Annulets in Almost Distributive Lattices

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Abstract. We introduce the concept of annulets in an Almost Distributive lattice (ADL) \( R \) with 0. We characterize both generalized stone ADL and normal ADL in terms of their annulets. We characterize \( \star \)-ADLs by means of their annulets. It is proved that the lattice \( \mathcal{A}_0(R) \) of all annulets of a generalized stone ADL \( R \) is a relatively complemented sublattice of the lattice \( \mathcal{I}(R) \) of all ideals of \( R \). Finally, it is proved that \( \mathcal{A}_0(R) \) is relatively complemented iff \( R \) is sectionally \( \star \)-ADL.

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Key words: Almost Distributive Lattice (ADL), Boolean algebra, dense elements, maximal element, Annihilator ideal, Annulet, normal ADL, \( \star \)-ADL, generalized stone ADL, Disjunctive ADL.

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy. U.M. and Rao. G.C [8] as a common abstraction to most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. Later a more general class called \( \star \)-ADLs was introduced in the paper [10]. The characterization of \( \star \)-ADL by means of its dense elements was studied in [11]. In [5], Mandelker studied the properties of relative annihilators and...
characterized the distributive lattice in terms of relative annihilators. In this paper the concept of Annulet as an ideal of the form \((x)^* = \{ a \in R \mid x \land a = 0 \}\) in an ADL \(R\) with 0 is introduced, analogous to that in a distributive lattice[4]. It is proved that the set \(\mathcal{A}_0(R)\) of all annulets of an ADL \(R\) with 0 can be made into a distributive lattice and sublattice of the Boolean algebra \(\mathcal{A}(R)\) of all annihilator ideals of \(R\).

We characterize the generalized stone ADL and normal ADL in terms of their annulets. We introduce a more general class of ADLs called disjunctive ADLs with suitable examples and prove that a disjunctive normal ADL is dually isomorphic to the lattice \(\mathcal{A}_0(R)\). We characterize \(\star\)-ADLs by means of their annulets. If \(R\) is a generalized stone ADL, then it is proved that the lattice \(\mathcal{A}_0(R)\) is a relatively complemented sublattice of the lattice \(\mathcal{A}(R)\) of all ideals of \(R\). Finally, it is proved that \(\mathcal{A}_0(R)\) is relatively complemented iff \(R\) is sectionally \(\star\)-ADL.

2. Preliminaries

An Almost Distributive Lattice (ADL) is an algebra \((R, \lor, \land)\) of type \((2,2)\) satisfying

1. \((x \lor y) \land z = (x \land z) \lor (y \land z)\)
2. \(x \land (y \lor z) = (x \land y) \lor (x \land z)\)
3. \((x \lor y) \land y = y\)
4. \((x \lor y) \land x = x\)
5. \(x \lor (x \land y) = x.\) for any \(x, y, z \in R.\)

If \(R\) has an element 0 and satisfies \(0 \land x = 0\) and \(x \lor 0 = x\) along with the above properties, then \(R\) is called an ADL with 0.

Every non-empty set \(X\) can be regarded as an ADL as follows. Let \(x_0 \in X\). Define two
binary operations $\lor, \land$ on $X$ by

$$x \lor y = \begin{cases} 
  x & \text{if } x \neq x_0 \\
  y & \text{if } x = x_0 
\end{cases} \quad x \land y = \begin{cases} 
  y & \text{if } x \neq x_0 \\
  x_0 & \text{if } x = x_0 
\end{cases}$$

Then $(X, \lor, \land, x_0)$ is an ADL with $x_0$ as zero element and is called a discrete ADL.

If $(R, \lor, \land, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \land b$ (or equivalently, $a \lor b = b$), then $\leq$ is a partial ordering on $R$.

**Theorem 2.1.** For any $a, b, c \in R$, we have the following:

1. $a \lor b = a \iff a \land b = b$
2. $a \lor b = b \iff a \land b = a$
3. $a \land b = b \land a$ whenever $a \leq b$
4. $\land$ is associative in $R$
5. $a \land b \land c = b \land a \land c$
6. $(a \lor b) \land c = (b \lor a) \land c$
7. $a \land b = 0 \iff b \land a = 0$
8. $a \lor b = b \lor a$ whenever $a \land b = 0$
9. $a \lor (b \land c) = (a \lor b) \land (a \lor c)$
10. $a \land (a \lor b) = a, (a \land b) \lor b = b, \text{ and } a \lor (b \land a) = a$
11. $a \leq a \lor b$ and $a \land b \leq b$
12. $a \land a = a$ and $a \lor a = a$
13. $0 \lor a = a$ and $a \land 0 = 0$
14. If $a \leq c$ and $b \leq c$ then $a \land b = b \land a$ and $a \lor b = b \lor a$
15. $a \lor b = a \lor b \lor a$.

An element $m \in R$ is called maximal if it is maximal in the partial ordered set $(R, \leq)$. That is, for any $x \in R, m \leq x \Rightarrow m = x$.

**Theorem 2.2.** Let $R$ be an ADL and $m \in R$. Then the following are equivalent:
1. \( m \) is a maximal element with respect to \( \leq \)
2. \( m \lor x = m \), for all \( x \in R \)
3. \( m \land x = x \), for all \( x \in R \)
4. \( x \lor m \) is maximal for all \( x \in R \).

A non-empty subset \( I \) of \( R \) is called an ideal(filter) of \( R \) if \( a \lor b \in I(a \land b \in I) \) and \( a \land x \in I(x \lor a \in I) \) whenever \( a, b \in I \) and \( x \in R \). If \( I \) is an ideal of \( R \) and \( a, b \in R \), then \( a \land b \in I \iff b \land a \in I \). The set \( \mathcal{I}(R) \) of all ideals of \( R \) is a complete distributive lattice with least element \( \{0\} \) and the greatest element \( R \) under set inclusion in which, for any \( I, J \in \mathcal{I}(R), I \cap J \) is the infimum of \( I, J \) and the supremum is given by \( I \lor J = \{ i \lor j \mid i \in I, j \in J \} \). For any \( a \in R \), \( \{a\} = \{a \land x \mid x \in R\} \) is the principal ideal generated by \( a \). Similarly, for any \( a \in R \), \( \{a\} = \{x \lor a \mid x \in R\} \) is the filter generated by \( a \). An ideal \( I \) of \( R \) is called a direct summand of \( R \) if there exists an ideal \( J \) in \( R \) such that \( I \cap J = \{0\} \) and \( I \lor J = R \).

**Theorem 2.3.** For any \( a, b \in R \), we have the following:

1. \((a) \lor (b) = (a \lor b) = (b \lor a)\)
2. \((a) \land (b) = (a \land b) = (b \land a)\)
3. \([a] \lor [b] = [a \lor b] = [b \lor a]\)
4. \([a] \land [b] = [a \land b] = [b \land a]\)

Thus the set \( \mathcal{P}\mathcal{I}(R) \) of all principal ideals of \( R \) is a sublattice of the distributive lattice \( \mathcal{I}(R) \) of ideals of \( R \). A proper ideal \( P \) of \( R \) is said to be prime if for any \( x, y \in R, x \land y \in P \Rightarrow \) either \( x \in P \) or \( y \in P \). It is clear that a subset \( P \) of \( R \) is a prime ideal iff \( R - P \) is a prime filter.

For any \( A \subseteq R, A^\ast = \{x \in R \mid a \land x = 0\ \text{for all } a \in A\} \) is an ideal of \( R \). We write \((a)^\ast\) for \( \{a\}^\ast \). Then clearly \((0)^\ast = R \) and \( R^\ast = \{0\} \). An element \( a \in R \) is called dense if \((a)^\ast = \{0\} \). The set of all dense elements of \( R \) is denoted by \( D \). An ideal \( I \) of \( R \) is called dense if \( I^\ast = \{0\} \). An ADL \( R \) with 0 is called a \$\ast\$-ADL [10], if for each \( x \in R \), there exists an element \( x' \in R \) such that \((x)^{**} = (x')^\ast \). \( R \) is a \$\ast\$-ADL iff to each \( x \in R \), there exists \( x' \in R \) such that \( x \land x' = 0 \) and \( x \lor x' \) is dense. Every \$\ast\$-ADL possesses a dense element. An ADL \( R \) with 0 is called relatively...
complemented if each interval \([a, b], a \leq b\), in \(R\) is a complemented lattice.

An ideal \(I\) of \(R\) is called an annihilator ideal if \(I = I^{**}\), or equivalently, \(I = S^{*} = \{y \in R \mid y \land s = 0 \text{ for all } s \in S\}\) for some non-empty subset \(S\) of \(R\). We denote the set of all annihilator ideals of \(R\) by \(\mathcal{A}(R)\). The set \(\mathcal{A}(R)\) forms a complete Boolean algebra with bounds \(\{0\}, R\) and the complement of any \(I \in \mathcal{A}(R)\) is \(I^{*}\) with respect to the operations \(\land\) and \(\lor\) given by \(I \land J = I \cap J\) and \(I \lor J = (I^{*} \cap J^{*})^{*}\).

3. Annulets

In this section, we introduce the concept of annulets in \(R\) and study some basic properties of these annulets. We prove characterization theorems of a few algebraic structures with the help of their annulets. We begin with the following definition.

**Definition 3.1.** Let \(R\) be an ADL with 0 and \(x \in R\). Then define the annulet \((x)^{*}\) as follows:
\[
(x)^{*} = \{y \in R \mid x \land y = 0\}
\]
Clearly \((x)^{*}\) is an ideal in \(R\) and hence an annihilator ideal.

Let us denote \(\mathcal{A}_{0}(R) = \{(x)^{*} \mid x \in R\}\).

Annulets have many important properties. We give some of them in the following lemma which can be proved directly.

**Lemma 3.2.** Let \(R\) be an ADL with 0 and \(x, y \in R\). Then we have:
1. \(x \leq y \Rightarrow (y)^{*} \subseteq (x)^{*}\)
2. \((x \land y)^{*} = (y \land x)^{*}\)
3. \((x \lor y)^{*} = (y \lor x)^{*}\)
4. \((x \lor y)^{*} = (x)^{*} \cap (y)^{*}\)
5. \((x)^{*} \lor (y)^{*} \subseteq (x \land y)^{*}\).

**Note:** Since each annulet is an annihilator ideal, we can have the following:
\[
(x)^{*} \lor (y)^{*} = [(x)^{*} \cap (y)^{*}]^{*} = [(x \land y)^{*}]^{*} = (x \land y)^{*}
\]
\[(x]^* \land (y]^* = (x]^* \land (y]^* = (x \lor y]^*).

Now we prove in the following theorem that the set \( \mathcal{A}_0(R) \) of all annulets of an ADL \( R \) forms a distributive lattice.

**Theorem 3.3.** Let \( R \) be an ADL with 0. Then \( (\mathcal{A}_0(R), \land, \lor) \) is a distributive lattice and a sublattice of the Boolean algebra \( (\mathcal{A}(R), \land, \lor, (0], R) \) of annihilator ideals of \( R \). \( \mathcal{A}_0(R) \) has the same greatest element \( R = (0]^* \) as \( \mathcal{A}(R) \) while \( \mathcal{A}_0(R) \) has the smallest element iff \( R \) possesses a dense element.

**Proof:** Let \((x]^*, (y]^* \in \mathcal{A}_0(R)\), where \( x, y \in R \). Then
1. \((x]^* \land (y]^* = (x]^* \land (y]^* = (x \lor y]^* \in \mathcal{A}_0(R)\) and
2. \((x]^* \lor (y]^* = (x \lor y]^* \in \mathcal{A}_0(R)\).

Hence \( \mathcal{A}_0(R) \) is a sublattice of \( \mathcal{A}(R) \). Since \( \mathcal{A}(R) \) is distributive, we have that \( \mathcal{A}_0(R) \) is also distributive. Clearly \((0]^* \) is the greatest element of \( \mathcal{A}(R) \). Now for any \((x]^* \in \mathcal{A}_0(R)\), we get \((x]^* \land (0]^* = (x \lor 0]^* = (x]^* \) and \((x]^* \lor (0]^* = (x \land 0]^* = (0]^*\). It shows that \((0]^* \) is the greatest element in \( \mathcal{A}_0(R) \). Now, it remains to prove the final condition of the theorem. Assume \( \mathcal{A}_0(R) \) has the smallest element, say \((d]^* where \( d \in R \). Suppose \( x \in (d]^* \). Then \( x \land d = 0 \). Since \((d]^* \) is the least element, we get \((x]^* = (x]^* \lor (d]^* = (x \land d]^* = (0]^* = R \). Hence \( x = 0 \). Thus \((d]^* = (0] \). Therefore \( d \) is a dense element in \( R \).

Conversely, suppose that \( R \) possesses a dense element, say \( d \). So \((d]^* = (0] \). Clearly \((d]^* \in A_0(R) \). Now for any \( x \in R \), consider \((x]^* \land (d]^* = (x]^* \land (0] = (0] \). Also \((x]^* \lor (d]^* = [(x]^* \land (d]^*] = [(x]^* \land (0] = [(x]^* \land R]^* = (x]^* = (x]^* \). Hence \((d]^* \) is the smallest element in \( \mathcal{A}_0(R) \).

The following definition of a normal ADL is taken from [7].

**Definition 3.4.** An ADL \( R \) with 0 is called normal ADL iff for all \( x, y \in R \)
\((x]^* \lor (y]^* = (x \land y]^* \).

Swamy U.M., Rao G.C., Nanaji Rao G. [9] and [10], have studied the properties of a psuedo-complemented ADL and later introduced the concept of stone ADL [10] as a psuedo-complemented
ADL $R$ with 0, in which $x^* \lor x^{**} = 0^*$, for all $x \in R$. Now we give the definition of a generalized stone ADL in the following.

**Definition 3.5.** An ADL $R$ with 0 is called a generalized Stone ADL iff

$$(x^*)^* \lor (x^{**})^* = R \text{ for each } x \in R.$$  

**Example 3.6.** Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs. Write $R = A \times B = \{(0,0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. Then $(R, \lor, \land, 0')$ is an ADL where $0' = (0,0)$, under point-wise operations.

Now \(((a,0))^* \lor ((a,0))^{**} = \{(0,0), (0, b_1), (0, b_2)\} \lor \{(0,0), (a, 0)\} = R.\)

\(((0,b_1))^* \lor ((0,b_1))^{**} = \{(0,0), (a, 0)\} \lor \{(0,0), (0, b_1), (0, b_2)\} = R.\)

Also \(((a,b_1))^* \lor ((a,b_1))^{**} = \{(0,0)\} \lor R = R.\)

Hence $(R, \lor, \land, 0')$ is a generalized stone ADL.

We now characterize normal ADL and the generalized stone ADL in terms of annulets.

**Theorem 3.7.** Let $R$ be an ADL with 0. Consider the following conditions:

1. Each annulet is a direct summand of $R$
2. $R$ is a generalized stone ADL
3. $R$ is normal
4. $\mathcal{A}_0(R)$ is a sublattice of the lattice $\mathcal{I}(R)$ of all ideals of $R$.

Then (1) is equivalent to (2), (3) is equivalent to (4), and (2) implies (3). If $R$ is a $\star$-ADL, then (4) implies (1).

**Proof:** (1) $\Rightarrow$ (2): Let $x \in R$. Then by (1), there exists an ideal $J$ of $R$ such that $(x^*)^* \cap J = \{0\}$ and $(x^*)^* \lor J = R$. Now $(x^*)^* \cap J = \{0\}$ implies that $J \subseteq (x)^{**}$. Hence $R = (x^*)^* \lor J \subseteq (x)^* \lor (x)^{**}$.

Thus $R = (x^*)^* \lor (x)^{**} \forall x \in R$.

(2) $\Rightarrow$ (1): Assume that $R$ is a generalized stone ADL. Let $x \in R$.

We have always $(x^*)^* \cap (x)^{**} = \{0\}$. By (2), we get $(x^*)^* \lor (x)^{**} = R$.

(2) $\Rightarrow$ (3): Assume that $R$ is a generalized stone ADL. Let $x, y \in R$. Always we have $(x^*)^* \lor (y^*)^* \subseteq (x \land y)^*$. Let $a \in (x \land y)^*$. Then $a \land x \land y = 0$.

$\Rightarrow (a \land x \land y) = \{0\}$
Thus \( (x] \cap (a \land y] = 0 \)
\( (a \land y] \subseteq [x]^* \)
\( (x]^{**} \subseteq (a \land y]\)
\( (x]^{**} \cap (a \land y] = 0 \)
\( (x]^{**} \cap [(a] \cap (y]) = 0 \)
\( [(x]^{**} \cap (a] \cap (y] = 0 \)
\( (x]^{**} \cap (a] \subseteq (y]^* \)

It is clear that \( (x]^* \cap (a] \subseteq (x]^* \)

Thus we get that \( [(x]^{**} \cap (a] \subseteq (x]^* \lor (y]^* \)
\( [(x]^{**} \cap (a] \subseteq (x]^* \lor (y]^* \)
\( R \cap (a] \subseteq (x]^* \lor (y]^* \) (since \( R \) is a generalized stone ADL)
\( (a] \subseteq (x]^* \lor (y]^* \)
\( a \in (x]^* \lor (y]^* \)

Hence \( (x \land y]^* \subseteq (x]^* \lor (y]^* \). Thus \( (x \land y]^* = (x]^* \lor (y]^* \). Therefore \( R \) is normal.

Now we prove the equivalency of (3) and (4).

(3) \( \Rightarrow \) (4): Assume that \( R \) is normal. Let \( x, y \in R \). We have always \( (x]^* \cap (y]^* = (x \lor y]^* \in \mathcal{A}_0(R) \). Since \( R \) is normal, we get \( (x]^* \lor (y]^* = (x \land y]^* \in \mathcal{A}_0(R) \). Therefore \( \mathcal{A}_0(R) \) is a sublattice of \( \mathcal{I}(R) \).

(4) \( \Rightarrow \) (3): Assume the condition (4). Let \( x, y \in R \). Then by (4), \( (x]^* \lor (y]^* = (z]^* \), for some \( z \in R \). Now \( (z]^* = [(x]^* \lor (y]^*]^* = (x]^{**} \cap (y]^{**} = (x \land y]^{**} \). Hence \( (x]^* \lor (y]^* = (x \land y]^* \).

Therefore \( R \) is normal.

(4) \( \Rightarrow \) (1): Suppose \( R \) is a \( *-ADL \). Assume that \( \mathcal{A}_0(R) \) is a sublattice of \( \mathcal{I}(R) \). Let \( x \in R \). Then there exists \( x' \in R \) such that \( (x]^{**} = (x']^* \). We have always \( (x]^* \cap (x]^{**} = 0 \). Now \( (x]^* \lor (x]^{**} = (x]^* \lor (x']^* = (z]^*, for some \( z \in R \)(by condition (4)). Hence \( (z]^{**} = [(x]^* \lor (x']^*[ = (x]^{**} \cap (x]^{**} = 0 \).

Thus \( (x]^* \lor (x]^{**} = (z]^* = 0]^* = R \). Thus \( (x]^* is a direct summand of \( R \).

**Definition 3.8.** An ADL \( R \) with 0, is called disjunctive iff for all \( a, b \in R \),
\( (a]^* = (b]^* \) implies \( a = b \).
**Example 3.9.** Let $R = \{0, a, b, c\}$ be a set. Define $\lor$ and $\land$ on $R$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>$0$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lor$</td>
<td>$0$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>$\land$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Then clearly $(R, \lor, \land, 0)$ is an ADL with 0.

Now, $(a)^* = \{0\}$, $(b)^* = \{0, c\}$ and $(c)^* = \{0, b\}$.

Thus $x \not= y$ implies that $(x)^* \not= (y)^*$ for all $x, y \in R$. Hence $R$ is disjunctive.

**Theorem 3.10.** A disjunctive ADL $R$ is dually isomorphic to $\mathcal{A}_0(R)$.

**Proof:** Let $R$ be a disjunctive ADL. Define a mapping $\Phi : R \rightarrow \mathcal{A}_0(R)$ by $\Phi(x) = (x)^*$, for all $x \in R$. Clearly $\Phi$ is well-defined.

(i). Let $x, y \in R$ be such that $\Phi(x) = \Phi(y)$. Then $(x)^* = (y)^*$. Since $R$ is disjunctive, we get that $x = y$. Therefore $\Phi$ is One-one.

(ii). Let $y \in \mathcal{A}_0(R)$. Then $y = (x)^*$, for some $x \in R$. Now for this $x$, $\Phi(x) = (x)^* = y$. Therefore $\Phi$ is onto.

(iii). Let $(x)^*, (y)^* \in \mathcal{A}_0(R)$, where $x, y \in R$.

Then $\Phi(x \land y) = (x \land y)^* = (x)^* \land (y)^* = \Phi(x) \land \Phi(y)$.

Again $\Phi(x \lor y) = (x \lor y)^* = (x)^* \lor (y)^* = \Phi(x) \lor \Phi(y)$.

Hence $\Phi$ is a dual isomorphism. \[\square\]

In an ADL $R$ with 0, we know that a maximal element is always a dense element. Now we prove the converse in disjunctive ADL.

**Theorem 3.11.** If $R$ is a disjunctive ADL, then every dense element of $R$ is a maximal element.

**Proof:** Assume that $R$ is disjunctive. Let $m$ be a dense element of $R$. That is $(m)^* = \{0\}$. For any $x \in R$, $(m \lor x)^* = (m)^* \lor (x)^* = (0)^* \lor (x)^* = (0) = (m)^*$. Since $R$ is disjunctive, we get that $m \lor
\( x = m \). Therefore \( m \) is a maximal element of \( R \).

\[ \square \]

We now characterize a \( \ast \)-ADL in terms of its lattice of annulets in the following theorem.

**Theorem 3.12.** Let \( R \) be an ADL with 0. Then \( R \) is a \( \ast \)-ADL iff \( \mathcal{A}_0(R) \) is a Boolean subalgebra of \( \mathcal{A}(R) \).

**Proof:** Assume that \( R \) is a \( \ast \)-ADL.

Then \( R \) has a dense element, say \( d \). Then \((d)^* = (0)\) is the least element and\((0)^*\) is the greatest element of the sublattice \( \mathcal{A}_0(R) \) of \( \mathcal{A}(R) \). Let \( x \in R \). Since \( R \) is a \( \ast \)-ADL, there exists \( x' \in R \) such that \((x)^* = (x')^*\).

We now show that \((x')^*\) is the complement of \((x)^*\) in \( \mathcal{A}_0(R) \), for each \( x \in R \).

Now \((x)^* \cap (x')^* = (x)^* \cap (x)^* = (0)\) \(\) and \(x^* \lor (x')^* = [(x)^* \cap (x')^*]^* = [(x)^* \cap (x')^*]^* = (0)\). Thus \( \mathcal{A}_0(R) \) is a Boolean subalgebra of \( \mathcal{A}(R) \). Conversely assume that \( \mathcal{A}_0(R) \) is a Boolean subalgebra of \( \mathcal{A}(R) \).

Let \( x \in R \). Then \((x)^* \in \mathcal{A}_0(R) \). Since \( \mathcal{A}_0(R) \) is a subalgebra of \( \mathcal{A}(R) \), there exists \((y)^* \in \mathcal{A}_0(R) \), with \( y \in R \) such that \((x)^* \cap (y)^* = (0)\) and \((x)^* \lor (y)^* = (0)^*\).

Now \((x)^* \lor (y)^* = (0)^* \Rightarrow (x \land y)^* = (0)^* \Rightarrow x \land y = 0 \). Again, \((x)^* \cap (y)^* = (0) \Rightarrow (x \lor y)^* = (0) \Rightarrow x \lor y \) is a dense element. Thus we proved that for each \( x \in R \), there exists \( y \in R \) such that \( x \land y = 0 \) and \( x \lor y \) is a dense element. Therefore \( R \) is a \( \ast \)-ADL.

\[ \square \]

**Definition 3.13.** An ADL \( R \) with 0 is called sectionally \( \ast \)-ADL iff for any \( x(\neq 0) \in R \), the interval \([0,x]\) is a \( \ast \)-ADL.

Before proving the next theorem, we need the following lemma.

**Lemma 3.14.** Let \( I,J \) be two ideals in an ADL \( R \). If \( I \cap J \) and \( I \lor J \) (i.e. The infimum and the supremum of \( I,J \) in the distributive lattice \( \mathcal{I}(R) \)) are both principal ideals, then \( I,J \) are also principal ideals.

**Proof:** Suppose \( I \lor J = (a) \) and \( I \cap J = (b) \), for some \( a,b \in R \).

Now \( a \in I \lor J \Rightarrow a = c \lor d \) for some \( c \in I \) and \( d \in J \). Then \( c \lor (b \land d) \in I \). So that
Again by the above lemma, we have that the infimum and the supremum of the ideals are $I \subseteq I$.

Let $x \in I$. Then $x \in I \lor J = \{ a \}$.

So $x = a \land x = (c \lor d) \land x = (c \land x) \lor (d \land x) \implies (1)$.

Now $x \in I$ and $d \in J \implies x \land d \in I \cap J = \{ b \} \implies d \land x \in \{ b \}$.

Hence $d \land x = b \land d \land x \implies (2)$.

From (1) and (2), we can obtain $x = (c \land x) \lor (b \land d \land x) = [c \lor (b \land d)] \land x$. Hence $x \in (c \lor (b \land d)]$. Therefore $I \subseteq (c \lor (b \land d)]$.

By symmetry, we get that $J$ is also a principal ideal. \qed

**Theorem 3.15.** Let $R$ be a generalized stone ADL. Then $\mathcal{A}_0(R)$ is a relatively complemented sublattice of the lattice $\mathcal{S}(R)$ of all ideals of $R$.

**Proof:** Let $R$ be a generalized stone ADL. By theorem 3.7, $\mathcal{A}_0(R)$ is a sublattice of $\mathcal{S}(R)$. So we can treat $\lor$ as $\lor$. Since $\mathcal{A}_0(R)$ is a distributive lattice with the greatest element $0^* = R$, it is enough to prove that each interval of the form $[I, R]$, where $I \in \mathcal{A}_0(R)$, is complemented.

Let $J = [\{ x \}^*, R]$ be an interval in $\mathcal{A}_0(R)$ and $(y)^* \in J$. We have clearly $(y)^* \cap (y)^* = 0$.

Since $R$ is generalized stone ADL, we have $(y)^* \lor (y)^* = R$ for all $y \in R$.

Now $\{ (x) \cap (y)^* \} \lor \{ (x) \cap (y)^* \} = (x) \cap \{ (y)^* \lor (y)^* \} = (x) \cap R = (x)$.

Also $\{ (x) \cap (y)^* \} \cap \{ (x) \cap (y)^* \} = (x) \cap \{ (y)^* \cap (y)^* \} = (x) \cap 0 = 0$.

Thus we have that the infimum and the supremum of the ideals $(x) \cap (y)^*$ and $(x) \cap (y)^*$ are the principal ideals $0^*$ and $(x)$.

Therefore, by the above lemma, $(x) \cap (y)^*$ and $(x) \cap (y)^*$ must be the principal ideals. Suppose $(x) \cap (y)^* = \{ a \}$ and $(x) \cap (y)^* = \{ b \}$ for some $a, b \in R$.

Now $a \in (x) \cap (y)^* \implies (a) \subseteq \{ x \} \implies (x) \subseteq (a)^*$. Therefore $(a)^* \in J$.

Also $(a) = (x) \cap (y)^* \subseteq (y)^* \implies (y)^* \subseteq (a)^*$. Hence $(y)^* \lor (y)^* \subseteq (y)^* \lor (a)^* \implies R \subseteq (a)^* \lor (y)^*$. Thus $R = (a)^* \lor (y)^* \implies (1)$

Again $(a)^* \cap (y)^* \cap (x) = (a)^* \cap (a) = 0$. Hence $(a)^* \cap (y)^* \subseteq (x)^*$.

But $(x)^* \subseteq (y)^*$ and $(x)^* \subseteq (a)^*$ imply that $(x)^* \subseteq (a)^* \cap (y)^*$.

Hence $(a)^* \cap (y)^* = (x)^*$ \implies (2)

From (1) and (2), $(a)^*$ is the required complement of $(y)^*$ in $J$. 

Hence $\mathcal{O}_0(R)$ is a relatively complemented sublattice of $\mathcal{O}(R)$. \hfill \square

**Definition 3.16.** Let $I = [0, x], 0 < x$, be an interval in an ADL $R$ with $0$. For $a \in I$, define the annihilator $(a)^+$ of $a$ with respect to $I$ as follows:

$$(a)^+ = \{ y \in I \mid y \wedge a = 0 \}.$$ 

Observe that $(a)^+ \cap I = (a)^+$.

**Lemma 3.17.** For $a \in I$, the annihilator $(a)^+$ is an ideal in $I$.

**Proof:** Since $0 \in I$ and $0 \wedge a = 0$, we get that $0 \in (a)^+$. Let $r, s \in (a)^+$. Then $r, s \in I$ and $r \wedge a = s \wedge a = 0$.

Since $r, s \in I$, we get $r \vee s \in I$, and $(r \vee s) \wedge a = (r \wedge a) \vee (s \wedge a) = 0 \vee 0 = 0$.

Hence $r \vee s \in (a)^+$. Let $y \in (a)^+$ and $t \in I$. Then $y \in I$ and $y \wedge a = 0$. Hence $y \wedge t \in I$. Now $(y \wedge t) \wedge a = t \wedge y \wedge a = t \wedge 0 = 0$, which implies that $y \wedge t \in (a)^+$. Thus $(a)^+$ is an ideal of $I$. \hfill \square

**Lemma 3.18.** Let $I = [0, x], 0 < x$, be an interval in an ADL $R$ with $0$. Then we have the following:

(i). For $a, b \in I$, $(a)^+ \subseteq (b)^+$ implies $(a)^+ \subseteq (b)^+$.

(ii). If $z \in R$, then $(z)^+ \cap I = (z \wedge x)^+$.

**Proof:**

(i). Let $a, b \in I$ and suppose $(a)^+ \subseteq (b)^+$. Let $t \in (a)^+$. Then $t \wedge a = 0$ and $t \in R \Rightarrow t \wedge x \wedge a = 0$ and $t \wedge x \in I$, since $x \in I$. Which implies $t \wedge x \in (a)^+ \subseteq (b)^+ \Rightarrow t \wedge x \wedge b = 0 \Rightarrow t \wedge b = 0$, since $t \in I = [0, x]$. Hence $t \in (b)^+$.

(ii). Let $t \in (z)^+ \cap I$. Then $t \in (z)^+$ and $t \in I$. Hence $t \wedge z = 0$ and $t \in I$. Thus $t \wedge z \wedge x = 0$ and $t \in I \Rightarrow t \wedge (z \wedge x) = 0$ and $t \in I \Rightarrow t \in (z \wedge x)^+$.

Therefore $(z)^+ \cap I \subseteq (z \wedge x)^+$. Again, let $t \in (z \wedge x)^+$, then $t \wedge z \wedge x = 0$ and $t \in I \Rightarrow z \wedge t \wedge x = 0$ and $t \in I \Rightarrow z \wedge t = 0$ and $t \in I \Rightarrow t \in (z)^+$ and $t \in I$. Hence $t \in (z)^+ \cap I$. Thus $(z \wedge x)^+ \subseteq (z)^+ \cap I$.

Therefore $(z)^+ \cap I = (z \wedge x)^+$. \hfill \square

We now prove the characterization theorem of a sectionally $*$-ADL in terms of its annulets. Before proving it, we can observe that if $R$ is an ADL with $0$ and $I = [0, x], 0 < x$ for some $x \in R$, then $\mathcal{O}_0(I)$ is a bounded distributive lattice (with respect to the operations given in the theorem 3.3) with the greatest element $I = (0)^+$ and the least element $(x)^+$. 

Theorem 3.19. Let R be an ADL with 0. Then $\mathcal{A}_0(R)$ is relatively complemented if and only if R is sectionally $\star$-ADL.

Proof: Assume that $\mathcal{A}_0(R)$ is relatively complemented.

We have to prove that each interval $I = [0, x]$ in R is a $\star$-ADL. By theorem 3.12, it is enough to prove that $\mathcal{A}_0(I)$ is relatively complemented.

Since $\mathcal{A}_0(I)$ is a distributive lattice with the greatest element $I = (0)^+$, it is enough to prove that each interval $[J, I], J \in \mathcal{A}_0(I)$ is complemented.

Choose $a, b \in I$ such that $(b)^+ \in [(a)^+, I] \subseteq \mathcal{A}_0(I)$. Then $(a)^+ \subseteq (b)^+ \subseteq I$.

By lemma 3.18(i), $(a)^+ \subseteq (b)^+ \subseteq R$.

Since $\mathcal{A}_0(R)$ is relatively complemented and $(b)^+ \in [(a)^+, R]$, there exists an element $c \in R$ such that $(c)^+ \in [(a)^+, R]$ and $(b)^+ \cap (c)^+ = (a)^+$ and $(b)^+ \cap (c)^+ = R$.

Now $(b)^+ \cap (c)^+ = (a)^+ \Rightarrow (b)^+ \cap (c)^+ \cap I = (a)^+ \cap I \Rightarrow [(b)^+ \cap I] \cap [(c)^+ \cap I] = (a)^+ \cap I \Rightarrow (b)^+ \cap (c)^+ = (a)^+$ → (1)

Secondly, $(b)^+ \cap (c)^+ = R \Rightarrow [(b)^+ \cap (c)^+] \cap I = R \cap I \Rightarrow [(b)^+ \cap I] \cap [(c)^+ \cap I] = I \Rightarrow (b)^+ \cap (c)^+ = I$ → (2)

From (1) and (2), we get that $(c)^+$ is the complement of $(b)^+$ in $[(a)^+, I]$.

Hence $[(a)^+, I]$ is relatively complemented.

Conversely assume that R is sectionally $\star$-ADL.

Since $\mathcal{A}_0(R)$ is a distributive lattice with the greatest element R, it is enough to prove that each interval $[(a)^+, R], (a)^+ \in \mathcal{A}_0(R)$ is complemented.

Let $(b)^+ \in [(a)^+, R]$. Therefore $(a)^+ \subseteq (b)^+ \subseteq R$.

Consider the interval $I = [0, b \lor a]$. Then by the hypothesis, I is a $\star$-ADL.

So by theorem 3.12, $\mathcal{A}_0(I)$ is complemented.

Hence each interval $[(a)^+, I], (a)^+ \in \mathcal{A}_0(I)$, where $a \in I$ is complemented.

We have by the lemma 3.18(ii), $(a)^+ \cap I = (a \land (b \lor a))^+$ and $(b)^+ \cap I = (b \land (b \lor a))^+ = (b)^+ \subseteq I$, that is $(b)^+ \in [(a \land (b \lor a))^+, I]$.

Since $\mathcal{A}_0(I)$ is complemented, there exists an element $c \in I$ such that $(b)^+ \cap (c)^+ = (a \land (b \lor a))^+$ and $(b)^+ \lor (c)^+ = I$ → (3)

Now our claim is $(b)^+ \cap (c)^+ = (a)^+$ and $(b)^+ \lor (c)^+ = R$. 

Let \( x \in (b]^* \cap (c]^* \). Then \( x \in (b]^* \) and \( x \in (c]^* \), implies \( b \land x = 0 \) and \( c \land x = 0 \)
\[ \Rightarrow x \land (b \lor a) \land b = 0 \text{ and } x \land (b \lor a) \land c = 0. \]
\[ \Rightarrow x \land (b \lor a) \in (b]^+ \text{ and } x \land (b \lor a) \in (c]^+ , \text{ since } x \land (b \lor a) \in I. \]
\[ \Rightarrow x \land (b \lor a) \in (b]^+ \land (c]^+ \]
\[ \Rightarrow x \land (b \lor a) \in (a \land (b \lor a)]^+ , \text{ by (3)} \]
\[ \Rightarrow x \land (b \lor a) \land a \land (b \lor a) = 0 \]
\[ \Rightarrow x \land a \land (b \lor a) \land (b \lor a) = 0 \]
\[ \Rightarrow (x \land a) \land (b \lor a) = 0 \]
\[ \Rightarrow (b \lor a) \land (x \land a) = 0 \]
\[ \Rightarrow x \land (b \lor a) \land a = 0 \]
\[ \Rightarrow x \land a = 0 \]
\[ \Rightarrow x \in (a]^* \]

Hence \( (b]^* \cap (c]^* \subseteq (a]^* ) \quad \rightarrow (4) \)

Conversely, let \( x \in (a]^* \). Then \( x \land a = 0 \)
\[ \Rightarrow x \land a \land (b \lor a) \land (b \lor a) = 0 \]
\[ \Rightarrow x \land (b \lor a) \land a \land (b \lor a) = 0 \]
\[ \Rightarrow x \land (b \lor a) \in (a \land (b \lor a)]^+ , \text{ since } x \land (b \lor a) \in I. \]
\[ \Rightarrow x \land (b \lor a) \in (b]^+ \land (c]^+ , \text{ by (3)} \]
\[ \Rightarrow x \land (b \lor a) \in (b]^+ \text{ and } x \land (b \lor a) \in (c]^+ \]
\[ \Rightarrow x \land (b \lor a) \land b = 0 \text{ and } x \land (b \lor a) \land c = 0. \]
\[ \Rightarrow x \land b = 0 \text{ and } x \land c = 0, \text{ since } c \in I = [0, b \lor a]. \]
\[ \Rightarrow x \in (b]^* \text{ and } x \in (c]^* \]
\[ \Rightarrow x \in (b]^* \cap (c]^* \]

Hence \( (a]^* \subseteq (b]^* \cap (c]^* ). \quad \rightarrow (5) \)

From (4) and (5), we can obtain \( (b]^* \cap (c]^* = (a]^* \)

Again from (3), we have \( (b]^+ \lor (c]^+ = I \)
\[ \Rightarrow (b \land c]^+ = (b]^+ \lor (c]^+ = I \]
\[ \Rightarrow (b \land c]^+ = I \]
\[ \Rightarrow b \land c = 0 \]
\[(b \land c)^* = (0)^* = R \Rightarrow (b)^* \lor (c)^* = R\]

Hence \((c)^*\) is the complement of \((b)^*\) in \([a]^*, R]\).

Thus \(\mathscr{A}_0(R)\) is relatively complemented. \(\square\)

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**References**


