \mathcal{R} b$. This leads to $a^0\mathcal{L}b^0\mathcal{R}0$ and so $Sa^0S = Sc^0S = Sb^0S$. Now, by (i), $(a, b) \in \tilde{\mathcal{J}}$ and hence $D \subseteq \tilde{\mathcal{J}}$. Therefore, $\tilde{\mathcal{J}} = D$.

(iii) Let $e, f$ be idempotents in $S$ with $e \leq f$. Since $S$ is completely $\tilde{\mathcal{J}}$-simple, $f \in Se_S$. Now by the first part of Exercise 3 in [14][§8.4], there exists an idempotent $g$ of $S$ such that $f \mathcal{D} g$ and $g \leq e$. Let $a \in S$ be such that $f \mathcal{L} a \mathcal{R} g$. Then $f \mathcal{L} a^0 \mathcal{R} g$ and since $g \leq f$, we have

$$a^0 = ga^0(gf)a^0 = gf = g.$$

Now by noting that $g \leq f$ and $g \mathcal{L} f$, we have $f = fg = g$. However, since $g \leq e$, we obtain $e = f$ and hence all idempotents of $S$ are primitive.

(iv) Let $a, b$ be regular elements of $S$. Since $S$ consists of a single $D$-class, by (ii) and by Lemma 2.6 (iii), there exists an element $c \in S$ such that $a \mathcal{L} \mathcal{R} c \mathcal{R} b$. Hence $a \mathcal{L} \mathcal{R} c \mathcal{R} b$. This leads to $c^0b = b$ and $a \mathcal{L} \mathcal{R} b$ since $a$ is regular. Now we have $ab \mathcal{L} b$ and so the regularity of $ab$ follows from the regularity of $b$.

We now establish the following theorem for $\tilde{\mathcal{H}}$-cryptogroups.

**Theorem 2.9** Let $S$ be an $\tilde{\mathcal{H}}$-cryptogroup. Then $S$ is a semilattice $Y$ of completely $\tilde{\mathcal{J}}$-simple semigroups $S_\alpha (\alpha \in Y)$ such that for every $\alpha \in Y$ and $a \in S_\alpha$, we have $L_a(S) = L_a(S_\alpha)$ and $R_a(S) = R_a(S_\alpha)$.

**Proof.** If $a \in S$, then $a \mathcal{H}a^2$ and so, $\tilde{\mathcal{J}}(a) = \tilde{\mathcal{J}}(a^2)$. Now for $a, b \in S$, we have $(ab)^2 \in Sba_S$, and hence, it follows that

$$\tilde{\mathcal{J}}(ab) = \tilde{\mathcal{J}}((ab)^2) \subseteq \tilde{\mathcal{J}}(ba).$$

Now, by symmetry, we obtain $\tilde{\mathcal{J}}(ab) = \tilde{\mathcal{J}}(ba)$. Since, by Lemma 2.8 (i), we have $\tilde{\mathcal{J}}(a) = Sa^0S$ and $\tilde{\mathcal{J}}(b) = Sb^0S$ so that if $c \in \tilde{\mathcal{J}}(a) \cap \tilde{\mathcal{J}}(b)$, then $c = xa^0y = zb^0t$ for some $x, y, z, t \in S$. Now $c^2 = zb^0txa^0y \in Sb^0txa^0S \subseteq \tilde{\mathcal{J}}(b^0txa^0)$ and hence, $\tilde{\mathcal{J}}(b^0txa^0) = \tilde{\mathcal{J}}(a^0b^0tx)$ by using previous arguments. Thus, $c^2 \in \tilde{\mathcal{J}}(a^0b^0)$ and since $\mathcal{H}c^2$, we have $c \in \tilde{\mathcal{J}}(a^0b^0)$. Since $a \mathcal{H}a^0$, $\tilde{\mathcal{J}}(ab) = \tilde{\mathcal{J}}(ba)$, and so $\tilde{\mathcal{J}}(ab) = \tilde{\mathcal{J}}(ba)$.
\( b\tilde{a}b^0 \) and \( \tilde{H} \) is a congruence on \( S \), we have \( ab\tilde{a}b^0 \). Consequently, \( c \in \tilde{J}(ab) \), and thereby \( \tilde{J}(a) \cap \tilde{J}(b) \subseteq \tilde{J}(ab) \). The converse containment is clear so that \( \tilde{J}(a) \cap \tilde{J}(b) = \tilde{J}(ab) \). We can easily see that the set \( Y \) of all \( \sim \)-ideals \( \tilde{J}(a)(a \in S) \) forms a semilattice under set intersection and that the mapping \( a \mapsto \tilde{J}(a) \) is a homomorphism from \( S \) onto \( Y \). The inverse image of \( \tilde{J}(a) \) is just the \( \tilde{J} \)-class \( \tilde{J}_a \) which is a subsemigroup of \( S \). Hence \( S \) is a semilattice \( Y \) of the semigroups \( \tilde{J}_a \). Now let \( a, b \) be elements of \( \tilde{J} \)-class \( \tilde{J} \) and suppose that \( (a, b) \in \tilde{E}(\tilde{J}) \). Then, \( a^0, b^0 \in \tilde{J} \) so that \( (a^0, b^0) \in \tilde{E}(\tilde{J}) \), that is, \( a^0b^0 = a^0, b^0a^0 = b^0 \) and \( (a^0, b^0) \in \tilde{L}(S) \). It follows that \( (a, b) \in \tilde{L}(S) \) and consequently, by \( \tilde{L}_a(S) \subseteq \tilde{J} \), we have \( \tilde{L}_a(S) = \tilde{L}_a(J) \). By using a similar argument, we can show that \( \tilde{R}_a(S) = \tilde{R}_a(J) \). From the above discussion, we can deduce that \( \tilde{H}_a(J) = \tilde{H}_a(S) \) and so \( J \) is indeed an \( \tilde{H} \)-abundant semigroup. Furthermore, if \( a, b \in \tilde{J} \), then by Lemma 2.8 (i), \( (a, b) \in \tilde{D}(S) \) and hence, by Lemma 2.6 (iii), there exists an element \( c \) in \( \tilde{L}_a(S) \cap \tilde{R}_b(S) = \tilde{L}_a(J) \cap \tilde{R}_b(J) \). Thus \( a, b \) are \( \tilde{D} \)-related in \( \tilde{J} \) and so \( J \) is \( \tilde{J} \)-simple.

For the \( \tilde{H} \)-cryptogroups, we have the following theorem.

**Theorem 2.10** Let \( S \) be an \( \tilde{H} \)-cryptogroup which is expressed by the semilattice of semigroups \( S = (Y; S_{\alpha}) \). Then the following statements hold:

1. For \( \alpha, \beta \) in the semilattice \( Y \) with \( \alpha \geq \beta \), if \( a \in S_{\alpha} \), then there exists \( b \in S_{\beta} \) with \( a \geq b \);
2. For \( a, b, c \in S \) with \( b\tilde{H}c \), if \( a \geq b \), \( a \geq c \) then \( b = c \);
3. For \( a \in E(S) \) and \( b \in S \), if \( a \geq b \) then \( b \in E(S) \).

**Proof.** (i) Let \( c \in S_{\beta} \). Then, by Lemma 2.6 (i), we see that \( a(acac)^0, (acac)^0a \) and \( (acac)^0 \) are all in the same \( \tilde{H} \)-class of the semigroup \( S \) and hence, \( a(acac)^0 = (acac)^0a(acac)^0 = (acac)^0a \). Write \( b = a(acac)^0 \). Then \( b \in S_{\beta} \) and \( a \geq b \). (ii) By the definition of \( \geq \), there exist \( e, f, g, h \in E(S) \) such that \( b = ea = af, c = ga = ah \). From \( eb = b \) and \( b\tilde{H}b^0 \), we have \( eb^0 = b^0 \). Similarly, \( c^0h = c^0 \). Thus \( ec = ec^0c = ebh \). Then \( bh = eah = ec = c \). By using similar arguments, we have \( bh = b \) and so, \( b = bh = eah = ec = c \), as required. (iii) We have \( b = ea = af \) for some \( e, f \in E(S) \), and whence \( b^2 = (ea)(af) = ea^2f = b \).

The following fact can be easily observed:

**Fact 2.11** Let \( \varphi \) be a homomorphism which maps an \( \tilde{H} \)-cryptogroup \( S \) into another \( \tilde{H} \)-cryptogroup \( T \). Then \( (a\varphi)^0 = a^0\varphi \).

### 3. Properties of regular \( \tilde{H} \)-cryptogroups

**Lemma 3.1** Let \( S \) be a regular \( \tilde{H} \)-cryptogroup (that is, \( \tilde{H} \) is a congruence on the \( \tilde{H} \)-abundant semigroup \( S \) such that \( S/\tilde{H} \) is a regular band). For every \( a \in S \), we define a relation \( \rho_a \) on \( S \) by \( (b_1, b_2) \in \rho_a \) if and only if \( (ab_1a)^0 = (ab_2a)^0, (b_1, b_2 \in S) \). Then the following properties hold on \( S \):
Lemma 3.2

(i) \( \rho_a \) is a band congruence on \( S \);

(ii) \( \forall a, a_1 \in S_a, \rho_a = \rho_{a_1} \), that is, \( \rho_a \) depends only on the component \( S_a \) containing the element \( a \) rather than on the element itself, hence we can write \( \rho_a = \rho_a \), for all \( a \in S_a \).

(iii) \( \forall \alpha, \beta \in Y \) with \( \alpha \geq \beta \), \( \rho_\alpha \subseteq \rho_\beta \) and \( \rho_\beta |_{S_\alpha} = \omega_{S_\alpha} \), where \( \omega_{S_\alpha} \) is the universal relation on \( S_\alpha \).

Proof. (i) It is easy to see that \( \rho_a \) is an equivalent relation on \( S \), for all \( a \in S \). We now prove that \( \rho_a \) is left compatible with the semigroup multiplication. For this purpose, we let \( (x, y) \in \rho_a \) and \( c \in S \). Then, by the definition of \( \rho_a \), we have \( (axa) = (aya) \). Since \( S \) is a regular \( \tilde{\mathcal{H}} \)-cryptogroup, by Lemma 2.6 (i) and the regularity of the band \( S/\tilde{\mathcal{H}} \), we obtain

\[
(axa)^0 = (ac(axa))^0 = ((ac)^0(axa))^0 = ((ac)^0(aya))^0 = (acya)^0.
\]

Hence, \( (cx, cy) \in \rho_a \). Dually, we can prove that \( \rho_a \) is right compatible with the semigroup multiplication. Thus \( \rho_a \) is a congruence on \( S \). Obviously, \( \tilde{\mathcal{H}} \subseteq \rho_a \) and so \( \rho_a \) is a band congruence on \( S \).

(ii) Let \( (x, y) \in \rho_a \). Then, by the definition of \( \rho_a \), we have \( (axa) = (aya) \), and so \( (axa)^0 = (aya)^0 = (a^0(aya))^0 = (a^0)^0 = (a_y)^0 \). This leads to \( (a_y^0(axa)^0)^0 = (a_y^0(aya)^0)^0 \). Since \( S/\tilde{\mathcal{H}} = (Y; S_\alpha/\tilde{\mathcal{H}}) \) is a regular band and by Lemma 2.6 (ii), we obtain \( (a_y^0(axa_y^0a))^0 = (a_y^0a^0ya^0a_y^0)^0 \). However, since \( a, a_1 \) are elements of the completely \( \tilde{\mathcal{J}} \)-simple semigroup \( S_a \), \( (a_y^0axa_y^0)^0 = a_y^0 \). Thereby, by Lemma 2.6 (i) again, we have \( (a_y^0axa_y^0)^0 = (a_y^0a_y^0)^0 \), that is, \( (x, y) \in \rho_a \). This shows that \( \rho_a \subseteq \rho_{a_1} \). Similarly, we also have \( \rho_{a_1} \subseteq \rho_a \). Thus, \( \rho_a = \rho_{a_1} \). Since this relation holds for all \( a \in S_a \), we usually write \( \rho_a = \rho_a \). (iii) Let \( a \in S_a, b \in S_b \) and \( \alpha \geq \beta \). We need to prove that \( \rho_a \subseteq \rho_b \). For this purpose, we let \( (x, y) \in \rho_a \), by (ii). Then, by the definition of \( \rho_a \), we have \( (axa)^0 = (aya)^0 \) and hence \( b(axa)^0b = (baya)^0b \). By Lemma 2.6 (i) and the regularity of the band, we have \( (babxbab)^0 = (babxbab)^0 \). Since \( \alpha \geq \beta \) in \( Y \) and \( a \in S_a, b \in S_b \), we have \( (bab)^0 = b^0 \). By using Lemma 2.6 (ii) again, we can show that \( (bab)^0 = (bab)^0 \), that is, \( (x, y) \in \rho_{a_1} = \rho_{a_2} \). Thus, \( \rho_a \subseteq \rho_b \) as required. Furthermore, it is trivial that \( \rho_b |_{S_a} = \omega_{S_a} \), which is the universal relation on the semigroup \( S_a \).

We now use the band congruence \( \rho_a \) defined in Lemma 3.1 to describe the structural homomorphisms for the \( \tilde{\mathcal{H}} \)-cryptogroup \( S = (Y; S_a) \), where each \( S_a \) is a completely \( \tilde{\mathcal{J}} \)-simple semigroup.

We first consider the congruence \( \rho_{a, \beta} = \rho_a |_{S_\beta} \) for \( \alpha, \beta \in Y \), which is a band congruence on the semigroup \( S_\beta \). Now, we denote all the \( \rho_{a, \beta} \)-classes of \( S_\beta \) by \( \{ S_{d(a, \beta)} : d(a, \beta) \in D(\alpha, \beta) \} \), where \( D(\alpha, \beta) \) is a non-empty index set. In particular, the set \( D(\alpha, \alpha) \) is a singleton and we can therefore write \( d(\alpha, \alpha) = D(\alpha, \alpha) \). We have the following lemma.

Lemma 3.2 Let \( S = (Y; S_a) \) be a regular \( \tilde{\mathcal{H}} \)-cryptogroup. Then, for all \( \alpha, \beta \in Y \) with \( \alpha \geq \beta \), the following statements hold for all \( d(\alpha, \beta) \in D(\alpha, \beta) \).

(i) For all \( a \in S_a \), there exists a unique \( a_{d(a, \beta)} \in S_{d(a, \beta)} \) satisfying \( a \geq a_{d(a, \beta)} \);

(ii) For all \( a \in S_a \) and \( x \in S_{d(a, \beta)} \), if \( a^0 \geq e \) for some idempotent \( e \in S_{d(a, \beta)} \) then \( eax = ax, xae = xa, ea = ae \) and \( (ea)^0 = e \);
(iii) Let \( a \in S_\alpha \). Define \( \varphi_{d(\alpha,\beta)} : S_\alpha \rightarrow S_{d(\alpha,\beta)} \) by \( a\varphi_{d(\alpha,\beta)} = a_{d(\alpha,\beta)}, \) where \( a_{d(\alpha,\beta)} \in S_{d(\alpha,\beta)} \) and \( a \geq a_{d(\alpha,\beta)} \). Then \( \varphi_{d(\alpha,\beta)} \) is a homomorphism and \( a_{d(\alpha,\beta)} = (aba)^0a \) for any \( b \in S_{d(\alpha,\beta)} \).

**Proof.** (i) We first show that for any \( a \in S_\alpha \) and \( b \in S_{d(\alpha,\beta)} \), we have \( ab \in S_{d(\alpha,\beta)} \), that is, \( (ab, b) \in \rho_{a,\beta} \). In fact, since \( S = (Y, S_\alpha) \) is a \( \mathcal{H} \)-cryptogroup, each \( S_\alpha \) is a completely \( \mathcal{J} \)-simple semigroup. Hence, we have \( (xax)^0 = x^0 \), for all \( x \in S_\alpha \). This leads to \( (xabx)^0 = (xbx)^0 \) by the regularity of the band \( S/\mathcal{H} \) and Lemma 2.6 (i). Thereby, \( (ab, b) \in \rho_{a,\beta} \). Similarly, we also have \( ba \in S_{d(\alpha,\beta)} \). Invoking the above results, we have \( ab \in S_{d(\alpha,\beta)} \) for any \( b \in S_{d(\alpha,\beta)} \). Since \( \mathcal{H} \) is a band congruence on \( S \), by Lemma 2.6 (i) again, we see that \( a_{d(\alpha,\beta)}^0 , (aba)^0 \) and \( (aba)^0 a \) are in the same \( \mathcal{H} \)-class of \( S \) so that \( a_{d(\alpha,\beta)}^0 = (aba)^0 a(aba)^0 = (aba)^0 a \). Let \( a_{d(\alpha,\beta)}^0 = a_{d(\alpha,\beta)} \). Then by the natural partial order imposed on \( S \), we have \( a \geq a_{d(\alpha,\beta)} \). In order to show the uniqueness of \( a_{d(\alpha,\beta)} \), we assume that there is another \( a_{d(\alpha,\beta)}^* \in S_{d(\alpha,\beta)} \) satisfying \( a \geq a_{d(\alpha,\beta)}^* \). Then, by the definition of \( "\leq\" \), we can write \( a_{d(\alpha,\beta)}^* = e\alpha = af \) for some \( e, f \in E(S) \) and so \( a_{d(\alpha,\beta)}^* a^0 = a_{d(\alpha,\beta)}^* = a^0 a_{d(\alpha,\beta)}^* \). By the fact \( a_{d(\alpha,\beta)}^* \mathcal{H} a^0 \), we have \( (a_{d(\alpha,\beta)}^*)^0 a^0 = (a_{d(\alpha,\beta)}^*)^0 a^0 \) and \( a^0 (a_{d(\alpha,\beta)}^*)^0 = (a_{d(\alpha,\beta)}^*)^0 a^0 \). Consequently, by the definition of \( "\leq\" \), we have \( a^0 \geq (a_{d(\alpha,\beta)}^*)^0 a^0 \). By Lemma 2.6 (i) again, we deduce that

\[
(a_{d(\alpha,\beta)}^*)^0 = (a^0 (a_{d(\alpha,\beta)}^*)^0 a^0)^0 = (a^0 a_{d(\alpha,\beta)}^*)^0 = (aba)^0.
\]

Hence, \( (a_{d(\alpha,\beta)}^*, a_{d(\alpha,\beta)}) \in \mathcal{H} \), and consequently, by Theorem 2.10 (ii), \( a_{d(\alpha,\beta)}^* = a_{d(\alpha,\beta)} \). This shows the uniqueness of \( a_{d(\alpha,\beta)} \). (ii) It is easy to see that, by the definition of \( "\leq\" \), \( a^0 \geq (a^0 (ax)^0 a^0)^0 \). Also, since \( a \in S_\alpha \) and \( x \in S_{d(\alpha,\beta)} \), we have \( ax \in S_{d(\alpha,\beta)} \) by (i). Moreover, since \( S_{d(\alpha,\beta)} \) is a \( \rho_{a,\beta} \)-congruence class, \( (ax)^0 \in S_{d(\alpha,\beta)} \). Thus, by (i) again, we have \( (a^0 (ax)^0 a^0)^0 \in S_{d(\alpha,\beta)} \) and \( e = (a^0 (ax)^0 a^0)^0 \). Thereby, we have \( eax = (a^0 (ax)^0 a^0)^0 a^0 (ax)^0 a^0 ax = ax \). Similarly, we have \( xae = xa \). Since \( x \) is arbitrarily chosen element in \( S_{d(\alpha,\beta)} \), we can particularly choose \( x = e \). In this way, we obtain \( ea = ae \) and consequently, by Lemma 2.6 (i), we have \( (ea)^0 = (ea)^0 = e \). (iii) By using the result in (i), we can define \( \varphi_{d(\alpha,\beta)} : S_\alpha \rightarrow S_{d(\alpha,\beta)} \) by \( a\varphi_{d(\alpha,\beta)} = a_{d(\alpha,\beta)} = a(aca)^0 = (aca)^0 a \), for any \( a \in S_\alpha \) and \( c \in S_{d(\alpha,\beta)} \). Then, for any \( a, b \in S_\alpha \), we have, by (ii),

\[
(a\varphi_{d(\alpha,\beta)}) (b\varphi_{d(\alpha,\beta)}) = a_{d(\alpha,\beta)} b_{d(\alpha,\beta)} = (aca)^0 ab(bcb)^0 = (aca)^0 (ab(bcb)^0) = ab(bcb)^0.
\]

Similarly, we can show that \( (a\varphi_{d(\alpha,\beta)}) ((b\varphi_{d(\alpha,\beta)}) = (aca)^0 ab \). Hence, \( ab \geq (a\varphi_{d(\alpha,\beta)}) (b\varphi_{d(\alpha,\beta)}) \).

Thus \( (ab)\varphi_{d(\alpha,\beta)} = (a\varphi_{d(\alpha,\beta)}) (b\varphi_{d(\alpha,\beta)}) \), by the definition of \( \varphi_{d(\alpha,\beta)} \). This shows that \( \varphi_{d(\alpha,\beta)} \) is indeed a homomorphism.

We now proceed to show that the homomorphisms given in Lemma 3.2 (iii) are the structural homomorphisms for the \( G \)-strong semilattice \( G[Y; S_\alpha, \varphi_{\alpha,\beta}] \) induced by the semigroup \( S = (Y; S_\alpha) \) under the band congruence \( \rho_\alpha \) on the semigroup \( S_\alpha \).
Lemma 3.3 Let $S = (Y; S_a)$ be an $\tilde{H}$-cryptogroup and $\varphi_{\alpha, \beta} = \{ \varphi_{d(\alpha, \beta)} | d(\alpha, \beta) \in D(\alpha, \beta) \}$ for $\alpha \geq \beta$ on $Y$, where $D(\alpha, \beta)$ is a non-empty index set. Then

(i) $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} \subseteq \varphi_{\alpha, \gamma}$ for $\alpha \geq \beta \geq \gamma$ on $Y$.

(ii) For $a \in S_\alpha$ and $\beta \in Y$,

$$a \varphi_{\alpha, \beta} = \{a \varphi_{d(\alpha, \beta)} | \forall d(\alpha, \alpha) \in D(\alpha, \alpha)\} \subseteq S_{d(\alpha, \beta)},$$

for some $\rho_{\beta, \alpha\beta}$-class $S_{d(\alpha, \beta)}$.

**Proof.** (i) Clearly, $\varphi_{d(\alpha, \alpha)}$ is an identity automorphism of $S_\alpha$. We now prove that $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} \subseteq \varphi_{\alpha, \gamma}$ for $\alpha \geq \beta \geq \gamma$ on $Y$. Pick $\varphi_{d(\alpha, \beta)} : S_\alpha \to S_{d(\alpha, \beta)} \subseteq S_\beta$ and $\varphi_{d(\beta, \gamma)} : S_\beta \to S_{d(\beta, \gamma)} \subseteq S_\gamma$. We show that $\varphi_{d(\alpha, \beta)} \varphi_{d(\beta, \gamma)} = \varphi_{d(\alpha, \gamma)}$ for some $\varphi_{d(\alpha, \gamma)} : S_\alpha \to S_{d(\alpha, \gamma)} \subseteq S_\gamma$. For this purpose, we let $a \in S_{d(\alpha, \gamma)}$, $b_1, b_2 \in S_{d(\alpha, \beta)}$ and $c \in S_{d(\beta, \gamma)}$. Then, because $S/\tilde{H}$ is a band, by Lemma 3.2, we have $b_1 \varphi_{d(\beta, \gamma)} = b_1 (b_1 c b_1) = b_2 (b_2 c b_2)$. Since $b_1, b_2 \in S_{d(\alpha, \beta)}$, by the definition of $\rho_{\alpha, \beta}$, $(b_1, b_2) \in \rho_{\alpha, \beta}$. This leads to $(ab_1 a) = (ab_2)$. Now, by the regularity of the band $S/\tilde{H}$, we can easily deduce that

$$a \varphi_{d(\alpha, \gamma)} = \{(ab_1 \varphi_{d(\beta, \gamma)} a) = (ab_2) = (ab_2) = (ab_2) \}.$$

Thus, by the definition of $\rho_{\alpha, \gamma}$, we have $(b_1 \varphi_{d(\beta, \gamma)}, b_2 \varphi_{d(\beta, \gamma)}) \in \rho_{\alpha, \gamma}$. In other words, there exists a $\rho_{\alpha, \gamma}$-class $S_{d(\alpha, \gamma)}$ satisfying $S_{d(\alpha, \beta)} \varphi_{d(\beta, \gamma)} \subseteq S_{d(\alpha, \gamma)}$. Also, $\varphi_{d(\alpha, \beta)} \varphi_{d(\beta, \gamma)}$ clearly maps $S_\alpha$ into $S_{d(\alpha, \gamma)}$ by the transitivity of “$\subseteq$”, and hence $\varphi_{d(\alpha, \beta)} \varphi_{d(\beta, \gamma)} = \varphi_{d(\alpha, \gamma)}$. This proves that $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} \subseteq \varphi_{\alpha, \gamma}$. (ii) It suffices to show that for any $\varphi_{d(\alpha, \beta)}$ and $\varphi_{d(\alpha, \beta)} \in \varphi_{\alpha, \beta}$, we have

$$a \varphi_{d(\alpha, \beta)} \varphi_{d(\alpha, \beta)} = a \varphi_{d(\alpha, \beta)} \varphi_{d(\alpha, \beta)} \in \rho_{\beta, \alpha\beta}.$$
Proof. Let \( c_1 \in S_{d(a,\alpha \beta)}, c_2 \in S_{d(\beta,\alpha \beta)} \). Then \((ac_1a)^0 \in S_{d(a,\alpha \beta)}\) because \( S_{d(a,\alpha \beta)} \) is a \( \rho_{\alpha \beta} \)-equivalence class of \( S_{\alpha \beta} \). Now, by Lemma 3.2, \( a\varphi_{d(a,\alpha \beta)} = (ac_1a)^0a \) and \( b\varphi_{d(\beta,\alpha \beta)} = (bc_2b)^0 \) for \( \varphi_{d(a,\alpha \beta)} \in \varphi_{\alpha \beta} \) and \( \varphi_{d(\beta,\alpha \beta)} \in \varphi_{\beta \alpha} \). Since we assume that \( a\varphi_{\alpha \beta} \subseteq S_{d(\beta,\alpha \beta)} \), we have \( a\varphi_{d(a,\alpha \beta)} = (ac_1a)^0a \in S_{d(\beta,\alpha \beta)} \). Similarly, we have \( b\varphi_{d(\beta,\alpha \beta)} \in S_{d(a,\alpha \beta)} \cap S_{d(\beta,\alpha \beta)} \). Thus, by Lemma 3.2 (ii), we have

\[
(a\varphi_{d(a,\alpha \beta)})(b\varphi_{d(\beta,\alpha \beta)}) = (ac_1a)^0(ab(bc_2b)^0) = ab(bc_2b)^0
\]

and also

\[
(a\varphi_{d(a,\alpha \beta)})(b\varphi_{d(\beta,\alpha \beta)}) = ((ac_1a)^0)(bc_2b)^0 = (ac_1a)^0ab.
\]

However, by the definition of the natural partial order “\( \leq \)”, we have \( ab \geq (a\varphi_{d(a,\alpha \beta)})(b\varphi_{d(\beta,\alpha \beta)}) \).

On the other hand, since every semigroup \( S_{\alpha \beta} \) is primitive, we obtain

\[
ab = (a\varphi_{d(a,\alpha \beta)})(b\varphi_{d(\beta,\alpha \beta)}).
\]

4. Structure of regular \( \tilde{H} \)-cryptogroups

In this section, we use the \( KG \)-strong semilattice to characterize regular \( \tilde{H} \)-cryptogroups. Also, we consider the question when will the Green \( \sim \)-relation \( \tilde{H} \) to be a right quasi-normal band congruence? By using the \( KG \)-strong semilattice, we are able to give a description for the normal \( \tilde{H} \)-cryptogroups. We note here that the orthodox regular \( \tilde{H} \)-cryptogroups with \( KG \)-strong semilattices have been studies in [10]. A construction theorem of orthodox regular \( \tilde{H} \)-cryptogroups was also given in [8].

**Theorem 4.1** An \( \tilde{H} \)-cryptogroup \( S \) is a regular \( \tilde{H} \)-cryptogroup if and only if \( S \) is an \( \tilde{H}G \)-strong semilattice of completely \( J \)-simple semigroups, that is, \( S = \tilde{H}G[Y; S_{\alpha}, \varphi_{\alpha \beta}] \).

**Proof.** By the definition of the \( KG \)-strong semilattice and the results obtained in §3, we have already proved the necessity part of Theorem 4.1 since it is obvious that \( \tilde{H}(S_{\alpha}) \subseteq \rho_{\alpha \beta} \) for \( \alpha \geq \beta \) on \( Y \). We now prove the sufficiency part of the theorem. To prove that \( S/\tilde{H} \) is a regular band, we use a result in [14]. What we need is to prove that the usual Green relations \( L \) and \( R \) are congruences on \( S/\tilde{H} \). In fact, we only need to verify that \( L \) is a left congruence on \( S/\tilde{H} \) since \( R \) is a right congruence on \( S/\tilde{H} \) can be proved in a similar fashion. Since \( S = (Y; S_{\alpha}) \) is an \( \tilde{H} \)-cryptogroup, we can let \( e\tilde{H}, f\tilde{H} \) and \( g\tilde{H} \in S/\tilde{H} \), where \( e, f \in S_{\alpha} \cap E(S) \), \( g \in S_{\beta} \cap E(S) \) with \( (e, f) \in L \). Then, we have \( ef = e \) and \( fe = f \). By the definition of \( \tilde{H}G \)-strong semilattice \( \tilde{H}G[Y; S_{\alpha}, \varphi_{\alpha \beta}] \), we can find the homomorphisms \( \varphi_{d(\beta,\alpha \beta)} \) and \( \varphi_{d(\alpha,\alpha \beta)} \) such that

\[
(gef)\tilde{H} = ([g(ef)](gf))\tilde{H} = [[(g\varphi_{d(\beta,\alpha \beta)})(ef)](g\varphi_{d(\alpha,\alpha \beta)})][g\varphi_{d(\beta,\alpha \beta)}(f\varphi_{d(\alpha,\alpha \beta)})] \tilde{H}
\]

\[
= (g\varphi_{d(\beta,\alpha \beta)})(f\varphi_{d(\alpha,\alpha \beta)})] \tilde{H}
\]
and

\[(ge)\tilde{H} = |g(ef)|\tilde{H} = |(g\varphi^f_d(\beta,\alpha,\beta))(ef)\varphi^g_d(\alpha,\alpha,\beta)]\tilde{H} = |(g\varphi^f_d(\beta,\alpha,\beta))(f\varphi^g_d(\alpha,\alpha,\beta)]\tilde{H}.

Thus, \((gef)\tilde{H} = (ge)\tilde{H} \) and \((gfg)\tilde{H} = (gf)\tilde{H}\). This proves that \(L \) is left compatible with the multiplication of \(S/\tilde{H}\). Since \(L \) is always right congruence, \(L \) is a congruence on \(S/\tilde{H}\), as required. Dually, \(R \) is also a congruence on \(S/\tilde{H}\). Thus by [14] (see II. 3.6 Proposition), \(S/\tilde{H}\) forms a regular band and hence \(S \) is indeed a regular \(\tilde{H}\)-cryptogroup. Our proof is completed.

Recall that a right quasi-normal band is a band satisfying the identity \(yxa = yaxa \) [6]. Also, a left quasi-normal band is a band satisfying the identity \(axy = axay \). Thus, we can easily observe that both the right quasi-normal bands and the left quasi-normal bands are special cases of the regular bands. Also, a normal band (that is, a band satisfies the identity \(axya = ayxa \)) is a special right quasi-normal band and a left quasi-normal band. Based on the above observation, we are able to establish the following theorem for right quasi-normal \(\tilde{H}\)-cryptogroups.

**Theorem 4.2** An \(\tilde{H}\)-abundant semigroup \(S \) is a right quasi-normal \(\tilde{H}\)-cryptogroup if and only if \(S \) is an \(\tilde{L}G\)-strong semilattice of completely \(\tilde{J}\)-simple semigroups, that is, \(S = \tilde{L}G[Y; S_\alpha, \varphi_{\alpha, \beta}] \).

**Proof.** (*Necessity*) Let \(S \) be a right quasi-normal \(\tilde{H}\)-cryptogroup. Then \(S/\tilde{H}\) is a right quasi-normal band. To show that \(S \) is an \(\tilde{L}G\)-strong semilattice, by invoking Lemma 3.3 and its proof, we only need to show that for any \(\delta \geq \gamma \) on \(Y \), \(\tilde{L}|S_\gamma \subseteq \rho_{\delta, \gamma} \). In fact, for \(a \in S_\delta, x, y \in S_\gamma \) with \((x, y) \in \tilde{L} \), we have \((axa)\tilde{H} = ((axy)a)\tilde{H} = (ayxa)\tilde{H} = (aya)\tilde{H} \) by the right quasi-normality of the band \(S/\tilde{H} \). Thus, by the definition of \(\rho_{\delta, \gamma} \), we have \(\tilde{L}|S_\gamma \subseteq \rho_{\delta, \gamma} \) as required. This shows that \(S = \tilde{L}G[Y; S_\alpha, \varphi_{\alpha, \beta}] \). (*Sufficiency*) Let \(a \in S_\alpha, x \in S_\beta \), and \(y \in S_\gamma \). Then, since \(S = \tilde{L}G[Y; S_\alpha, \varphi_{\alpha, \beta}] \) is an \(\tilde{H}G\)-strong semilattice of \(S_\alpha \) and by Theorem 4.1, \(\tilde{H} \) is a congruence on \(S \). Moreover, we have \(xa = (x\varphi^a_d(\beta,\alpha,\beta))(ax\varphi^a_d(\alpha,\alpha,\beta)) \) and thereby, \(axa = ((x\varphi^a_d(\beta,\alpha,\beta))(ax\varphi^a_d(\alpha,\alpha,\beta)))(a\varphi^x_d(\alpha,\alpha,\beta)) \). By the fact \((xa)^0, (axa)^0 \in L \), we can easily see that \((xa, axa) \in \tilde{L}|S_\alpha, \) and so, by our hypothesis, \(S = \tilde{L}G[Y; S_\alpha, \varphi_{\alpha, \beta}] \). This implies that there exist some homomorphisms \(\varphi^E_d(\alpha,\beta,\gamma) \in \varphi_{\alpha,\beta,\gamma} \) and \(\varphi^E_d(\gamma,\alpha,\beta) \in \varphi_{\gamma,\alpha,\beta} \) satisfying the conditions \(y(xa) = ((ya)\varphi^E_d(\gamma,\alpha,\beta))((xa)\varphi^E_d(\alpha,\alpha,\beta)) \) and \(y(axa) = ((ya)\varphi^E_d(\gamma,\alpha,\beta))((xa)\varphi^E_d(\alpha,\alpha,\beta)) \). Hence, it follows that

\[(y(xa))\tilde{H} = [(ya)\varphi^E_d(\gamma,\alpha,\beta)](xa)\varphi^E_d(\alpha,\alpha,\beta)\tilde{H} = (ya)\varphi^E_d(\gamma,\alpha,\beta)(xa)\varphi^E_d(\alpha,\alpha,\beta)\tilde{H} = (ya)\varphi^E_d(\gamma,\alpha,\beta)(xa)\varphi^E_d(\alpha,\alpha,\beta)\tilde{H} \]
and
\[
(y(axa))\tilde{H} = \{(y\varphi_d^{\gamma}(\alpha,\alpha\beta\gamma))\{(a\varphi_d^{\rho}(\alpha,\alpha\beta\gamma))(a\varphi_d^{\rho}(\alpha,\alpha\beta\gamma))\}^\tilde{H}
\]
\[
= \{(y\varphi_d^{\gamma}(\alpha,\alpha\beta\gamma))(a\varphi_d^{\rho}(\alpha,\alpha\beta\gamma))\}^\tilde{H}.
\]
This leads to \((yxa)\tilde{H} = (yaxa)\tilde{H}\) and so \(S/\tilde{H}\) is a right quasi-normal band. Thus, \(S\) is indeed a right quasi-normal \(\tilde{H}\)-cryptogroup.

Since we have already mentioned that a band \(B\) is a normal band if for all elements \(e, f, g\) in \(B\), the identity \(efge = egfe\) holds in \(B\) (see [6]). In closing this paper, we characterize the normal \(\tilde{H}\)-cryptogroups. In fact, this result gives a modified version of the theorem of Petrich and Reilly in [11] on normal cryptogroups, in particular, the theorem on normal cryptogroups in [11] and also the theorem of Fountain on superabundant semigroups in [4] is now refined and amplified in the class of quasiabundant semigroups.

**Theorem 4.3** An \(\tilde{H}\)-abundant semigroup \(S\) is a normal \(\tilde{H}\)-cryptogroup if and only if \(S\) is a \(\tilde{D}G\)-strong semilattice of completely \(\tilde{F}\)-simple semigroups, that is, \(S = \tilde{D}G[Y; S_\alpha, \alpha, \beta]\).

**Proof.** (*Necessity*) The proof is similar to the necessity part given in Theorem 4.2, that is, we only need to prove that \(\tilde{D}|S_{\alpha} \subseteq \rho_{\alpha, \beta}\) for all \(\alpha, \beta \in Y\) with \(\alpha \geq \beta\). Since every semigroup \(S_\alpha\) can be regarded as a \(\tilde{D}\)-class of \(S\), we can just let \(a \in S_\alpha, x, y \in S_\beta\). Recall that \(S = (Y; S_\alpha)\) is a normal \(\tilde{H}\)-cryptogroup. \(S/\tilde{H}\) is a normal band. Now, by the normality of the band \(S/\tilde{H}\), we have
\[
(axa)\tilde{H} = (a(axy)a)\tilde{H} = (ayxa)\tilde{H} = (aya)\tilde{H}.
\]
Thus, by Lemma 3.1, we see that \((x, y) \in \rho_{\alpha, \beta}\) and whence \(\tilde{D}|S_\alpha \subseteq \rho_{\alpha, \beta}\). This proves that \(S = \tilde{D}G[Y; S_\alpha, \alpha, \beta]\). (*Sufficiency*) Let \(S = \tilde{D}G[Y; S_\alpha, \alpha, \beta]\), where each \(S_\alpha\) is a completely \(\tilde{F}\)-simple semigroup, for all \(\alpha \in Y\). Then by definition, \(S\) is an \(\tilde{D}G\)-strong semilattice of semigroups \(S_\alpha\) and also \(S\) is an \(\tilde{D}G\)-strong semilattice of semigroups \(S_\alpha\). By applying Theorem 4.2 and its dual, we immediately deduce that \(\tilde{H}\) is a congruence on \(S\) and for all \(a, x, y \in S\), we have
\[
[(axy)a]\tilde{H} = [ay(xya)]\tilde{H} = (ayxya)\tilde{H} = (aya)\tilde{H}.
\]
This shows that \(S/\tilde{H}\) is a normal band. Moreover, since each \(S_\alpha\) is a \(\tilde{D}\)-class of \(S\), for every \(\alpha, \beta \in Y\) with \(\alpha \geq \beta\), the set \(D(\alpha, \beta)\) is just a singleton. This means that \(S\) is a strong semilattice of completely \(\tilde{F}\)-simple semigroups \(S_\alpha\). Our proof is completed.

References

REFERENCES