EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 8, No. 3, 2015, 343-356
ISSN 1307-5543 – www.ejpam.com

An Algorithm for Explicit Form of Fundamental Units of Certain Real Quadratic Fields and Period Eight

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Abstract. In this paper, we have given an explicit formulation to determine the form of the fundamental units of certain real quadratic number fields. This new algorithm for such quadratic fields is first in the literature and it gives us a more practical way to calculate the fundamental unit. Where, the period in the continued fraction expansion of the quadratic irrational number of the certain real quadratic fields is equal to 8.

2010 Mathematics Subject Classifications: 11A55, 11R27
Key Words and Phrases: real quadratic field, continued fraction, fundamental unit

1. Introduction and Notation

Determination of the fundamental units of quadratic fields has a great importance at many branches in number theory. Although the fundamental units of real quadratic fields of Richaut-Degert type are well-known, explicit form of the fundamental units are not known very well and these determinations were very limited except for these type an K. therefore, Tomita has described explicitly the form of the fundamental units of the real quadratic fields $Q(\sqrt{d})$ such that d is a square-free positive integer congruent to 1 modulo 4 and the period k_d in the continued fraction expansion of the quadratic irrational number $\omega_d = (\frac{1+\sqrt{d}}{2})$ in $Q(\sqrt{d})$ is equal to 3 and 4, 5 respectively in [4] and [5]. Later, explicit form of the fundamental units for all real quadratic fields $Q(\sqrt{d})$ such that the period k_d in the continued fraction expansion of the quadratic irrational number ω_d is equal to 6, has been described in [3].

In this paper, we will deal with all real quadratic fields $Q(\sqrt{d})$ such that d is a square free positive integer congruent to 1 modulo 4 and the period k_d in the continued fraction expansion of the quadratic irrational number $\omega_d = (\frac{1+\sqrt{d}}{2})$ in $Q(\sqrt{d})$ is equal to 8 and describe explicitly T_d, U_d in the fundamental unit $\varepsilon_d = (\frac{T_d+U_d\sqrt{d}}{2}) > 1$ of $Q(\sqrt{d})$ and d itself by using at most five parameters appearing in the continued fraction expansion of ω_d.

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Let $I(d)$ be the set of all quadratic irrational numbers in $Q(\sqrt{d})$. For an element ξ of $I(d)$ if $\xi > 1$, $-1 < \xi^{'} < 0$ then ξ is called reduced, where $\xi^{'}$ is the conjugate of ξ with respect to Q. More information on reduced irrational numbers may be found in [2]. We denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$. It is well known that if an element ξ of $I(d)$ is in $R(d)$ then the continued fractional expansion of ξ is purely periodic. Moreover, the denominator of its modular automorphism is equal to fundamental unit ε_d of $Q(\sqrt{d})$ and the norm of ε_d is $(-1)^{k_d}$ [3]. In this paper $[x]$ means the greatest integer less than or equal to x and continued fraction with period k is generally denoted by $[a_0, \overline{a_1, a_2, \ldots, a_k}]$.

2. Preliminaries and Lemmas

In this section some of the important required preliminaries and lemmas are given.

Now, for any square-free positive integer d, we can put $d = a^2+b$ with $a, b \in Z$, $0 < b \leq 2a$. Here, since $\sqrt{d} - 1 < a < \sqrt{d}$ the integers a and b are uniquely determined by d.

Let d be a square-free positive integer congruent to 1 modulo 4, then we consider the following two cases:

Case 1. If a is even, then $b = 4\ell + 1$ with $l \in Z$, $\ell \geq 0$.

Case 2. If a is odd, then $b = 4\ell$ with $l \in Z$, $\ell \geq 1$.

Let denote by D the set of all positive square-free integers and by ${D_t}^k$ the set of all positive square-free integer d such that $d \equiv k(8)$ and $b \equiv t(8)$. Hence, we have
${D_t}^k = \{d \in Z \mid d \equiv k(8), b \equiv t(8)\}$. Then, we get some remarks as follows:

Remark 1. *d can be congruent to 1 or 5 modulo 8 since d is congruent to 1 modulo 4.*

In the case of $d \equiv 1(8)$, b can be congruent to 0, 1 or 5 modulo 8. Therefore, the set of all positive square-free integers congruent to 1 modulo 8 is equal ${D_0}^1 \cup {D_1}^1 \cup {D_5}^1$. Thus the set of all positive square free integers congruent to1 modulo 8 is the union of ${D_0}^1, {D_1}^1, {D_5}^1$.

In the case of $d \equiv 5(8)$, b can be congruent to 1, 4 or 5 modulo 8. So the set of all positive square-free integers congruent to 5 modulo 8 is equal to ${D_1}^5 \cup {D_4}^5 \cup {D_5}^5$.

Remark 2. *Let d be a square-free positive integer congruent to 1 modulo 4, then:*

- *If a is even; b can only be congruent to 1 or 5 modulo 8 since $b \equiv 1(mod4)$ when a is even. Then, d belongs to ${D_1}^5 \cup {D_5}^5 \cup {D_5}^1 \cup {D_1}^1$ in the case of a is even.*
- *If a is odd; b can be only be congruent to 0 or 4 modulo 8 since $b \equiv 0(mod4)$ when a is odd. Then, d belongs to ${D_0}^1 \cup {D_4}^5$ in the case of a is odd.*

Remark 3. *The sets ${D_0}^1, {D_1}^1, {D_5}^1, {D_1}^5, {D_4}^5$ and ${D_5}^5$ are represented as follows:*

$$\begin{aligned}
{D_0}^1 =& \{d \in D \mid d = a^2 + 8m, a \equiv 1(mod2), 0 < 4m < a\} \\
{D_1}^1 =& \{d \in D \mid d = a^2 + 8m + 1, a \equiv 0(mod4), 0 \leq 4m < a\} \\
{D_5}^1 =& \{d \in D \mid d = a^2 + 8m + 5, a \equiv 2(mod4), 0 \leq 4m < a - 2\}
\end{aligned}$$

$$D_1{}^5 = \{d \in D \mid d = a^2 + 8m + 1, a \equiv 2(mod4), 0 \leq 4m < a\}$$
$$D_4{}^5 = \{d \in D \mid d = a^2 + 8m + 4, a \equiv 1(mod2), 0 \leq 4m < a-2\}$$
$$D_5{}^5 = \{d \in D \mid d = a^2 + 8m + 5, a \equiv 0(mod4), 0 \leq 4m < a-2\}$$

Now in order to prove our theorems we need the following lemmas.

Lemma 1. *For a square-free positive integer $d > 5$ congruent to 1 modulo 4, we put $\omega_d = (\frac{1+\sqrt{d}}{2})$, $q_0 = [\omega_d]$ $\omega_R = q_0 - 1 + \omega$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period k of ω_R, we get $\omega_R = [\overline{2q_0 - 1, q_1, \ldots, q_{k-1}}]$ and $\omega_d = [q_0, \overline{q_1, \ldots, q_{k-1}, 2q_0 - 1}]$. Furthermore, let $\omega_R = \frac{(P_{k-1}\omega_R + P_{k-2})}{(Q_{k-1}\omega_R + Q_{k-2})} = [2q_0 - 1, q_1, \ldots, q_{k-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ε_d of $Q(\sqrt{d})$ is given by the following formula:*

$$\varepsilon_d = (\frac{T_d + U_d\sqrt{d}}{2}) > 1,$$

where $T_d = (2q_0 - 1)Q_{k-1} + 2Q_{k-2}$, $U_d = Q_{k-1}$, and Q_i is determined by $Q_{-1} = 0$, $Q_0 = 1$, $Q_{i+1} = q_{i+1}Q_i + Q_{i-1}$, $(i \geq 0)$.

Proof. See [5, Lemma 1]. □

Lemma 2. *For a square-free positive integer d, we put $d = a^2 + b$ $(0 < b \leq 2a)$, $a, b \in Z$. Moreover let $\omega_i = \ell_i + \frac{1}{\omega_{i+1}}$ $(\ell_i = [\omega_i], i \geq 0)$ be the continued fraction expansion of $\omega = \omega_0$ in $R(d)$. Then each ω_i is expressed in the form $\omega_i = \frac{a - r_i + \sqrt{d}}{c_i}$ $(c_i, r_i \in Z)$, and ℓ_i, c_i, r_i can be obtained from the following recurrence formula:*

$$\omega_0 = \frac{a - r_0 + \sqrt{d}}{c_0},$$
$$2a - r_i = c_i\ell_i + r_{i+1},$$
$$c_{i+1} = c_{i-1} + (r_{i+1} - r_i)\ell_i \quad (i \geq 0), \text{ where } 0 \leq r_{i+1} < c_i, c_{-1} = \frac{(b + 2ar_0 - r_0{}^2)}{c_0}.$$

Moreover for the period $k \geq 1$ of ω_0, we get

$$\ell_i = \ell_{k-i} \quad (1 \leq i \leq k-1),$$
$$r_i = r_{k-i+1}, c_i = c_{k-i} \quad (1 \leq i \leq k).$$

Proof. See [1, Proposition 1]. □

Lemma 3. *For a square-free positive integer d congruent to 1 modulo 4, we put $\omega_d = (\frac{1+\sqrt{d}}{2})$, $q_0 = [\omega_d]$ and $\omega_R = q_0 - 1 + [\omega_d]$.*

If we put $\omega = \omega_R$ in Lemma 2. , then we have the following recurrence formula:

$$r_0 = r_1 = a - l_0 = a - 2q_0 + 1,$$
$$c_0 = 2, c_1 = c_{-1} = \frac{(b + 2ar_0 - r_0{}^2)}{c_0},$$
$$\ell_0 = 2q_0 - 1, \ell_i = q_i \quad (1 \leq i \leq k-1).$$

Proof. It can be easily proved by using Lemma 2. □

3. Theorems

Theorem 1. *Let $d = a^2 + b \equiv 1mod(4)$ is a square free integer for positive integers a and b satisfying $0 < b \leq 2a$. Let the period k_d of the integral basis element of $\omega_d = (\frac{1+\sqrt{d}}{2})$ in $Q(\sqrt{d})$ be 8. If a is odd then,*

$$w_d = [\frac{a+1}{2}, l_1, l_2, l_3, \frac{s_1(C + l_3A) - 2l_2B - rA}{C(r - s_1l_3) + B^2}, l_3, l_2, l_1, a],$$

where $(i = 1, 2, 3, 4)$, $l_i \geq 1$. Then the coefficients T_d and U_d of ε_d

$$(T_d, U_d) = ([(Ar + s_1l_1)(C^2l_4 + 2AC) + 2(C(Bl_4 + \ell_2) + B)], C(Cl_4 + 2A))$$

and

$$d = (Ar + s_1l_1)^2 + 4rl_2 + 4s_1$$

hold. Where A, B, C are determined by $A = l_1l_2 + 1$, $B = l_2l_3 + 1$, and $C = l_1 + Al_3$, moreover r and s are uniquely determined with the equalities $a = Ar + s_1l_1$ and

$$B(Bl_4 + 2l_2) = s_1[C(1 + l_3l_4) + Al_3] - r(A + Cl_4).$$

Proof. Let a be an odd integer then $d \in D_0^1 \cup D_4^5$. Since $q_0 = \omega_d = \frac{a+1}{2}$ then from Lemma 3 we obtain

$$\begin{aligned} r_0 =& r_1 = a - 2q_0 + 1 = 0, \\ c_0 =& 2, \\ l_0 =& a \end{aligned} \tag{1}$$

and from the Lemma 2: $l_1 = 7$, $l_2 = l_6$, $l_3 = l_5$, $w_d = [\frac{a+1}{2}, \overline{l_1, l_2, l_3, l_4, l_3, l_2, l_1, a}]$ hold for $k_d = 8$. Furthermore we have $r_1 = r_8$, $r_2 = r_7$, $r_3 = r_6$, $r_4 = r_5$, $c_1 = c_7$, $c_2 = c_6$, $c_3 = c_5$.

Let $d \in D_0^1$, where

$$\begin{aligned} D_0^1 =& \{d \in D | d \equiv 1(mod8), b \equiv 0(mod8)\} \\ =& \{d \in D | d = a^2 + 8m, a \equiv 1(mod2), 0 < 4m < a\} \end{aligned}$$

In this case we have $b = 8m \ni m > 0$. From Lemma 2, it can easily seen that $c_1 = c_{-1} = 4m$ and $2a = 4ml_1 + r_2$ (for $i = 1$).

Since $r_2 = 2a - 4ml_1$ is even then there exists a positive integer r such that $r_2 = 2r$. Therefore

$$2r = 2a - 4ml_1 \Rightarrow r = a - 2ml_1$$

and so r is an odd integer. From Lemma 2, we have $2a = c_2l_2 + r_2 + r_3$ for $i = 2$ and

$$2a = (2 + 2rl_1) \cdot l_2 + r_2 + r_3 \tag{2}$$

for $c_2 = c_0 = (r_2 - r_1)l_1$, $c_2 = 2 + 2rl_1$. If we put $4m = 2rl_2 + s$ in (1) then we obtain

$$4ml_1 = 2l_2(rl_2 + rl_1 + 1) + r_3. \tag{3}$$

Where $2l_2 + r_3 \equiv 0(modl_1)$ and so there exists positive integer s such that $2l_2 + r_3 = sl_1$ then we can obtain

$$r_3 = sl_1 - 2l_2. \tag{4}$$

Here if the value r_3 is written in (3) then it is immediately seen that $4m = 2rl_2 + s$ and s is even. If we put $4m = 2rl_2 + s$ in (1) then

$$\begin{aligned}2a =& (2rl_2 + s)l_1 + 2r \Rightarrow 2a = 2rl_1l_2 + 2r + sl_1\\ &\Rightarrow 2a = r(2l_1l_2 + 2) + sl_1\\ &\Rightarrow 2a = 2r(l_1l_2 + 1) + sl_1\end{aligned}$$

hold.

If we take $A = l_1 \cdot l_2 + 1$ then we have $a = rA + s_1l_1$ because of $s = 2s_1$ is even, $s_1 > 0$, $s_1 \in Z$. On the other hand, for $i = 2$ we have

$$\begin{aligned}c_3 =& c_1 + (r_3 - r_2)l_2 = 4m + (r_2 - r_2)l_2\\ c_4 =& c_2 + (r_4 - r_2)l_3 = (2 + 2rl_1) + (r_4 - r_3)l_3\\ c_5 =& c_3 + (r_5 - r_4)l_4 = 4m + (r_3 - r_2)l_2 + (r_5 - r_4)l_4) = c_3,\end{aligned}$$

for $(c_3 = c_1 + (r_3 - r_2)l_2 \Rightarrow c_3 = 4m + (r_3 - r_2)l_2)$.

From Lemma 2: $(i = 3)$,

$$\begin{aligned}2a =& c_3l_3 + r_3 + r_4,\\ r_3 =& 2a - c_3l_3 - r_4 \Rightarrow r_3 = 2a - (4m + (r_3 - r_2)l_2)l_3 - r_4\\ &\Rightarrow r_4 = 2a - 4ml_3 + (2r - r_3)l_2l_3 - r_3) \text{ for } i = 4\\ 2a =& c_4l_4 + r_5 + r_4,\\ r_4 =& r_5 \Rightarrow 2a = (2 + 2rl_1)l_4 + (r_4 - r_3)l_3l_4 + 2r_4\\ r_4 =& 2rA + 2s_1l_1 - 2rl_2l_3 - 2s_1l_3 + 2rl_2l_3 - (sl_1 - 2l_2)(l_2l_3 + 1)\\ &\Rightarrow r_4 = 2rA + 2s_1l_1 - 2rl_2l_3 - 2s_1l_3 + 2rl_2l_3 - sl_1l_2l_3 - sl_1 + 2l_2^2l_3 + 2l_2\\ &\Rightarrow r_4 = 2rA - 2rl_2l_3 - 2s_1l_3 + 2rl_2l_3 - 2s_1l_1l_2l_3 + 2l_2^2l_3 + 2l_2\\ r_4 =& 2rl_1l_2 + 2r - 2rl_2l_3 - 2s_1l_3 + 2rl_2l_3 + 2s_1l_1l_2l_3 + 2l_2^2l_3 + 2l_2\end{aligned}$$

hold. Therefore we obtain the value r_4, as

$$r_4 = 2[(r - s_1l_3)A + l_2B] = (r - s_1l_3)A + l_2B \text{ for } B = l_2l_3 + 1, A = l_1l_2 + 1. \tag{5}$$

Furthermore

$$c_4 = (2 + r_2l_1) + (r_4 - r_3)l_3 = (2 + 2rl_1) + [(2r - sl_3)A + 2l_2B - sl_1 + 2l_2]l_3$$

$$
\begin{aligned}
&=(2+2rl_1)+2rAl_3-sl_3{}^2A+2l_2l_3B-sl_1l_3+2l_3\\
&=2rC-sl_3C+2(1+l_2l_3)+2l_2l_3B\\
&=C(2r-sl_3)+2B+2Bl_2l_3\\
&=C(2r-sl_3)+2B(1+l_2l_3)
\end{aligned}
$$

and so we have

$$c_4 = C(2r-sl_3)+2B^2 \quad \text{(for the values } A, B \text{ and } C = l_1 + Al_3). \tag{6}$$

If the equalities $a = rA + sl_1$, $r_4 = (r - s_1 l_3)A + l_2 B$ and $c_4 = C(2r - sl_3) + 2B^2$ are written in l_4 then

$$l_4 = \frac{2a-2r_4}{c_4} = \frac{2rA+sl_1-2(2r-sl_3)A-4l_2B}{C(2r-sl_3)+2B^2}$$

holds. By taking $s = 2s_1$ we obtain

$$l_4 = \frac{s_1(C+l_3A)-2l_2B-rA}{C(r-s_1l_3)+B^2}. \tag{7}$$

From this equation

$$
\begin{aligned}
C(r-s_1l_3)l_4+B^2l_4 =&s_1(C+l_3A)-2l_2B-rA\\
&\Rightarrow B^2l_4+2l_2B = s_1C+s_1l_3A-rA-rCl_4+s_1Cl_3l_4\\
&\Rightarrow B^2l_4+2l_2B = s_1(C+l_3A+Cl_3l_4)-rA+Cl_4\\
&\Rightarrow B^2l_4+2l_2B = s_1[C(1+l_3l_4)+Al_3]-r(A+Cl_4)
\end{aligned}
$$

hold and this proves that r and s_1 are uniquely determined by $a = rA + s_1 l_1$.

Now, let's determine the coefficients T_d and U_d of the fundamental unit $\varepsilon_d = (\frac{T_d+U_d\sqrt{d}}{2}) > 1)$ for $d \equiv 1(mod 4)$ and the period $k_d = 8$. Since

$$
\begin{aligned}
Q\_1 =&0\\
Q_0 =&1\\
Q_{i+1} =&q_{i+1}Q_i + Q_{i-1}, \quad (i \geq 0)\\
Q_1 =&q_1Q_0 + Q_{-1} \Rightarrow Q_1 = l_1 1 + 0 \Rightarrow Q_1 = l_1\\
Q_2 =&q_2Q_1 + Q_0 \Rightarrow Q_2 = l_2l_1 + 1 \Rightarrow Q_2 = A\\
Q_3 =&q_3Q_2 + Q_1 \Rightarrow Q_3 = l_3A + l_1 \Rightarrow Q_3 = C\\
Q_4 =&q_4Q_3 \Rightarrow Q_4 = l_4C + A\\
Q_5 =&q_5Q_4 + Q_3 \Rightarrow Q_5 = l_3(l_4C + A) + C \Rightarrow Q_5 = l_3l_4C + C + l_3A\\
&\Rightarrow Q_5 = C(l_3l_4 + 1) + l_3A.\\
Q_6 =&\ell_2Q_5 + Q_4 \Longrightarrow Q_6 = \ell_2[C(\ell_3\ell_4 + 1) + \ell_3A] + \ell_4C + A = C\ell_4(\ell_2\ell_3 + 1) + C\ell_2 + A(1 + \ell_2\ell_3)
\end{aligned}
$$

holds, where if we take $B = (\ell_2\ell_3 + 1)$

$$Q_6 = C\ell_4B + C\ell_2 + AB = C(\ell_4B + \ell_2) + AB$$

$$\begin{aligned}Q_7 =&\ell_1 Q_6 + Q_5 \Longrightarrow Q_7 = \ell_1[C(B\ell_4+\ell_2)+AB] + C(\ell_3\ell_4+1)+A\ell_3\\ &\Longrightarrow Q_7 = CB\ell_4\ell_1 + C\ell_1\ell_2 + AB\ell_1 + C\ell_3\ell_4 + C + \ell_3 A\\ =&(C\ell_4+A)(\ell_1\ell_2\ell_3+\ell_1+\ell_3)+CA\\ =&(C\ell_4+A)(A\ell_3+\ell_1)+CA\end{aligned}$$

and so we can obtain $Q_7 = C^2\ell_4 + 2AC$ for $C = (A\ell_3+\ell_1)$. Therefore we can determine that

$$(T_d, U_d) = ([(A_r + s_1\ell_1)(C^2\ell_4+2AC)+2(C(B\ell_4+\ell_2)+AB)], C(C\ell_4+2A))$$

and $d = (A_r + s_1\ell_1)^2 + 4r\ell_2 + 4s_1$.

Now let $d \in D_4^5$ where,

$$\begin{aligned}D_4^5 =&\{d \in D | d \equiv 5(mod 8), b \equiv 4(mod 8)\}\\ =&\{d \in D \mid d = a^2 + 8m + 4, a \equiv 1(mod 2), 0 \le 4m < a-2\}\end{aligned}$$

therefore $b = 8m + 4$ and $m > 0$ hold. Besides we have the following equations from Lemma 2:

$$\begin{aligned}c_{-1} =&\frac{b}{2} = 4m+2\\ c_1 =&c_{-1} + (r_1 - r_0)\ell_0 \Longrightarrow c_1 = c_{-1} \Longrightarrow c_1 = 4m+2\end{aligned}$$

and

$$\begin{aligned}2a - r_i =&c_i\ell_i + r_{i+1} \Longrightarrow (i=1)\\ 2a - r_1 =&c_1\ell_1 + r_2 \Longrightarrow 2a = (4m+2)\ell_1 + r_2,\\ r_2 =&2a - 2(m+1)\ell_1 \Longrightarrow r_1 = r_8,\\ c_1 =&c_7, r_2 = r_7, c_2 = c_6, r_3 = r_6, c_3 = c_5, r_4 = r_5.\end{aligned}$$

Since r_2 is even number then $\exists r \ni r_2 = 2r$. And so r is defined as

$$r = \begin{cases} odd & \ell_1 \text{ even number}\\ even & \ell_1 \text{ odd number}\end{cases}$$

If we take $i = 2$, then from Lemma 2

$$\begin{aligned}2a =&C2\ell_2 + r_2 + r_3 \Longrightarrow 2a = (2+2r\ell_1)\ell_2 + r_2 + r_3\\ c_2 =&c_0 + (r_2 - r_1)\ell_1 \Longrightarrow c_2 = 2 + 2r\ell_1.\end{aligned} \tag{8}$$

By using the value $2a = (4m+2)\ell_1 + r_2$ and (8) we can write;

$$\begin{aligned}(4m+2)\ell_1 + r_2 =&(2+2r\ell_1)\ell_2 + r_2 + r_3\\ &\Longrightarrow (4m+2)\ell_1 = (2+2r\ell_1)\ell_2 + r_3\end{aligned}$$

$$\Longrightarrow (4m+2)\ell_1 = 2\ell_2 + 2r\ell_1\ell_2 + r_3$$
$$\Longrightarrow 2\ell_2 + r_3 \equiv 0(mod\ell_1)$$
$$\Longrightarrow \exists s \in Z \ni 2\ell_2 + r_3 = s\ell_1$$
$$\Longrightarrow r_3 = s\ell_1 - 2\ell_2.$$

For the value $r_3 = s\ell_1 - 2\ell_2$ we obtain $4m+2 = 2r\ell_2 + s$ where $s = 4m+2-2r\ell_2$ is even number and so there exists $s_1 \in Z^+$ such that $s = 2s_1$. If we write $4m+2 = 2r\ell_2 + s$ instead of $2a = (4m+2)\ell_1 + r_2$ then we obtain $2a = 2r(\ell_1\ell_2+1)s\ell_1$ and we can write $a = rA + s_1\ell_1$ by taking $A = \ell_1\ell_2 + 1$. In the same way, we have

$$\begin{aligned} r_4 =& 2a - (4m+2)\ell_3 + (2r - r_3)\ell_2\ell_3 - r_3 \\ &\Longrightarrow r_4 = 2rA - 2r\ell_2\ell_3 - 2s_1\ell_3 + 2r\ell_2\ell_3 - 2s_1\ell_1\ell_2\ell_3 + 2\ell_2^2\ell_3 + 2\ell_2 \\ =& 2(r - s_1\ell_3)A + \ell_2 B \end{aligned}$$

for $A = \ell_1\ell_2 + 1$ and $B = \ell_2\ell_3 + 1$. Furthermore

$$\begin{aligned} c_4 =& (2 + 2r\ell_1) + [(2r - s\ell_3)A + 2\ell_2 B - s\ell_1 + 2\ell_2]\ell_3 \\ &\Longrightarrow c_4 = C(2r - 2s_1\ell_3) + 2(1 + \ell_2\ell_3) + 2\ell_2\ell_3 B \\ =& 2C(r - s_1\ell_3) + 2B^2 \end{aligned}$$

and

$$\ell_4 = \frac{2a - 2r_4}{c_4} = \frac{s_1(C + \ell_3 A) - 2\ell_2 B - rA}{C(r - s_1\ell_3) + B^2}$$

for $C = \ell_1 + A\ell_3$ and $(1 + \ell_2\ell_3) = B$. This is completed the proof of the theorem. □

Example 1. *Let a is odd,* $d \in D_4^5$ *and* $d = 869 \equiv 5(mod 8)$*. Since* $a = 29$*,* $b = 28$*,* $b = 3 \cdot 8 + 4$*,* $m = 3$ *then we can determine that* $\ell_1 = 4$*,* $\ell_2 = 5$*,* $\ell_3 = 1$*,* $c_1 = 14$ *and*

$$r_2 = 2a - (4m+2)\ell_1 \Longrightarrow r_2 = 58 - 14 \cdot 4 = 58 - 56 = 2 \Longrightarrow r = 1,$$

$c_2 = 10$ *because of* ℓ_1 *is even.*

$$(4m+2)\ell_1 = (2 + 2r\ell_1) \cdot \ell_2 + r_3 \Longrightarrow 14 \cdot 4 = 10 \cdot 5 + r_3 \Longrightarrow r_3 = 6, s = 2s_1 = 4.$$

Therefore we obtain $A = 21$*,* $B = 6$*,* $C = 25$ *and* $r_4 = 18$*,* $c_4 = 22$*,* $\ell_4 = 1$*. If it is taken above values then the coefficients of the fundamental units of* $Q(\sqrt{869})$ *is easily determined as* $T_d = 49377$*,* $U_d = 1675$ *and so* $\varepsilon_d = \frac{49377+1675\sqrt{869}}{2} > 1$ *holds.*

Theorem 2. *Let* $d = a^2 + b \equiv 1mod(4)$ *is a square free integer for positive integers a and b satisfying* $0 < b \leq 2a$*. Let the period* k_d *of the integral basis element of* $\omega_d = (\frac{1+\sqrt{d}}{2})$ *in* $Q(\sqrt{d})$ *be 8. If a is even then,*

$$w_d = [\frac{a}{2}; \overline{\ell, \ell_2, \ell_3, \frac{BC + AD - 2\ell_3}{\ell_3^2 - CD}, \ell_3, \ell_2, \ell_1, a-1}], \quad 1 \leq \ell_i, (i = 2, 3)$$

and then the coefficients T_d and U_d of ε_d

$$T_d = [(A(r+1)+B-2)\cdot C + 2(C-\ell_3)](C\ell_4+2A)+2C\ell_2$$
$$U_d = C(C\ell_4+2A)$$

and

$$d = [A(r+1)+B-1]^2 + 2[A(r+1)+B-2s-2]-1$$

hold. Where $A=\ell_2+1$, $B=2s-r$, $C=1+A\ell_3=1+\ell_3+\ell_2\ell_3$, $D=B\ell_3-r-1$, $E=\ell_3+1$ and r and s are uniquely determined with the equations $a=A(r+1)+B-1$ and $\ell_3(\ell_3\ell_4+2)=BC+AD+2CD\ell_4$.

Proof. Let a be even and $k_d=8$. If $d\equiv 1(mod 8)$ then $b\equiv 1(mod 8)$ or $b\equiv 5(mod 8)$ hold. Furthermore in the case when a is even, d can belong to $D_1^5\cup D_5^5\cup D_5^1\cup D_1^1$. $q_0=[w_d]=\frac{a}{2}$ and from Lemma 3 we can write $r_0=r_1=a-2q_0+1 \Rightarrow r_0=r_1=1, c_0=2, \ell_0=a-1$ and because of $k_d=8$ and from Lemma 2 we have $\ell_1=\ell_7$, $\ell_2=\ell_6$, $\ell_3=\ell_5$ and $r_1=r_8$, $r_2=r_7$, $r_3=r_6$, $r_4=r_5$ and $c_1=c_7$, $c_2=c_6$, $c_3=c_5$.

We first assume that d is in $D_1^1\cup D_1^5$. Then we get $b\equiv 1(mod 8)$ and so $\exists m\in\mathbb{Z}^+ \ni b=8m+1$ holds. From Lemma ??? $c_{-1}=\frac{(8m+1+2a-1)}{2}\Rightarrow c_{-1}=4m+a$ and $c_1=c_{-1}+(r_1-r_0)\ell_0\Rightarrow c_1=4m+a$ hold.

By taking equation $2a-r_i=c_i\ell_i+r_{i+1}$ in Lemma 2 for $i=1$, we obtain

$$\begin{aligned} 2a-r_1 =& c_1\ell_1+r_2 \Rightarrow (r_1=1 \text{ and } c_1=4m+a)\\ &\Rightarrow 2a-1=(4m+a)\ell_1+r_2\\ &\Rightarrow 2a-1=4m\ell_1+a\ell_1+r_2\\ &\Rightarrow (2-\ell_1)a=4m\ell_1+r_2+1>0\\ &\Rightarrow 2-\ell_1>0\\ &\Rightarrow \ell_1<2 \text{ and } \ell_1\geq 1\\ &\Rightarrow \ell_1=1\end{aligned}$$

and so we have

$$w_d=[\frac{a}{2};\overline{1,\ell_2,\ell_3,\ell_4,\ell_3,\ell_2,1,a-1}].$$

Since $\ell_1=1$ then $a=4m+1+r_2$ and if $r_2=a-4m-1$ then a, $4m$ are even and $r_2\geq 1$ is odd and so there exists $r\geq 0$ such that $r_2=2r+1$ and we can obtain $r_2<a$.

If we use $c_{i+1}=c_{i-1}+(r_{i+1}-r_i)\ell_i$ for $i\geq 0$ then we obtain

$$c_2=c_0+(r_2-r_1)\ell_1=2+(r_2-1)1=2r+2.$$

Furthermore we have obtain the following equalities from Lemma 2 $c_2=2r+2$,

$$2a=c_2\ell_2+r_2+r_3\Rightarrow 2a=(2r+2)\ell_2+2r+1+r_3$$

and by taking $a=4m+1+r_2$ and $a=4m+2r+2$ we have

$$8m+4r+4=(2r+2)\ell_2+2r+1+r_3=(2r+2)\ell_2+r_3-2r-3.$$

Since a is even then

$$\begin{aligned}2a =&(2r+2)\ell_2+(2r+1)+r_3 \equiv 0(mod4)\\ &\Rightarrow r_3+2r+1\equiv 0(mod4) \text{ and } \ell_2\equiv 0(mod2)\\ &\Rightarrow \exists s\in\mathbb{Z}^+ \quad \ni r_3+2r+1=4s, \quad r_3\geq 0\\ &\Rightarrow 4s-2r-1\geq 0\\ &\Rightarrow 4s>2r+1\end{aligned}$$

hold, where $r_3=4s-2r-1$ is odd number because of $4s$ is even and $2r+1$ is odd.

From Lemma 2. we have

$$\begin{aligned}c_3 =&c_1+(r_3-r_2)\ell_2=4m+a+(4s-2r-1-2r-1)\ell_2\\ =&a-2r-2+a+(4s-4r-2)\ell_2\\ =&2a-2r-2+(4s-4r-2)\ell_2\end{aligned}$$

and if the value $2a$ is written instead of c_3 then c_3 is obtained as

$$\begin{aligned}c_3 =&(2r+2)\ell_2+4s-2r-2+4s\ell_2-4r\ell_2-2\ell_2\\ =&(4s-2r)(\ell_2+1)-2.\end{aligned} \tag{9}$$

By using the values c_3, $A=\ell_2+1$ and $B=2s-r$ we have

$$\begin{aligned}2a =&(2r+2)\ell_2+r_3+2r+1=(2r+2)\ell_2+4s-2r-1+2r+1\\ =&2[(r+1)\ell_2+2s]\\ a =&(r+1)\ell_2+2s\end{aligned} \tag{10}$$

and $2a=c_3\ell_3+r_3+r_4$ ise $r_4=2a-c_3\ell_3-r_3$

$$\begin{aligned}r_4 =&2[(r+1)\ell_2+2s]-[(4s-2r)(\ell_2+1)-2]\ell_3-(4s-2r-1)\\ =&2(r+1)\ell_2+4s-[(4s-2r)(\ell_2+1)-2]\ell_3-4s+2r+1\\ =&2(r+1)\ell_2+2r+1+[(4s-2r)(\ell_2+1)-2]\ell_3\end{aligned} \tag{11}$$

hold from Lemma 2. Moreover we have

$$a=(r+1)\ell_2+2s=A(r+1)+B-1 \tag{12}$$

for $A=\ell_2+1$, $B=2s-r$ and

$$\begin{aligned}r_3 =&4s-2r-1\Rightarrow r_3=2B-1\\ r_4 =&2a-c_3\ell_3-r_3\Rightarrow r_4=2a-2(BA-1)\ell_3-2B+1\\ &\Rightarrow r_4=2r\ell_2+\ell_2-2BC+2\ell_3+4s+A.\end{aligned}$$

If $C=1+A\ell_3=1+\ell_3+\ell_2\ell_3$, $B=2s-r$,

$$4s =2B+2r\Rightarrow r_4=2r\ell_2+\ell_2-2BC+2\ell_3+2B+2r+A$$

$$\Rightarrow r_4 = 2rA + 2A - 2BC + 2B + 2\ell_3 - 1$$

hold. For $\ell_1 = 1$, $c_3 = 2BA - 2 = 2(BA - 1)$. If we take $c_4 = (2r+2) + (2rA + 2A - 2BC + 2\ell_3)\ell_3$ then

$$c_4 = 2r + 2 + 2rA\ell_3 + 2A\ell_3 - 2BC\ell_3 + 2\ell_3^2 \Rightarrow c_4 = 2C[r + 1 - B\ell_3] + 2\ell_3^2$$

hold. Where if we take $r + 1 - B\ell_3 = -D$ then we obtain $c_4 = 2\ell_3^2 - 2CD = 2(\ell_3^2 - CD)$ and $r_4 = 2rA + 2A - 2BC + 2B + 2\ell_3 - 1$ for $A = \ell_2 + 1$, $B = 2s + r$, $C = A\ell_3 + 1$. Finally r_4 is determined as $r_4 = 2rA + 2A - 2B - 2AB\ell_3 + 2B + 2\ell_3 - 1 = 2\ell_3 - 2A(B\ell_3 - r - 1) - 1 = 2E - 2AD - 3$ for the values $D = B\ell_3 - r - 1$ and $E = \ell_3 + 1$. On the other hand we can write

$$\begin{aligned} a - r_4 =& (r+1)\ell_2 + 2s - 2E + 2AD + 3 \\ =& (r+1)\ell_2 + B + r - 2E + 2AD + 3 = r(\ell_2 + 1) + \ell_2 + B - 2E + 2AD + 3 \\ =& B - 2E + AB\ell_3 + A(B\ell_3 - r - 1) + 2 \end{aligned}$$

where $D = B\ell_3 - r - 1$,

$$\begin{aligned} a - r_4 =& B - 2E + AB\ell_3 + AD + 2 = B + AB\ell_3 + AD + 2 - 2\ell_3 - 2 \\ =& B + AB\ell_3 + AD - 2\ell_3 \\ =& B(1 + A\ell_3) + AD - 2\ell_3 = BC + AD - 2\ell_3 \\ & \Rightarrow a - r_4 = BC + AD - 2\ell_3 \end{aligned}$$

holds and so we have l_4 as $\ell_4 = \frac{2(a-r_4)}{c_4} = \frac{2(BC+AD-2\ell_3)}{2(\ell_3^2-CD)} = \frac{BC+AD-2\ell_3}{\ell_3^2-CD} \Rightarrow \ell_4 = \frac{BC+AD-2\ell_3}{\ell_3^2-CD}$. Besides s and r are uniquely determined by $\ell_3^2\ell_4 + 2\ell_3 = BC + AD + 2CD\ell_4$ and (9).

Now, let's determine the coefficients T_d and U_d of the fundamental unit ε_d. Since $d \equiv 1(mod4)$ then we know that $w_d = [q_0; \overline{q_1, \ldots, q_{k-1}, 2q_0 - 1}]$ and $\varepsilon_d = (\frac{T_d + U_d\sqrt{d}}{2}) > 1$. Furthermore

$$\begin{aligned} Q_{-1} =& 0 \\ Q_0 =& 1 \\ Q_{i+1} =& q_{i+1}Q_i + Q_{i-1} \quad (i \geq 0) \\ Q_1 =& q_1Q_0 + Q_{-1} \Rightarrow Q_1 = \ell_1, \quad \ell_1 = 1 \Rightarrow Q_1 = 1 \\ Q_2 =& q_2Q_1 + Q_0 \Rightarrow Q_2 = \ell_2 \cdot 1 + 1, \quad (A = \ell_2 + 1) \Rightarrow Q_2 = A \\ Q_3 =& q_3Q_2 + Q_1 \Rightarrow Q_3 = \ell_3A + 1, \quad (C = \ell_3A + 1) \Rightarrow Q_3 = C \\ Q_4 =& q_4Q_3 + Q_2 \Rightarrow Q_4 = \ell_4C + A \\ Q_5 =& q_5Q_4 + Q_3 \Rightarrow Q_5 = \ell_3(\ell_4C + A) + C \Rightarrow Q_5 = C(\ell_3\ell_4 + 1) + \ell_3A \\ Q_6 =& q_6Q_5 + Q_4 = \ell_2[C(\ell_3\ell_4 + 1) + \ell_3A] + \ell_4C + A = C[\ell_4(\ell_2\ell_3 + 1) + \ell_2] + A(1 + \ell_2\ell_3) \\ =& C[((A-1)(E-1)+1)\ell_4 + \ell_2] + A[(A-1)(E-1)+1] \text{ for } ((1+\ell_2\ell_3) = (A-1)(E-1)+1 \text{ or} \\ Q_6 =& (C - \ell_3)(C\ell_4 + A) + C\ell_2 \\ Q_7 =& \ell_1Q_6 + Q_5 = 1Q_6 + Q_5 = (C - \ell_3)(C\ell_4 + A) + C\ell_2 + (C\ell_4 + A)\ell_3 + C = C[C\ell_4 + 2A] \end{aligned}$$

and so we obtain that

$$\begin{aligned}T_d =&(A(r+1)+B-2)C(C\ell_4+2A)+2(C-\ell_3)(C\ell_4+A)+2C\ell_2,\\U_d =&C(C\ell_4+2A)\end{aligned}$$

and

$$\begin{aligned}d =&a^2+b=[A(r+1)+B-1]^2+2A(r+1)+2B-4s-5\\=&[A(r+1)+B-1]^2+2[A(r+1)+B-2s-2]-1.\end{aligned}$$

Now let $d \in D_5^1 \cup D_5^5$ then $b \equiv 5(mod8)$ and $\exists m \in \mathbb{Z}^+ \ni b = 8m+5$. From Lemma 2 we can obtain the following equalities:

$$\begin{aligned}c_{-1} =&\frac{(8m+5+2a-1)}{2} \Rightarrow c_{-1} = 4m+4+a\\c_1 =&c_{-1}+(r_1-r_0)\ell_0 \quad (r_1-r_0=0) \Rightarrow c_1 = 4m+4+a\\2a-r_1 =&c_1\ell_1+r_2 \Rightarrow (r_1=1, \quad c_1=4m+4+a)\\&\Rightarrow 2a-1=(4m+4+a)\cdot\ell_1+r_2\\&\Rightarrow (2-\ell_1)a = 4m\ell_1+4\ell_1+r_2+1>0\\&\Rightarrow 2-\ell_1>0, \quad \ell_1 \geq 1\\&\Rightarrow \ell_1=1\end{aligned}$$

and then we get $a = 4(m+1)+r_2+1$

$$r_2 = a-4(m+1)-1 \Rightarrow r_2 \text{ is an odd integer}$$

so $r_2 < a$ holds and $\exists r \geq 0$, $r \in \mathbb{Z} \ni r_2 = 2r+1$. For $i \geq 0$ we have

$$c_2 = c_0+(r_2-r_1)\ell_1 = 2+(r_2-1)1 = 2r+2$$

from relation $c_{i+1} = c_{i-1}+(r_{i+1}-r_i)\ell_i$ and $2a = c_2\ell_2+r_2+r_3$. $2a = (2r+2)\ell_2+2r+1+r_3$ and

$$\begin{aligned}a =&4(m+1)+1+r_2 \Rightarrow a = 4m+2r+6\\&\Rightarrow 2(4m+2r+6) = (2r+2)\ell_2+2r+1+r_3\\&\Rightarrow 2a = (2r+2)\ell_2+r_3+2r+1 \equiv 0(mod4)\\&\Rightarrow r_3+2r+1 \equiv 0(mod4) \text{ and } \ell_2 \equiv 0(mod2)\\&\Rightarrow \exists s \in \mathbb{Z}^+ \ni r_3+2r+1=4s, \quad r_3 \geq 0\\&\Rightarrow 4s-2r-1 \geq 0,\end{aligned}$$

therefore

$$r_3 = 4s-2r-1 \text{ is odd.} \tag{13}$$

From lemma 2 we know that $c_3 = c_1 + (r_3 - r_2)\ell_2 = 4m + 4 + a + (4s - 2r - 1 - 2r - 1)\ell_2$ then we obtain

$$\begin{aligned} c_3 =& a - 2r - 2 + a + (4s - 4r - 2)\ell_2 \Rightarrow c_3 = 2a - 2r - 2 + (4s - 4r - 2)\ell_2 \\ =&(2r+2)\ell_2 + 4s - 2r - 2 + (4s\ell_2 - 4r - 2)\ell_2 \\ =&(4s - 2r)(\ell_2 + 1) - 2 \end{aligned}$$

hold for $a = 4m + r_2 + 5 \Rightarrow 4m + 4 = a - r_2 - 1 = a - 2r - 2$.

If we take the values $A = \ell_2 + 1$ and $B = 2s - r$ then

$$2a = (2r+2)\ell_2 + r_3 + 2r + 1 = (2r+2)\ell_2 + 4s - 2r - 1 + 2r + 1 = 2[(r+1)\ell_2 + 2s]$$

and $a = (r+1)\ell_2 + 2s$. If $2a = c_3\ell_3 + r_3 + r_4$ then

$$\begin{aligned} r_4 =& 2a - c_3\ell_3 - r_3 \\ &\Rightarrow r_4 = 2[(r+1)\ell_2 + 2s] - [(4s-2r)(\ell_2+1) - 2]\ell_3 - (4s - 2r - 1) \\ r_4 =& 2(r+1)\ell_2 + 2r + 1 + [(4s-2r)(\ell_2+1) - 2]\ell_3 \end{aligned} \tag{14}$$

If $a = (r+1)\ell_2 + 2s$, $A = \ell_2 + 1$, $B = 2s - r$ then

$$a = r\ell_2 + \ell_2 + 2s = r\ell_2 + \ell_2 + 2s + r - r = r(\ell_2 + 1) + \ell_2 + (2s - r) = Ar + A - 1 + B$$

and so

$$a = A(r+1) + B - 1 \tag{15}$$

holds. Since $A = \ell_2 + 1$, and $B = 2s - r$ then

$$\begin{aligned} c_3 =& 2BA - 2 = 2(BA - 1) \\ r_3 =& 4s - 2r - 1 \Rightarrow r_3 = 2B - 1 \\ r_4 =& 2a - c_3\ell_3 - r_3 \Rightarrow r_4 = 2a - 2(BA-1)\ell_3 - 2B + 1 \\ &\Rightarrow r_4 = 2r\ell_2 + \ell_2 - 2B(1 + A\ell_3) + 2\ell_3 + 4s + A \end{aligned}$$

and if we take $C = 1 + A\ell_3 = 1 + \ell_3 + \ell_2\ell_3$, $D = B\ell_3 - r - 1$ and $E = \ell_3 + 1$ then we obtain

$$\begin{aligned} r_4 =& 2r\ell_2 + \ell_2 - 2BC + 2\ell_3 + 2B + 2r + A = 2\ell_3 - 2A(B\ell_3 - r - 1) - 1 \\ =& 2(\ell_3 + 1) - 2AD - 3 \\ &\Rightarrow r_4 = 2(\ell_3 + 1) - 2AD - 3 \\ &\Rightarrow r_4 = 2E - 2AD - 3. \end{aligned}$$

At the same way we can determine c_4 as

$$\begin{aligned} c_4 =& (2r+2) + (2rA + 2A - 2BC + 2\ell_3)\ell_3 = 2r(1 + A\ell_3) + 2(1 + A\ell_3) - 2BC\ell_3 + 2\ell_3^2 \\ =& -2C(B\ell_3 - r - 1) + 2\ell_3^2 = 2(\ell_3^2 - CD). \end{aligned}$$

We know that $\ell_4 = \frac{2(a-r_4)}{c_4}$ from Lemma ??? then we can determine the value of $a - r_4$ in the following:

$$\begin{aligned} a - r_4 =&(r+1)\ell_2 + B + r - 2E + 2AD + 3 \\ &\Rightarrow a - r_4 = B - 2E + AB\ell_3 + A(B\ell_3 - r - 1) + 2 \\ &\Rightarrow a - r_4 = B(1 + A\ell_3) + AD - 2\ell_3 \\ &\Rightarrow a - r_4 = BC + AD - 2\ell_3. \end{aligned}$$

Moreover we can easily seen that $\ell_4 = \frac{2(a-r_4)}{c_4} = \frac{2(BC+AD-2\ell_3)}{2(\ell_3^2-CD)} = \frac{BC+AD-2\ell_3}{\ell_3^2-CD} = \frac{BC+AD-2\ell_3}{\ell_3^2-CD}$ and s and r are uniquely determined from the relations $a = A(r+1) + B - 1$ and $\ell_3^2\ell_4 + 2\ell_3 = BC + AD + 2CD\ell_4$. □

Example 2. *Let a is even, $d \equiv 1(mod4)$. If we choose $D = 501 \equiv 5(mod8)$ then we can practically determine that $a = 22$, $b = 17 \equiv 1(mod8)$, $17 = 8m + 1 \Rightarrow m = 2$, $c_1 = 4m + a \Rightarrow c_1 = 8 + 22 \Rightarrow c_1 = 30$, $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 4$, $a = 4m + 1 + r_2 \Rightarrow 22 = 9 + r_2 \Rightarrow r_2 = 13$, $r_2 = 2r + 1 = 13 \Rightarrow r = 6$, $c_2 = 2r + 2 \Rightarrow c_2 = 14$, $2a = c_2\ell_2 + r_2 + r_3 \Rightarrow r_3 = 44 - 28 - 13 = 3$, $r_3 + 2r + 1 = 4s \Rightarrow 3 + 13 = 4s \Rightarrow s = 4$, $c_3 = (4s - 2r)(\ell_2 + 1) = 2 \Rightarrow c_3 = 4 \cdot 3 - 2 = 10$, $r_4 = 2a - c_3\ell_3 - r_3 \Rightarrow r_4 = 44 - 40 - 3 = 1$, $A = \ell_2 + 1 \Rightarrow A = 3$, $B = 2s - r \Rightarrow B = 2$, $C = 1 + A\ell_3 = 1 + \ell_2 + \ell_2\ell_3 \Rightarrow C = 13$, $D = -r + B\ell_3 - 1 \Rightarrow D = 1$, $E = \ell_3 + 1 \Rightarrow E = 5$, $r_4 = 1$, $a = 22$, $c_4 = 6 \Rightarrow \ell_4 = 7$. Therefore the fundamental unit of $Q(\sqrt{5}01)$ is obtained as $\varepsilon_d = \frac{28225+1261\sqrt{501}}{2}$ for $T_d = 28225$, $U_d = 1261$.*

ACKNOWLEDGEMENTS This work was supported by the research scientific project of Kırklareli University with the number KLUBAP-044.

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