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Baer Elements In Lattice Modules

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Abstract. Let L be a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact and M be a module over L. In this paper we generalize the concepts of Baer elements,$*$-elements and closed elements and obtain the relation between $*$-elements and Baer elements and also closed elements and Baer elements. Some characterization are also obtain for closed elements of M and minimal prime elements of M.

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1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $a \in L$ is called proper if $a < 1$. A proper element p of L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. If $a \in L$, $b \in L$, $(a : b)$ is the join of all elements c in L such that $cb \leq a$. A proper element p of L is said to be primary if $ab \leq p$ implies $a \leq p$ or $b^n \leq p$ for some positive integer n. If $a \in L$ then $\sqrt{a} = \vee\{x \in L \mid x^n \leqslant a, n \in Z_+\}$. An element $a \in L$ is called a radical element if $a = \sqrt{a}$. An element $a \in L$ is called compact if $a \leqslant \bigvee_{\alpha} b_\alpha$ implies $a \leqslant b_{\alpha 1} \vee b_{\alpha 2} \vee \ldots \vee b_{\alpha n}$ for some finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact and every finite product of compact elements is compact. We shall denote by L_* the set compact elements of L. A nonempty subset F of L_* is called a filter of L_* if the following conditions are satisfied,

(i) $x, y \in F$ *implies* $xy \in F$

(ii) $x \in F, x \leqslant y$ *implies* $y \in F$.

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Let $F(L_*)$ denote the set of all filters of L. For a nonempty subset $\{F_\alpha\} \subseteq F(L_*)$, define $\Cup F_\alpha = \{x \in L_* \mid x \geq f_1 f_2 \cdots f_n \in F_{\alpha_i}$, for some $i = 1, 2, \ldots, n\}$. Then it is observed that, $F(L_*) = \langle F(L_*), \Cup, \cap \rangle$ is a complete distributive lattice with $\Cup$ as the supremum and the set theroretic $\bigcap$ as the infimum. For $a \in L_*$ the smallest filter containing a is denoted by [a) and it is given by $[a) = \{x \in L* \mid x \geq a^n$ for some nonnegative integer $n\}$. For a filter $F \in F(L_*)$ we denote,$0_F = \vee\{x \in L_* \mid xs = 0$, for $s \in F\}$.

Let M be a complete lattice and L be a multiplicative lattice. Then M is called L-module or module over L if there is a multiplication between elements of L and M written as aB where $a \in L$ and $B \in M$ which satisfies the following properties,

(i) $(\bigvee_\alpha a_\alpha)A = \bigvee_\alpha a_\alpha A \quad \forall a_\alpha \in L,\ A \in M$

(ii) $a(\bigvee_\alpha A_\alpha) = \bigvee_\alpha a A_\alpha \quad \forall a \in L,\ A_\alpha \in M$

(iii) $(ab)A = a(bA) \quad \forall a, b \in L,\ A \in M$

(iv) $1B = B$

(v) $0B = 0_M$ for all $a, a_\alpha, b \in L$ and $A, A_\alpha \in M$, where 1 is the supremum of L and 0 is the infimum of L. We denote by 0_M and I_M the least element and the greatest element of M. Elements of L will generally be denoted by $a, b, c, \ldots$ and elements of M will generally be denoted by $A, B, C \ldots$.

Let M be a L-module. If $N \in M$ and $a \in L$ then $(N : a) = \vee\{X \in M \mid aX \leqslant N\}$. If $A, B \in M$, then $(A : B) = \vee\{x \in L \mid xB \leqslant A\}$. An L-module M is called a multiplication L-module if for every element $N \in M$ there exists an element $a \in L$ such that $N = aI_M$ see [2]. In this paper a lattice module M will be a multiplication lattice module, which is compactly generated with the largest element I_M compact. A proper element N of M is said to be prime if $aX \leqslant N$ implies $X \leqslant N$ or $aI_M \leqslant N$ that is $a \leqslant (N : I_M)$ for every $a \in L$, $X \in M$. If N is a prime element of M then $(N : I_M)$ is prime element of L [4]. An element $N < I_M$ in M is said to be primary if $aX \leqslant N$ implies $X \leqslant N$ or $a^n I_M \leqslant N$ that is $a^n \leqslant (N : I_M)$ for some integer n. An element N of M is called a radical element if $(N : I_M) = \sqrt{(N : I_M)}$. If $aN = 0_M$ implies $a = 0$ or $N = 0_M$ for any $a \in L$ and $N \in M$ then M is called a torsion free L-module.

2. Residuation properties

We state some elementary properties of residuation in the following theorem.

Theorem 1. *Let L be a multiplicative lattice and M be a multiplication lattice module over L.For $x, y \in L$ and $Z, A, B \in M$, where $(0_M : I_M)$ is a radical element. We have the following identities,*

(i) $x \leqslant y$ *implies* $(0_M : y) \leqslant (0_M : x)$ *and* $0_M : (0_M : x) \leqslant 0_M : (0_M : y)$

(ii) $x \leqslant 0_M : (0_M : x)$

(iii) $0_M : [0_M : (0_M : x)] = (0_M : x)$

(iv) $(0_M : x) = (0_m : x^n)$ *for every* $n \in Z_+$

(v) $0_M : (0_M : x) \wedge 0_M : (0_M : y) = 0_M : (0_M : xy) = 0_M : [0_M : (x \wedge y)]$

(vi) $(0_M : a) = 0_M$ *implies* $(0_M : a^n) = 0$ *for every* $n \in Z_+$

(vii) $x \vee y = 1$ *implies* $(0_M : x) \vee (0_M : y) = 0_M : (x \wedge y) = 0_M : xy$

(viii) *For* Z *in* M, $Z \leqslant 0_M : (0_M : Z)$

(ix) $A \leqslant B$ *implies* $(0_M : B) \leqslant (0_M : A)$

(x) $0_M : [0_M : (0_M : A)] = 0_M : A$

(xi) $0_M : xI_M = 0_M : x^n I_M$ *for some positive integer n.*

We define, $0_{FM} = \vee\{X \in M_* \mid sX = 0_M$ for some $s \in F\}$, where M_* is the set of compact elements of M.

The proofs of the following theorems are simple

Theorem 2. *Let* $F \subseteq L$ *be a filter of* $F(L*)$ *and let* X *be a compact element of* M. *Then* $X \leqslant 0_{FM}$ *if and only if* $sX = 0_M$ *for some* $s \in F$.

Theorem 3. *For* $F \in F(L_*)$, $0_{FM} = \vee\{(0_M : x) \mid x \in F\}$.

Theorem 4. *For* $F_1, F_2 \in F(L_*)$

(i) $F_1 \subseteq F_2$ *implies* $0_{F_1M} \leqslant 0_{F_2M}$.

(ii) $0_{F_1M} \wedge 0_{F_2M} = 0_{(F_1 \bigcap F_2)M}$

3. Baer Elements

A study of Baer elements, $*$-elements and closed elements carried out by D D Anderson, *et al.* [1]. We generalize these concepts for lattice modules.

Definition 1. *An element* $A \in M$ *is said to be Baer element if for* $x \in L*$, $xI_M \leqslant A$ *implies* $0_M : (0_M : xI_M) \leqslant A$.

Definition 2. *An element A of M is said to be* $*$*-element if* $A = 0_{FM}$ *for some filter* $F \in F(L_*)$ *such that zero does not belong to F.*

Definition 3. *An element A of M is said to be closed element if* $A = 0_M : (0_M : A)$.

The next result establishes the relation between closed element and Baer element.

Theorem 5. *Every closed element is a Baer element.*

Proof. Let A be a closed element of M and x be a compact element of L_* such that $xI_M \leqslant A$. Then $0_M : (0_M : xI_M) \leqslant 0_M : (0_M : A) = A$ as A is a closed. This shows that A is a Baer element. □

Definition 4. *An element P of M is called a minimal prime element over* $A \in M$ *if* $A \leqslant P$ *and there is no other prime element Q of M such that* $A \leqslant Q < P$.

The following result gives the characterization of a minimal prime element over an element.

Theorem 6. *Let a be proper element of L and P be a prime element of M with* $aI_M \leqslant P$. *Then the following statements are equivalent,*

(i) P is minimal prime element over aI_M.

(ii) For each compact element x in L, $xI_M \leqslant P$, *there is compact element y in L such that* $yI_M \notin P$ *and* $x^n yI_M \leqslant aI_M = A$ *for some positive integer n.*

Proof. $(i) \Rightarrow (ii)$
Let P be a minimal prime over aI_M and suppose $xI_M \leqslant P$. Let

$$S = \{x^n y \mid y \nleqslant (P : I_M) \text{ and } n \text{ is a positive integer } \}.$$

It is clear that, S is a multiplicatively closed set. Suppose $x^n y \nleqslant aI_M$ for any integer n and for any $yI_M \notin P$, where y is compact in L. By the separation lemma (see [5]), there is a prime element $(Q : I_M)$ of L such that $(P : I_M) \leqslant (Q : I_M)$ and $t \nleqslant (Q : I_M)$ for all $t \in S$. Then we have $(Q : I_M) \leqslant (P : I_M)$ since otherwise $x^n(Q : I_M) \in S$ and $x^n(Q : I_M) \nleqslant (Q : I_M)$ a contradiction. Hence $(P : I_M) = (Q : I_M)$. It follows that $P = Q$ (see [3]. But then for $t \in S$, $t \leqslant x \leqslant (P : I_M) = (Q : I_M)$ a contraduction.
$(ii) \Rightarrow (i)$
Suppose for any x in L, $xI_M \leqslant P$, there is y in L such that $yI_M \nleqslant P$ and $x^n yI_M \leqslant aI_M$ for some positive integer n. Also suppose that there is a prime element Q of M with $aI_M \leqslant Q < P$. Choose, $xI_M \leqslant P$ and $xI_M \nleqslant Q$. By hyphothesis, there is a compact element y in L such that $yI_M \nleqslant P$ and integer n such that $x^n yI_M \leqslant aI_M \leqslant Q$. As $xI_M \notin Q$, $x \notin (Q : I_M)$. Since Q is a prime element of M, $(Q : I_M)$ is also prime element of L (see [4]). Hence $x^n \notin (Q : I_M)$. Thus, $x^n \nleqslant (Q : I_M)$ and $y \nleqslant (Q : I_M)$ where $(Q : I_M)$ is a prime element of L, which is a contradiction. □

In the next result, we prove the important property of a minimal prime element.

Theorem 7. *Let M be an lattice module. Every minimal prime element of M is a* $*$*-element where* 0_{FM} *is prime element.*

Proof. Let p be a minimal prime element of M. Define the set $F = \{x \in L_* \mid xI_M \notin P\}$. We first show that F is a filter of $F(L_*)$. Let x and y be compact element of L such that $x, y \in F$. So $xI_M \notin P$ and $yI_M \notin P$. As P is prime, $xyI_M \notin P$. This shows that $xy \in F$. Now let $x \in F$ and $x \leqslant y$. Hence $xI_M \notin P$ implies $yI_M \notin P$ and $y \in F$. If $0 \in F$ then we have $0I_M \notin P$ that is $0_M \notin P$ a contradction. Thus $F \in F(L_*)$ and $0 \notin F$. Now we show that $P = 0_{FM}$. Let x be a compact element of L such that $xI_M \leqslant P$. By Theorem 6 it follows that there exist a compact element $y \in L$ such that $yI_M \notin P$ and $x^n yI_M = 0_M$ for some positive integer n. We have $y \in F$ and $x^n I_M \leqslant 0_{FM}$. As 0_{FM} is prime element, so $xI_M \leqslant 0_{FM}$ implies $P \leqslant 0_{FM}$. Now let x be a compact element of L such that $xI_M \leqslant 0_{FM}$. Then by Theorem 2, $rxI_M = 0_M$ for some $r \in F$. So we have $rxI_M \leqslant P$ and $rI_M \nleq P$. As P is prime, $xI_M \leqslant P$ and $0_{FM} \leqslant P$ which shows that $P = 0_{FM}$. Thus every minimal prime element of M is $*$-element. □

The relation between $*$-element and Baer element is proved in the next result.

Theorem 8. *Each $*$-element of M is a Baer element.*

Proof. Suppose an element A of M is $*$-element. Hence $A = 0_{FM}$ for some filter $F \in F(L_*)$ such that $0 \notin F$. Let $x \in L_*$ such that $xI_M \leqslant A$. Then we have $rxI_M = 0_M$ that is $xI_M \leqslant (0_M : r)$ for some $r \in F$ by Theorem 2. Therefore by *(i)* and *(iii)* of Theorem 1 we get

$$0_M : (0_M : xI_M) \leqslant 0_M : [0_M : (0_M : r)] = (0_M : r).$$

Hence by Theorem 3, $0_M : (0_M : xI_M) \leqslant \bigvee_{s \in F} (0_M : s) = 0_{FM} = A$. This shows that A is a Baer element. □

The next result we prove the existence of closed and Baer elements.

Theorem 9. *Let M be multiplication lattice module. For any $x \in L, (0_M : x)$ is both Baer and closed element.*

Proof. For an element $x \in L_*$, let $xI_M \leqslant (0_M : x)$, then

$$0_M : (0_M : xI_M) \leqslant 0_M : [0_M : (0_M : x)] = (0_M : x)$$

by *(i)* and *(iii)* of Theorem 1. Thus $(0_M : x)$ is a Baer element. Again from *(iii)* of Theorem 1, $(0_M : x) = 0_M : (0_M : (0_M : x))$. This shows that $(0_M : x)$ is a closed element. □

In the following theorem we prove the characterization of closed element in terms of Baer element.

Theorem 10. *For $a \in L_*, aI_M$ is closed if and only if aI_M is a Baer element.*

Proof. Let L_* be the set of all compact element of L and aI_M be a Baer element of M. We show that $aI_M = 0_M : (0_M : aI_M)$. As $aI_M \leqslant aI_M$, we have $[0_M : (0_M : aI_M)] \leqslant aI_M$. But $aI_M(0_M : aI_M) \leqslant 0_M$ implies $aI_M \leqslant 0_M : (0_M : aI_M)$. Therefore $0_M : (0_M : aI_M) = aI_M$. Thus aI_M is closed. The converse is proved in Theorem 5. □

Theorem 11. *For a nonzero compact element a in L, $0_M : a = 0_{[a)}$.*

Proof. We note that $F = [a) = \{z \in L_* \mid z \geq a^n$ for some $n \in Z_+\} \in F(L_*)$ and $0_{FM} = \vee\{X \in M_* \mid sX = 0_M$ for some $s \in F\}$. Now let z be compact element of L such that $z \in F \cap \{0\}$. Then $z \in F$ and $z = 0$. As $z \in F, z \geq a^n$ for some $n \in Z_+$. Hence $a \leqslant \sqrt{z} = 0$ which shows that $a = 0$. This contradiction implies that $0 \notin F$. Now we show that $0_M : a = 0_{FM}$. As a is a compact element in L, $a \in F$. So we have $0_M : a \leqslant 0_{FM} = \vee\{(0_M : x) \mid x \in F\}$. Let Z be a compact element in M and $Z \leqslant 0_{FM}$. Then by Theorem 2 $sZ = 0_M$ for some $s \in F$. So $s \geq a^n$ for some $n \in Z_+$. We note that $0_M : a^n = 0_M : a$. Consequently, we have $a^n Z \leqslant sZ = 0_M$. This implies that $Z \leqslant (0_M : a^n) = (0_M : a)$. Consequently, $0_F \leqslant (0_M : a)$ and $(0_M : a) = 0_F$. □

The following theorem establishes the property of Baer, closed and $*$-element.

Theorem 12. *Suppose L has no divisors of zero then the element 0_M is always a Baer, closed and $*$-element whereas 1_M is Baer and closed.*

Proof. Let x be a nonzero element of L. From Theorem 9,for any $x \in L, 0_M : x$ is both Baer and closed and by Theorem 11 for a nonzero compact element x of L, $0_M : x = 0_{[x)}$. To show that 0_M a is Baer element,take $x \in L_*$ such that $xI_M \leqslant 0_M$. We have

$$0_M : (0_M : xI_M) \leqslant O_M : (0_M : 0_M) = 0_M.$$

Hence 0_M is a Baer element. As $0_M = 0_M : (0_M : 0_M)$, 0_M is closed. Every Baer element is a $*$-element. To show that 1_M is a Baer element. Take any $x \in L_*$ such that $xI_M \leqslant 1_M$. We have $0_M : (0_M : xI_M) = 0_M : [\vee\{a \in L \mid axI_M = 0_M\}] = 0_M : 0 = 1_M$. So 1_M is a Baer element. Now $0_M : (0_M : 1_M) = 0_M : [\vee\{a \in L \mid aI_M = 0_M\}] = 1_M$ and 1_M is closed. □

Remark 1. *For defining the $*$-element, the condition $0 \notin F$ is necessary.*

Suppose if possible X is a $*$-element. Hence $X = 0_{FM}$, for some filter F such that $0 \notin F$. Then we have $X = \vee\{(0_M : r) \mid r \in F\}$. Now $0_M : 0 = \vee\{A \in M \mid 0A = 0_M\} = 1_M$. Thus only 1_M will be a $*$-element. Hence, for defining a $*$-element we take F such that $0 \notin F$.

Theorem 13. *If $\{A_{\alpha\}_\alpha}$ is a family of Baer elements then $\bigwedge\limits_{\alpha} A_\alpha$ is a Baer element.*

Proof. Let $x \in L_*$ such that $xI_M \leqslant \bigwedge\limits_{\alpha} A_\alpha$. Then for each $\alpha, xI_M \leqslant A_\alpha$. As each A_α is a Baer element, $0_M : (0_M : xI_M) \leqslant A_\alpha$. Hence $0_M : (0_M : xI_M) \leqslant \bigwedge\limits_{\alpha} A_\alpha$. Thus $\bigwedge\limits_{\alpha} A_\alpha$ is a Baer element. □

The next result we prove the relation between minimal prime element and Baer element.

Theorem 14. *If A is a meet of minimal prime elements then A is a Baer element.*

Proof. From Theorem 7, every minimal prime element of M is a $*$-element and by Theorem 8, each $*$-element of M is a Baer element. From these two results,every minimal prime element is a Baer element. So meet of all minimal prime elements is a Baer element, by Theorem 13. □

Theorem 15. *If $\{A_\alpha\}_\alpha$ is a family of closed elements then $\bigwedge\limits_{\alpha} A_\alpha$ is a closed element.*

Proof. We have $\bigwedge\limits_{\alpha} A_\alpha \leqslant A_\alpha$ for each α. As each A_α is a closed element we have $0_M : [0_M : (\wedge A_\alpha)] \leqslant 0_M : (0_M : A_\alpha) = A_\alpha$. This gives $0_M : [0_M : (\bigwedge\limits_{\alpha} A_\alpha)] \leqslant \bigwedge\limits_{\alpha} A_\alpha$. Now let Z be an element of M such that $Z \leqslant \bigwedge\limits_{\alpha} A_\alpha$. Then we have $Z \leqslant 0_M : (0_M : Z) \leqslant 0_M : (0_M : \bigwedge\limits_{\alpha} A_\alpha)$, by *(ix)* of Theorem 1. This gives $\bigwedge\limits_{\alpha} A_\alpha \leqslant 0_M : [0_M : (\bigwedge\limits_{\alpha} A_\alpha)]$. Thus we get $0_M : [0_M : (\bigwedge\limits_{\alpha} A_\alpha)] = \bigwedge\limits_{\alpha} A_\alpha$. □

Here is an important property of largest element of M which is compact.

Theorem 16. *1_M is never a $*$-element where 1_M is compact and M is torsion free L-module.*

Proof. Suppose that 1_M is a $*$-element. Then there exist some filter $F \in F(L_*)$ such that $1_M = 0_{FM}$, where $0 \notin F$. Then as 1_M is compact and $1_M = 0_{FM} = \vee\{(0_M : x) \mid x \in F\}$, $1_M = (0_M : x_1) \vee (0_M : x_2) \vee \ldots \vee (0_M : x_n)$ for some $x_1, x_2, \ldots, x_n \in F$. Consequently, as 1_M is closed,

$$\begin{aligned} 1_M =& 0_M : (0_M : 1_M) = 0_M : [0_M : ((0_M : x_1) \vee (0_M : x_2) \vee \ldots \vee (0_M : x_n))] \\ =& 0_M : [0_M : (0_M : x_1) \wedge 0_M : (0_M : x_2) \wedge \ldots \wedge 0_M : (0_M : x_n)]. \end{aligned}$$

Therefore $1_M = 0_M : [0_M : (0_M : (x_1 x_2 \ldots x_n)] = 0_M : (x_1 x_2 \ldots x_n)$, by *(iii)* and *(v)* of Theorem 1. This implies that $x_1 x_2 \ldots x_n = 0$. Since $x_1, x_2, \ldots, x_n$ are in F. We have $0 = x_1 x_2 \ldots x_n \in F$. Which is a contradiction as $0 \notin F$. □

The next result we prove the characterization of a Baer element.

Theorem 17. *The following statements are equivalent,*

(i) An element $A \in M$ is a Baer element.

(ii) For any element $x, y \in L$ such that x is compact $0_M : xI_M = 0_M : yI_M$ and $xI_M \leqslant A$ implies $yI_M \leqslant A$.

(iii) For any element $x, y \in L_*, 0_M : x = 0_M : y$ and $xI_M \leqslant A$ implies $yI_M \leqslant A$.

Proof. $(i) \Rightarrow (ii)$
Assume that A is a Baer element of M. Let $x, y \in L$ be such that x is compact, $xI_M \leqslant A$, and $0_M : xI_M = 0_M : yI_M$. Then by Theorem 1, $yI_M \leqslant 0_M : (0_M : yI_M) = 0_M : (0_M : xI_M) \leqslant A$, since A is a Baer element.
$(ii) \Rightarrow (iii)$
Obvious.
$(iii) \Rightarrow (i)$
Assume that for any element $x, y \in L_*, 0_M : xI_M = 0_M : yI_M$ and $xI_M \leqslant A$ implies $yI_M \leqslant A$. We show that $A \in M$ is a Baer element. Let $x \in L_*$ be such that $xI_M \leqslant A$. We have $0_M : xI_M = 0_M : [0_M : (0_M : xI_M)]$. Hence by *(iii)*, we have $0_M : (0_M : xI_M) \leqslant A$. Hence, A is a Baer element. □

In the following theorem we prove the relation between Baer element of a lattice module and radical element of a multiplicative lattice.

Theorem 18. *If A is Baer element of M then $A : I_M$ is a radical element.*

Proof. Let A be Baer element of a lattice module M. We show that $(A : I_M) = \sqrt{(A : I_M)}$. Assume that x is compact element such that $x^n I_M \leqslant A$ for some positive integer n. We have $0_M : xI_M = 0_M : x^n I_M$, by *(xii)* of Theorem 1 and hence by above theorem $xI_M \leqslant A$ that is $x \leqslant (A : I_M)$. Hence $\sqrt{(A : I_M)} \leqslant (A : I_M)$ and we have $\sqrt{(A : I_M)} = (A : I_M)$ i.e.$(A : I_M)$ is a radical element. □

Theorem 19. *If A is a Baer element then every minimal prime element over A is a Baer element.*

Proof. Let A be a Baer element and P be a minimal prime in M over A. Assume that $0_M : x = 0_M : z$ for some $x, z \in L$ such that x is compact and $xI_M \leqslant P$. There exists a compact element $y \in L$ such that $yI_M \notin P$ and $x^n yI_M \leqslant A \leqslant P$ for some positive integer n, by Theorem 14. Note that $0_M : yx = (0_M : x) : y = (0_M : x^n) : y = 0_M : x^n y = 0_M : yx^n = 0_M : yz$. As A is a Baer element. By Theorem 17, $xyI_M \leqslant A$ implies $yzI_M \leqslant A \leqslant P$. Hence $zI_M \leqslant P$ as P is prime. So again by Theorem 17, P is a Baer element. □

The characterization of minimal prime element of M is proved in the next theorem.

Theorem 20. *Let L be a lattice module and P be a prime element of M. Then P is a minimal prime element if and only if for $x \in L_*$, P contains precisely one of xI_M and $0_M : x$.*

Proof. If part:
Assume that for $x \in L_*$,P contains precisely one of xI_M and $0_M : x$. First assume that P contains xI_M. But $0_M : x \notin P$. Therefore there exists a compact element y in L such that $yI_M \leqslant 0_M : x$ but $yI_M \notin P$. Thus $xyI_M \leqslant 0_M$. This shows that for each compact element x in L,$xI_M \leqslant P$, there exist a compact element y in L such that $yI_M \notin P$ and $xyI_M \leqslant 0_M$. By Theorem 6, it follows that P is a minimal prime element of M. Next assume that $0_M : x \leqslant P$ but $xI_M \notin P$. Let z be a compact element of L such that $zI_M \leqslant (0_M : x) \leqslant P$. But $xI_M \notin P$ and $xzI_M \leqslant 0_M$. Consequently, by Theorem 6 P is a minimal prime element. Thus the condition is sufficient.
Only if part:
Assume that P is a minimal prime element of M. Let x be a compact element of L. Suppose if possible $xI_M \leqslant P$. Then by Theorem 6, there exist a compact element y in L such that $yI_M \notin P$ and $x^n yI_M = 0_M$ for some positive integer n. Consequently, $yI_M \leqslant 0_M : x^n = 0_M : x$. This implies that $0_M : x \notin P$. Now suppose if possible $xI_M \notin P$ and $0_M : x \notin P$. Then there exist a compact element y in L such that $yI_M \leqslant 0_M : x$ but $yI_M \notin P$. Hence we have $xyI_M \leqslant 0_M$ and so $xyI_M \leqslant P$. But $xI_M \notin P$ and $yI_M \notin P$ which contradicts the fact that P is prime element of M. This shows that P contains precisely one of xI_M and $(0_M : x)$. □

The relation between $*$-element of M and a minimal prime element over it is established in the next theorem.

Theorem 21. *If A is a $*$-element of M then every minimal prime over A is a minimal prime.*

Proof. Let P be a minimal prime element of M over A. We know by Theorem 8 and Theorem 18, a $*$-element A is a Baer element and $(A : I_M)$ is a radical element. Let $x \in L_*$ be such that $xI_M \leqslant P$. But P is a minimal prime over A. Then by Theorem 2 there exists $y \in L_*$ such that $yI_M \notin P$ and $x^n yI_M \leqslant A$ i.e. $x^n y \leqslant A : I_M$. So $x^n y^n \leqslant A : I_M$ i.e. $xy \leqslant \sqrt{(A : I_M)} = (A : I_M)$. By hyphothesis, xy is compact and $xyI_M \leqslant A = 0_{FM}$, for some filter F of L_* such that $0 \notin F$. Hence $xyI_M d = 0_M$ for some $d \in F$. We show that there is no compact element x in F such that $xI_M \leqslant P$. Suppose there is compact element z in L such that $zI_M \leqslant P$ and $z \in F$. Then by Theorem 3, $0_M : z \leqslant 0_F = A \leqslant P$. This contradict the fact that P contains precisely one of zI_M and $0_M : z$ where $z \in L_*$. Hence there is no compact element x in F such that $xI_M \leqslant P$. This implies that $dI_M \nleqslant P$. As P is prime, $dI_M \nleqslant P$ and $yI_M \nleqslant P$ implies $ydI_M \nleqslant P$. Thus $xydI_M = 0_M \leqslant P$ and $ydI_M \nleqslant P$. Therefore by Theorem 6, P is minimal prime. □

Remark 2. *By Theorem 7, we infer that every minimal prime element is a $*$-element and it is a Baer element. Therefore by Theorem 21, if A is the meet of all minimal prime elements containing it, A is a Baer element.*

Notation: For a family $\{A_\alpha\}$ of Baer elements of L we define,

$$\underline{\vee} A_\alpha = \vee\{xI_M, x \in L_* \mid 0_M : (x_1 \vee x_2 \ldots \vee x_n)I_M \leqslant 0_M : xI_M,$$

for some compact elements $x_j I_M \leqslant A_{\alpha j}$ and some $j = 1, 2, \ldots, n\}$.

The important property of a family of Baer elements is established in the next theorem.

Theorem 22. *If $\{A_\alpha\}$ is a family of Baer elements of L, $\underline{\vee} A_\alpha$ is the smallest Baer element greater than each A_α.*

Proof. We first show that $\underline{\vee} A_\alpha$ is a Baer element greater than each A_α. Let x be a compact element of L such that $xI_M \leqslant \underline{\vee} A_\alpha$. Then there exist compact elements $x_1, x_2, \ldots, x_n$ such that $0_M : (x_1 \vee x_2 \vee \ldots \vee x_n)I_M \leqslant 0_M : xI_M$ and $x_j I_M \leqslant A_{\alpha j}$ $j = 1, 2, \ldots, n$. Next we show that $0_M : (0_M : xI_M) \leqslant \underline{\vee} A_\alpha$. Let z be compact element in L such that $zI_M \leqslant 0_M : (0_M : xI_M)$. Then $0_M : zI_M \geq 0_M : [0_M : (0_M : xI_M)]$. That is $0_M : xI_M \leqslant 0_M : zI_M$ (by Theorem 1, *(x)* and *(xi)*). Therefore $0_M : (x_1 \vee x_2 \vee \ldots \vee x_n)I_M \leqslant 0_M : zI_M$. This implies that $zI_M \leqslant \underline{\vee} A_\alpha$. Thus $0_M : (0_M : xI_M) \leqslant \underline{\vee} A_\alpha$. This shows that $\underline{\vee} A_\alpha$ is a Baer element. Let z be a compact element in L such that $zI_M \leqslant A_\alpha$ for some α. But $0_M : zI_M \leqslant 0_M : zI_M$. Thus $zI_M \leqslant \underline{\vee} A_\alpha$. Hence each $A_\alpha \leqslant \underline{\vee} A_\alpha$. Let B be a Baer element such that $A_\alpha \leqslant B$ for each α and let x be a compact element in L such that $0_M : (x_1 \vee x_2 \vee \ldots \vee x_n)I_M \leqslant 0_M : xI_M$ for some compact elements $x_j I_M \leqslant A_{\alpha j}$, $j = 1, 2, \ldots, n$ so that $xI_M \leqslant \underline{\vee} A_\alpha$. Note that B is a Baer element and the compact element $(x_1 \vee x_2 \vee \ldots \vee x_n)I_M \leqslant B$. Hence $0_M : [0_M : (x_1 \vee x_2 \vee \ldots \vee x_n)I_M] \leqslant B$. Again note that $0_M : (0_M : xI_M) \leqslant 0_M : [0_M : (x_1 \vee x_2 \vee \ldots \vee x_n)I_M]$ and $xI_M \leqslant 0_M : (0_M : xI_M)$. Therefore $xI_M \leqslant B$ and hence $\underline{\vee} A_\alpha \leqslant B$. Consequently $\underline{\vee} A_\alpha$ is the smallest Baer element greater than each A_α. □

Theorem 23. *For any proper element $A \in M$, $\underline{\vee}\{0_M : (0_M : xI_M) \mid x \in L_*$ and $xI_M \leqslant A\}$ is the smallest Baer element greater than A.*

Proof. First we show that $0_M : (0_M : xI_M)$ is a Baer element i.e. we show that for any $x \in L_*$, $xI_M \leqslant 0_M : (0_M : xI_M)$ implies $0_M : (0_M : xI_M) \leqslant 0_M : (0_M : xI_M)$ which holds obviously. Hence by Theorem 22, $B = \underline{\vee}\{0_M : (0_M : xI_M) \mid x \in L_*$ and $xI_M \leqslant A\}$ is the smallest Baer element containing each $0_M : (0_M : xI_M)$ for $xI_M \leqslant A$. Let a compact element x in L be such that $xI_M \leqslant A$. Then we have $xI_M \leqslant 0_M : (0_M : xI_M) \leqslant B$. Thus $A \leqslant B$. Let zI_M be a Baer element in M such that $A \leqslant zI_M$ and let y be compact element in L such that $yI_M \leqslant B$. Then $0_M : (z_1 \vee z_2 \vee \ldots \vee z_n)I_M \leqslant 0_M : yI_M$, for some compact elements $z_i I_M \leqslant 0_M : (0_M : x_i I_M)$, where $i = 1, 2, \ldots, n$. Thus $0_M : x_i I_M \leqslant 0_M : z_i I_M$ for each i. This gives

$$\begin{aligned}0_M :&(x_1 \vee x_2 \vee \ldots \vee x_n)I_M = 0_M : x_1 I_M \wedge 0_M : x_2 I_M \wedge \ldots 0_M : x_n I_M \\ &\leqslant 0_M : z_1 I_M \wedge 0_M : z_2 I_M \wedge \ldots \wedge 0_M : z_n I_M \\ &= 0_M : (z_1 \vee z_2 \vee \ldots \vee z_n)I_M \leqslant 0_M : yI_M.\end{aligned}$$

Thus if $x = x_1 \vee x_2 \vee \ldots \vee x_n$ is compact element such that $xI_M = (x_1 \vee x_2 \vee \ldots \vee x_n)I_M \leqslant A \leqslant zI_M$, we get $0_M : xI_M \leqslant 0_M : yI_M$. As zI_M is a Baer element we have

$$yI_M \leqslant 0_M : (0_M : yI_M) \leqslant 0_M : (0_M : xI_M) \leqslant zI_M.$$

Therefore $B \leqslant zI_M$. This shows that $\underline{\vee}\{0_M : (0_M : xI_M) \mid x \in L_* \text{ and } xI_M \leqslant A\}$ is the smallest Baer element greater than A. □

Notation : For a family $\{A_\alpha\}$ of closed elements of M we define,

$$A \bigtriangledown B = \vee\{zI_M, z \in L_* \mid 0_M : (x \vee y)I_M \leqslant 0_M : zI_M$$

for some $xI_M \leqslant A$ and $yI_M \leqslant B\}$. Then we have the following important result.

The property of closed elements is proved in the next theorem.

Theorem 24. *If A and B are closed elements of M $A \bigtriangledown B$ is the smallest closed element greater than A as well as B.*

Proof. We show that $A \bigtriangledown B$ is closed greater than A as well as B. Let $C = A \bigtriangledown B$. We always have $C \leqslant 0_M : (0_M : C)$ where $C \in M$. Let x be compact element in L such that $xI_M \leqslant 0_M : (0_M : C)$. Then $0_M : C \leqslant 0_M : xI_M$. This implies that

$$0_M : (y \vee z)I_M \leqslant 0_M : C \leqslant 0_M : xI_M$$

where $y, z \in L_*$, $yI_M \leqslant A$ and $zI_M \leqslant B$. But $yI_M \leqslant A \bigtriangledown B$, $zI_M \leqslant A \bigtriangledown B$. Hence $0_M : (r \vee s)I_M \leqslant 0_M : yI_M$ and $0_M : (u \vee v)I_M \leqslant 0_M : zI_M$ where $rI_M, uI_M \leqslant A$ and $sI_M, vI_M \leqslant B$. Therefore $0_M : (r \vee s)I_M \wedge 0_M : (u \vee v)I_M \leqslant 0_M : yI_M \wedge 0_M : zI_M$. Consequently

$$0_M : (r \vee s \vee u \vee v)I_M \leqslant 0_M : (y \vee z)I_M \leqslant 0_M : xI_M,$$

where $(r \vee u)I_M \leqslant A$ and $(s \vee v)I_M \leqslant B$. This implies that $xI_M \leqslant C$. Hence $0_M : (0_M : C) \leqslant C$. This gives $0_M : (0_M : C) = C$ and C is closed. As $0_M ; sI_M \leqslant 0_M : sI_M$ for any element s in L, it follows that $A, B \leqslant A \bigtriangledown B$. Suppose that W is closed element such that $A, B \leqslant W$ and let $x \in L_*$ be such that $0_M : (u \vee v)I_M \leqslant 0_M : xI_M$ for some $uI_M \leqslant A$ and $vI_M \leqslant B$. Note that W is a closed element and $(u \vee v)I_M \leqslant W$. Hence we have $0_M : [0_M : (u \vee v)I_M] \leqslant 0_M : (0_M : W) = W$. Again note that $0_M : (0_M : xI_M) \leqslant 0_M : [0_M : (u \vee v)I_M] \leqslant W$ and $xI_M \leqslant 0_M : (0_M : xI_M)$. Therefore $xI_M \leqslant W$ and hence $A \bigtriangledown B \leqslant W$. Consequently, it proves that $A \bigtriangledown B$ is the smallest closed element greater than A as well as B. □

Theorem 25. *If A and B are closed elements of M then $A \bigtriangledown B = 0_M : [0_M : (A \vee B)]$.*

Proof. By Theorem 24, we have $A \vee B \leqslant A \bigtriangledown B$. Hence $0_M : [0_M : (A \vee B)] \leqslant A \bigtriangledown B$ as $A \bigtriangledown B$ is a closed element. Let $xI_M \leqslant A \bigtriangledown B, x \in L_*$. Then $0_M : (u \vee v)I_M \leqslant 0_M : xI_M$, for some $uI_M \leqslant A$ and $vI_M \leqslant B$. Consequently, we have

$$xI_M \leqslant 0_M : (0_M : xI_M) \leqslant 0_M : [0_M : (u \vee v)I_M] \leqslant 0_M : [0_M : (A \vee B)].$$

Hence $A \bigtriangledown B \leqslant (0_M : 0_M : (A \vee B))$. Thus $A \bigtriangledown B = 0_M : [0_M : (A \vee B)]$. □

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