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Simpler Proof of the Ringrose's Characterization of Compact Operators

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Abstract. The aim of this note is to give short, simpler and elementary proof of one characterization of compact operators via orthonormal sequences, which, hopefully will make this fact more accessible to nonspecialists and to a wide audience (especially to students). Beside this, the proof given here shows that this fact holds true for operator acting from Hilbert space to some Banach (not necessarily Hilbert) space.

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1. Introduction

The aim of this note is to give short, simpler and elementary proof of the following

Theorem 1. *Linear (not necessarily bounded) operator acting on a Hilbert space H is compact if and only if it satisfies $\|Ae_n\| \to 0$ for each orthonormal sequence $\{e_n\}$ in H.*

The proof of this theorem for bounded linear operators can be found in [3]. It is shown in [2] that continuity assumption on A is superfluous.

Note that, the following, in some sense more general fact is also valid: Linear (not necessarily bounded) operator acting on a Hilbert space H is compact if and only if it satisfies $(Ae_n, e_n) \to 0$ for each orthonormal sequence $\{e_n\}$ in H. For bounded operator this proposition can be found in [1, 4, 5]. It is shown in [2] that this fact remains valid without boundedness condition. Note that this proposition can be proved (which is seen from the cited references) by reducing it to the above theorem.

Note that the proof of the theorem given below shows that the proposition of this theorem is also true for an operator, acting from a Hilbert space to some Banach space.

Hopefully the exposition given in this short note will make the theorem and its generalization (mentioned above) more accessible to a wide audience (especially to students).

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2. Main result and its proof

Theorem 2. *Linear (not necessarily bounded) operator acting from a Hilbert space H to some Banach space is compact if and only if it satisfies $\|Ae_n\| \to 0$ for each orthonormal sequence $\{e_n\}$ in H.*

Proof. One has to prove only sufficiency of the above condition. The reverse implication is obvious (it is a well-known fact from almost all university textbooks on functional analysis, which states that compact operator takes weakly convergent sequence to convergent one).

Let B be a unit ball of the space H: $B = \{x : \|x\| \leq 1\}$. To prove the theorem it suffices to show that $A(B)$ is compact. Assume the contrary: $A(B)$ is not compact. By Hausdorff criterion there exists such an $\epsilon_0 > 0$ that there is not any compact ϵ_0-net for $A(B)$. Then there exists $x_1 \in B$ such that $\|Ax_1\| \geq \epsilon_0$ (otherwise 0 is ϵ_0-net for $A(B)$). Without loss of generality we can take $\|x_1\| = 1$. If the elements $x_1, x_2, \ldots, x_n$ are already chosen, then (n+1)th element is determined as follows: every element $x \in B$ can be represented in the form

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + \psi(x),$$

where $|\alpha_k| \leq 1$, for all $k = 1, 2, \ldots, n$ and $\psi(x)$ is perpendicular to $x_1, x_2, \ldots, x_n$. The set

$$A_n = \left\{\alpha_1 Ax_1 + \alpha_2 Ax_2 + \ldots + \alpha_n Ax_n : |\alpha_k| \leq 1, k = 1, 2, \ldots, n\right\}$$

is compact(since it is bounded subset of finite dimensional subspace). Therefore there exists $\xi_{n+1} \in B$ such that $\|A\psi(\xi_{n+1})\| \geq \epsilon_0$; otherwise the relation

$$\|Ax - (\alpha_1 Ax_1 + \alpha_2 Ax_2 + \ldots + \alpha_n Ax_n)\| = \|A\psi(x)\| < \epsilon_0$$

would show that the compact set A_n is ϵ_0-net of $A(B)$ that contradicts the definition of ϵ_0. Define x_{n+1} to be $\frac{\psi(\xi_{n+1})}{\|\psi(\xi_{n+1})\|}$: $x_{n+1} = \frac{\psi(\xi_{n+1})}{\|\psi(\xi_{n+1})\|}$. So, there exists orthonormal sequence $x_1, x_2, \ldots, x_n$ such that $\|Ax_n\| \geq \epsilon_0$, for all $n \in \mathbb{N}$. Contradiction. The theorem is proved. □

Remark 1. *Since every orthonormal sequence of a separable Hilbert space can be made an orthonormal basis by adding new elements, it is easy to see that the following equivalent formulations of the above theorems holds:*

Proposition 1. *Linear (not necessarily bounded) operator acting from a separable Hilbert space H to a Banach space B is compact if and only if it satisfies $\|Ae_n\| \to 0$ for each orthonormal basis $\{e_n\}$ in H.*

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