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Fourier Coefficients of Some Eta Quotients of Weight 8

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Abstract. Recently, Williams[12] and then Yao, Xia, Jin [13] discovered the explicit formulas of the coefficients of Fourier series expansions of a class of eta quotients. Williams expressed all coefficients of 126 eta quotients in terms of $\sigma(n)$, $\sigma(\frac{n}{2})$, $\sigma(\frac{n}{3})$, $\sigma(\frac{n}{6})$ and Yao, Xia, Jin expressed only even coefficients of 104 eta quotients in terms of $\sigma_3(n)$, $\sigma_3(\frac{n}{2})$, $\sigma_3(\frac{n}{3})$, $\sigma_3(\frac{n}{6})$. Here, we will express the odd Fourier coefficients of 64 eta quotients in terms of $\sigma_7(2n-1)$, $\sigma_7\left(\frac{2n-1}{3}\right)$ and even Fourier coefficients of 130 eta quotients in terms of $\sigma_7(n)$, $\sigma_7\left(\frac{n}{2}\right)$, $\sigma_7\left(\frac{n}{3}\right)$, $\sigma_7\left(\frac{n}{4}\right)$, $\sigma_7\left(\frac{n}{6}\right)$, $\sigma_7\left(\frac{n}{12}\right)$.

2010 Mathematics Subject Classifications: 11F20, 11F30

Key Words and Phrases: Dedekind eta function; eta quotients; Fourier series.

1. Introduction

The divisor function $\sigma_i(n)$ is defined by

$$\sigma_i(n) := \begin{cases} \sum\limits_{d \text{ positive integer}, d|n} d^i & \text{if } n \text{ is a positive integer} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Dedekind eta function is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n), q := e^{2\pi i z}, \tag{2}$$

and an eta quotient of level n is defined by

$$f(z) := \prod_{m|n} \eta(mz)^{a_m}, n, m \in \mathbb{N}, a_m \in Z. \tag{3}$$

It is interesting and important to determine explicit formulas of the Fourier coefficients of eta quotients since they are the building blocks of modular forms of level n and weight k. The book of Kohler [10] describes such expansions by means of Hecke Theta series and develops

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algorithms for the determination of suitable eta quotients. One can find more information in [4, 5, 11, 14, 15]. I have determined the Fourier coefficients of the theta series associated to some quadratic forms, see [6–9].

Recently, Williams, see [12] discovered the explicit formulas of the coefficients of Fourier series expansions of a class of 126 eta quotients in terms of $\sigma(n)$, $\sigma(\frac{n}{2})$, $\sigma(\frac{n}{3})$, $\sigma(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)}.$$

Then Yao, Xia, Jin, see [13] expressed even Fourier coefficients of 104 eta quotients in terms of $\sigma_3(n)$, $\sigma_3(\frac{n}{2})$, $\sigma_3(\frac{n}{3})$, $\sigma_3(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^{25}(2z)\eta^4(3z)}{\eta^{12}(z)\eta^5(4z)\eta^3(6z)\eta(12z)},$$

where the even coefficients are obtained. Motivated by these two results, we find that we can express the odd Fourier coefficients of 64 eta quotients in terms of $\sigma_7(2n-1)$, $\sigma_7(\frac{2n-1}{3})$, see Table 1 in the appendix. One example is as follows:

$$\frac{\eta^{36}(2z)\eta^{14}(12z)}{\eta^{18}(4z)\eta^{16}(6z)},$$

where the odd coefficients are obtained.We can also express the even Fourier coefficients of 130 eta quotients in terms of

$$\sigma_7(n), \sigma_7\left(\frac{n}{2}\right), \sigma_7\left(\frac{n}{3}\right), \sigma_7\left(\frac{n}{4}\right), \sigma_7\left(\frac{n}{6}\right), \sigma_7\left(\frac{n}{12}\right),$$

see Table 2, also in the appendix. One example is as follows:

$$\frac{\eta^{24}(4z)\eta^8(12z)}{\eta^{12}(2z)\eta^4(6z)}.$$

Now we can state our main Theorem:

Theorem 1. *Let $b_1, b_2, \ldots, b_5$ be non-negative integers satisfying*

$$b_1 + b_2 + \ldots + b_5 \leq 16. \tag{4}$$

Define the integers a_1, a_2, a_3, a_4, a_6, a_{12} by

$$a_1 := -b_1 + 2b_2 - 2b_3 - 4b_4 - b_5 + 16 \tag{5}$$
$$a_2 := 3b_1 + b_2 + 3b_3 + 10b_4 + b_5 - 40 \tag{6}$$
$$a_3 := 3b_1 + 2b_2 + 6b_3 + 4b_4 + 3b_5 - 48 \tag{7}$$
$$a_4 := -2b_1 - b_2 - b_3 - 4b_4 + 2b_5 + 16 \tag{8}$$
$$a_6 := -9b_1 - 7b_2 - 9b_3 - 10b_4 - 7b_5 + 120 \tag{9}$$
$$a_{12} := 6b_1 + 3b_2 + 3b_3 + 4b_4 + 2b_5 - 48. \tag{10}$$

$$f_1 := \sum_{n=0}^{\infty} f_1(n) = \frac{\eta^{10}(4z)\,\eta^{28}(6z)}{\eta^{8}(2z)\,\eta^{14}(12z)},$$
$$f_2 := \sum_{n=0}^{\infty} f_2(n) = \frac{\eta^{5}(4z)\,\eta^{33}(6z)}{\eta^{7}(2z)\,\eta^{15}(12z)},$$
$$f_3 := \sum_{n=0}^{\infty} f_3(n) = \frac{\eta^{9}(2z)\,\eta^{17}(6z)}{\eta^{3}(4z)\,\eta^{7}(12z)},$$
$$f_4 := \sum_{n=0}^{\infty} f_4(n) = \frac{\eta^{17}(2z)\,\eta^{9}(6z)}{\eta^{7}(4z)\,\eta^{3}(12z)},$$
$$f_5 := \sum_{n=0}^{\infty} f_5(n) = \frac{\eta^{25}(2z)\,\eta(6z)\,\eta(12z)}{\eta^{11}(4z)},$$
$$f_6 := \sum_{n=0}^{\infty} f_6(n) = \frac{\eta^{33}(2z)\,\eta^{5}(12z)}{\eta^{15}(4z)\,\eta^{7}(6z)}$$
$$f_7 := \sum_{n=0}^{\infty} f_7(n) = \frac{\eta^{6}(4z)\,\eta^{26}(6z)}{\eta^{6}(2z)\,\eta^{10}(12z)},$$
$$f_8 := \sum_{n=0}^{\infty} f_8(n) = \frac{\eta^{2}(2z)\,\eta^{2}(4z)\,\eta^{18}(6z)}{\eta^{6}(12z)},$$
$$f_9 := \sum_{n=0}^{\infty} f_9(n) = \frac{\eta^{10}(2z)\,\eta^{10}(6z)}{\eta^{2}(4z)\,\eta^{2}(12z)},$$
$$f_{10} := \sum_{n=0}^{\infty} f_{10}(n) = \frac{\eta^{18}(2z)\,\eta^{2}(6z)\,\eta^{2}(12z)}{\eta^{6}(4z)},$$
$$f_{11} := \sum_{n=0}^{\infty} f_{11}(n) = \frac{\eta^{26}(2z)\,\eta^{6}(12z)}{\eta^{10}(4z)\,\eta^{6}(6z)}.$$

Now define integers k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}, k_{12}, k_{13}, k_{14}, k_{15} and k_{16} by

$$\frac{1}{2^{b_1+b_5}} x^{b_1}(1-x)^{b_2}(1+x)^{b_3}(1+2x)^{b_4}(2+x)^{b_5} \tag{11}$$
$$\begin{aligned}=&k_0+k_1x+k_2x^2+k_3x+k_4x^4+k_5x^5+k_6x^6+k_7x^7+k_8x^8\\ &+k_9x^9+k_{10}x^{10}+k_{11}x^{11}+k_{12}x^{12}+k_{13}x^{13}+k_{14}x^{14}+k_{15}x^{15}+k_{16}x^{16}.\end{aligned} \tag{12}$$

Define the rational numbers c_1, c_2, c_3, c_4, c_6, c_{12}, r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10} and r_{11} by

$$\begin{aligned}c_1 =&-21872/94095k_0+730/6273k_1-5488/94095k_2+2758/94095k_3\\ &-464/31365k_4+142/18819k_5-368/94095k_6+22/10455k_7\\ &-112/94095k_8+14/18819k_9-16/31365k_{10}+38/94095k_{11}\\ &-32/94095k_{12}+2/6273k_{13}-28/94095k_{14}+28/94095k_{15},\\ c_2 =&5614304/94095k_0-2807152/94095k_1+82564/5535k_2\end{aligned}$$

$$
\begin{aligned}
&-701806/94095k_3+350912/94095k_4-175462/94095k_5\\
&+87728/94095k_6-43846/94095k_7+21872/94095k_8\\
&-10822/94095k_9+304/5535k_{10}-2086/94095k_{11}\\
&+32/94095k_{12}+2018/94095k_{13}-5092/94095k_{14}\\
&+10724/94095k_{15}-28672/94095k_{16},\\
c_3=&198992/94095k_0-12538/6273k_1+182608/94095k_2-179878/94095k_3\\
&+59504/31365k_4-35566/18819k_5+177488/94095k_6-19702/10455k_7\\
&+177232/94095k_8-35438/18819k_9+59056/31365k_{10}-177158/94095k_{11}\\
&+177152/94095k_{12}-11810/6273k_{13}+177148/94095k_{14}-177148/94095k_{15},\\
c_4=&-5599232/94095k_0+2796032/94095k_1-93184/6273k_2+698368/94095k_3\\
&-4096/1107k_4+57344/31365k_5-16384/18819k_6+32768/94095k_7\\
&-32768/94095k_9+16384/18819k_{10}-57344/31365k_{11}+4096/1107k_{12}\\
&-698368/94095k_{13}+93184/6273k_{14}-2796032/94095k_{15}+5599232/94095k_{16},\\
c_6=&-5968544/94095k_0+2984272/94095k_1-1580708/94095k_2+878926/94095k_3\\
&-528032/94095k_4+352582/94095k_5-264848/94095k_6+12998/5535k_7\\
&-198992/94095k_8+187942/94095k_9-182288/94095k_{10}+179206/94095k_{11}\\
&-177152/94095k_{12}+175102/94095k_{13}-172028/94095k_{14}+9788/5535k_{15}\\
&+45371392/94095k_{16},\\
c_{12}=&50941952/94095k_0-2796032/94095k_1+93184/6273k_2-698368/94095k_3\\
&+4096/1107k_4-57344/31365k_5+16384/18819k_6-32768/94095k_7\\
&+32768/94095k_9-16384/18819k_{10}+57344/31365k_{11}-4096/1107k_{12}\\
&+698368/94095k_{13}-93184/6273k_{14}+2796032/94095k_{15}-50941952/94095k_{16}
\end{aligned}
$$

$$
\begin{aligned}
r_1=&8324272/10455k_0-1604737/1394k_1+21432061/20910k_2-8074478/10455k_3\\
&+1855009/3485k_4-1445785/4182k_5+4490171/20910k_6-445011/3485k_7\\
&+757742/10455k_8-160223/4182k_9+122227/6970k_{10}-58468/10455k_{11}\\
&-21683/10455k_{12}+7615/1394k_{13}-92542/10455k_{14}+92542/10455k_{15},\\
r_2=&-9365552/10455k_0+3452381/2788k_1-45845977/41820k_2+69110609/83640k_3\\
&-7952071/13940k_4+3105151/8364k_5-2415053/10455k_6+3834183/27880k_7\\
&-1632419/20910k_8+344483/8364k_9-260849/13940k_{10}+481609/83640k_{11}\\
&+53621/20910k_{12}-34879/5576k_{13}+415943/41820k_{14}-415943/41820k_{15},\\
r_3=&3765428/31365k_0-1968431/16728k_1+24665831/250920k_2-9439003/125460k_3\\
&+1132177/20910k_4-1855277/50184k_5+6009511/250920k_6-202957/13940k_7\\
&+252718/31365k_8-184453/50184k_9+67157/83640k_{10}+127237/125460k_{11}\\
&-279703/125460k_{12}+47555/16728k_{13}-216811/62730k_{14}+216811/62730k_{15},\\
r_4=&-1300196/31365k_0+619175/16728k_1-7731047/250920k_2+764689/31365k_3
\end{aligned}
$$

$$
\begin{aligned}
&-385399/20910k_4+665093/50184k_5-2240767/250920k_6+37917/6970k_7\\
&-85036/31365k_8+32269/50184k_9+73171/83640k_{10}-123107/62730k_{11}\\
&+343771/125460k_{12}-53657/16728k_{13}+115271/31365k_{14}-115271/31365k_{15},\\
r_5=&57262/10455k_0-53537/11152k_1+678533/167280k_2-276769/83640k_3\\
&+72257/27880k_4-64253/33456k_5+218083/167280k_6-10389/13940k_7\\
&+10343/41820k_8+6329/33456k_9-31429/55760k_{10}+73261/83640k_{11}\\
&-94159/83640k_{12}+14645/11152k_{13}-31379/20910k_{14}+31379/20910k_{15},\\
r_6=&-23182/94095k_0+21773/100368k_1-280913/1505520k_2+58637/376380k_3\\
&-31307/250920k_4+28193/301104k_5-94003/1505520k_6+2611/83640k_7\\
&+7/376380k_8-9413/301104k_9+31369/501840k_{10}-8822/94095k_{11}\\
&+94099/752760k_{12}-15683/100368k_{13}+70573/376380k_{14}-70573/376380k_{15},\\
r_7=&-5968/31365k_0+559211/3690k_1-9498581/62730k_2+7125649/62730k_3\\
&-2370607/31365k_4+1480451/31365k_5-1768421/62730k_6\\
&+1026679/62730k_7-290002/31365k_8+9388/1845k_9-172211/62730k_{10}\\
&+85609/62730k_{11}-21547/31365k_{12}+6041/31365k_{13}-2038/31365k_{14}\\
&-5968/31365k_{15}+11936/31365k_{16},\\
r_8=&8170304/31365k_0-553454/1845k_1+8087054/31365k_2-6100906/31365k_3\\
&+4250186/31365k_4-2796223/31365k_5+1754294/31365k_6\\
&-1053046/31365k_7+597956/31365k_8-18419/1845k_9+139724/31365k_{10}\\
&-32926/31365k_{11}-24124/31365k_{12}+65132/31365k_{13}-74056/31365k_{14}\\
&+91904/31365k_{15}-183808/31365k_{16},\\
r_9=&-1116896/6273k_0+1011709/6273k_1-836363/6273k_2+649939/6273k_3\\
&-479906/6273k_4+336985/6273k_5-223349/6273k_6+137143/6273k_7\\
&-74396/6273k_8+30469/6273k_9-575/6273k_{10}-19181/6273k_{11}\\
&+30910/6273k_{12}-38834/6273k_{13}+39148/6273k_{14}-39776/6273k_{15}\\
&+79552/6273k_{16},\\
r_{10}=&1067584/31365k_0-933638/31365k_1+784454/31365k_2-635366/31365k_3\\
&+494146/31365k_4-364693/31365k_5+14642/1845k_6-147746/31365k_7\\
&+61556/31365k_8+9607/31365k_9-66056/31365k_{10}+108554/31365k_{11}\\
&-138764/31365k_{12}+160012/31365k_{13}-9608/1845k_{14}+169984/31365k_{15}\\
&-339968/31365k_{16},\\
r_{11}=&-185456/94095k_0+326609/188190k_1-280927/188190k_2\\
&+234563/188190k_3-93929/94095k_4+70492/94095k_5-5531/11070k_6\\
&+47033/188190k_7-14/94095k_8-23483/94095k_9+93923/188190k_{10}\\
&-140797/188190k_{11}+93751/94095k_{12}-116933/94095k_{13}+8222/535k_{14}\\
&-185456/94095k_{15}+370912/94095k_{16}.
\end{aligned}
$$

Here $\{f_1, f_2, \ldots, f_{11}\}$ *are in* $S_8(\Gamma_0(12))$ *and*

$$\eta^{a_1}(z)\,\eta^{a_2}(2z)\,\eta^{a_3}(3z)\,\eta^{a_4}(4z)\,\eta^{a_5}(6z)\,\eta^{a_6}(12z) = \delta(b_1) + \sum_{n=1}^{\infty} c(n)q^n,$$

where for $n \in \mathbb{N}$,

$$\begin{aligned} c(n) =& c_1\sigma_7(n) + c_2\sigma_7\left(\frac{n}{2}\right) + c_3\sigma_7\left(\frac{n}{3}\right) + c_4\sigma_7\left(\frac{n}{4}\right) + c_6\sigma_7\left(\frac{n}{6}\right) + c_{12}\sigma_7\left(\frac{n}{12}\right) \\ &+ r_1f_1(n) + r_2f_2(n) + r_3f_3(n) + r_4f_4(n) + r_5f_5(n) + r_6f_6(n) + r_7f_7(n) \\ &+ r_8f_8(n) + r_9f_9(n) + r_{10}f_{10}(n) + r_{11}f_{11}(n). \end{aligned}$$

In particular,

$$\begin{aligned} c(2n) =& c_1\sigma_7(2n) + c_2\sigma_7(n) + c_4\sigma_7\left(\frac{n}{2}\right) + \left(129c_3 + c_6\right)\sigma_7\left(\frac{n}{3}\right) \\ &+ (c_{12} - 128c_3)\sigma_7\left(\frac{n}{6}\right) + r_7f_7(2n) + r_8f_8(2n) + \ldots + r_{11}f_{11}(2n), \\ c(2n-1) =& c_1\sigma_7(2n-1) + c_3\sigma_7\left(\frac{2n-1}{3}\right) \\ &+ r_1f_1(2n-1) + r_2f_2(2n-1) + r_3f_3(2n-1) + \ldots + r_6f_6(2n-1), \end{aligned}$$

for $n \in \mathbb{N}$.

Proof. It follows from equations (5)-(10) that

$$a_1 := -b_1 + 2b_2 - 2b_3 - 4b_4 - b_5 + 16 \tag{13}$$
$$a_2 := 3b_1 + b_2 + 3b_3 + 10b_4 + b_5 - 40 \tag{14}$$
$$a_3 := 3b_1 + 2b_2 + 6b_3 + 4b_4 + 3b_5 - 48 \tag{15}$$
$$a_4 := -2b_1 - b_2 - b_3 - 4b_4 + 2b_5 + 16 \tag{16}$$
$$a_6 := -9b_1 - 7b_2 - 9b_3 - 10b_4 - 7b_5 + 120 \tag{17}$$
$$a_{12} := 6b_1 + 3b_2 + 3b_3 + 4b_4 + 2b_5 - 48. \tag{18}$$

$$\begin{gathered} a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 12a_{12} = 24b_1, \\ a_1 + a_2 + a_3 + a_4 + a_6 + a_{12} = 16, \\ \frac{a_1}{6} + \frac{a_2}{3} + \frac{a_3}{6} + 2\frac{a_4}{3} + \frac{a_6}{3} + 2\frac{a_{12}}{3} = b_1 + b_5. \end{gathered} \tag{19}$$

Now we will use $p - k$ parametrization of Alaca, Alaca and Williams, see [1]:

$$p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, k(q) := \frac{\varphi^3(q^3)}{\varphi(q)},$$

where the theta function $\varphi\left(q\right)$ is defined by

$$\varphi\left(q\right)=\sum_{-\infty}^{\infty}q^{n^2}.$$

Setting $x=p$ in (11), and multiplying both sides by k^8, we obtain

$$\begin{aligned}&\frac{k^8}{2^{b_1+b_5}}p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}\\&\quad=(k_0+k_1p+k_2p^2+k_3p^3+k_4p^4+k_5p^5+k_6p^6+k_7p^7\\&\qquad+k_8p^8+k_9p^9+k_{10}p^{10}+k_{11}p^{11}+k_{12}p^{12}+k_{13}p^{13}+k_{14}p^{14}+k_{15}p^{15}+k_{16}p^{16})k^8.\end{aligned}$$

Alaca, Alaca and Williams [2] have established the following representations in terms of p and k:

$$\eta\left(q\right)=2^{-1/6}p^{1/24}(1-p)^{1/2}(1+p)^{1/6}(1+2p)^{1/8}(2+p)^{1/8}k^{1/2},\tag{20}$$
$$\eta\left(q^2\right)=2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2},\tag{21}$$
$$\eta\left(q^3\right)=2^{-1/6}p^{1/8}(1-p)^{1/6}(1+p)^{1/2}(1+2p)^{1/24}(2+p)^{1/24}k^{1/2},\tag{22}$$
$$\eta\left(q^4\right)=2^{-2/3}p^{1/6}(1-p)^{1/8}(1+p)^{1/24}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2},\tag{23}$$
$$\eta\left(q^6\right)=2^{-1/3}p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2},\tag{24}$$
$$\eta\left(q^{12}\right)=2^{-2/3}p^{1/2}(1-p)^{1/24}(1+p)^{1/8}(1+2p)^{1/24}(2+p)^{1/6}k^{1/2}.\tag{25}$$

Since $E_8=E_4^2$, we have

$$\begin{aligned}E_8\left(q\right):=&1+480\sum_{n=1}^{\infty}\sigma_7\left(n\right)q^n\\=&(1+248p+17304p^2+244648p^3+1628540p^4+6350520p^5\\&16004776p^6+27416744p^7+32723334p^8+27416744p^9\\&+16004776p^{10}+6350520p^{11}+1628540p^{12}+244648p^{13}\\&+17304p^{14}+248p^{15}+p^{16})k^8,\\E_8(q^2)=&(1+8p+144p^2+868p^3+5990p^4+25020p^5\\&+63316p^6+106964p^7+126819p^8+106964p^9+63316p^{10}\\&+25020p^{11}+5990p^{12}+868p^{13}+144p^{14}+8p^{15}+p^{16})k^8,\\E_8(q^3)=&(1+8p+24p^2+88p^3+380p^4+840p^5\\&+1576p^6+4184p^7+6534p^8+4184p^9+1576p^{10}\\&+840p^{11}+380p^{12}+88p^{13}+24p^{14}+8p^{15}+p^{16})k^8,\\E_8(q^4)=&(1+8p+24p^2+28p^3+20p^4+120p^5\\&+\frac{647}{2}p^6+\frac{463}{2}p^7-\frac{63}{8}p^8+\frac{943}{2}p^9+\frac{1937}{2}p^{10}\end{aligned}$$

$$+\frac{1155}{4}p^{11}-\frac{5515}{16}p^{12}-\frac{871}{8}p^{13}+\frac{1713}{32}p^{14}-\frac{29}{32}p^{15}+\frac{1}{256}p^{16})k^8,$$

$$\begin{aligned}E_8(q^6)=&(1+8p+24p^2+28p^3-10p^4-60p^5\\&-44p^6+44p^7+99p^8+44p^9-44p^{10}\\&-60p^{11}-10p^{12}+28p^{13}+24p^{14}+8p^{15}+p^{16})k^8,\\E_8(q^{12})=&(1+8p+24p^2+28p^3-10p^4-60p^5\\&-\frac{103}{2}p^6+\frac{13}{2}p^7+\frac{297}{8}p^8+\frac{43}{2}p^9+p^{10}\\&-\frac{15}{4}p^{11}-\frac{25}{16}p^{12}-\frac{1}{8}p^{13}+\frac{3}{32}p^{14}+\frac{1}{32}p^{15}+\frac{1}{256}p^{16})k^8.\end{aligned}$$

It is easy to check the following expressions by (20)-(25)

$$\begin{aligned}f_1:=&\sum_{n=0}^{\infty}f_1(n)=\frac{\eta^{10}(4z)\eta^{28}(6z)}{\eta^8(2z)\eta^{14}(12z)}=(\frac{1}{2}p+\frac{15}{4}p^2+\frac{91}{8}p^3+\frac{273}{16}p^4\\&+\frac{41}{4}p^5-\frac{99}{16}p^6-\frac{65}{4}p^7-\frac{213}{16}p^8-\frac{23}{4}p^9-\frac{21}{16}p^{10}-\frac{1}{8}p^{11})k^8,\\f_2:=&\sum_{n=0}^{\infty}f_2(n)=\frac{\eta^{5}(4z)\eta^{33}(6z)}{\eta^7(2z)\eta^{15}(12z)}=(\frac{1}{2}p+\frac{15}{4}p^2+\frac{45}{4}p^3+\frac{65}{4}p^4\\&+\frac{33}{4}p^5-\frac{33}{4}p^6-\frac{65}{4}p^7-\frac{45}{4}p^8-\frac{15}{4}p^9-\frac{1}{2}p^{10})k^8,\\f_3:=&\sum_{n=0}^{\infty}f_3(n)=\frac{\eta^{9}(2z)\eta^{17}(6z)}{\eta^3(4z)\eta^{7}(12z)}=(\frac{1}{2}p+\frac{15}{4}p^2+\frac{37}{4}p^3+\frac{13}{4}p^4\\&-\frac{87}{4}p^5-\frac{121}{4}p^6+\frac{19}{4}p^7+\frac{135}{4}p^8+\frac{65}{4}p^9-\frac{17}{2}p^{10}-9p^{11}-2p^{12})k^8,\\f_4:=&\sum_{n=0}^{\infty}f_4(n)=\frac{\eta^{17}(2z)\eta^{9}(6z)}{\eta^7(4z)\eta^{3}(12z)}=(\frac{1}{2}p+\frac{15}{4}p^2+\frac{33}{4}p^3-\frac{13}{4}p^4-\frac{135}{4}p\\&^5-\frac{99}{4}p^6+\frac{171}{4}p^7+\frac{225}{4}p^8-\frac{63}{4}p^9-44p^{10}-6p^{11}+12p^{12}+4p^{13})k^8,\\f_5:=&\sum_{n=0}^{\infty}f_5(n)=\frac{\eta^{25}(2z)\eta(6z)\eta(12z)}{\eta^{11}(4z)}=(\frac{1}{2}p+\frac{15}{4}p^2+\frac{29}{4}p^3-\frac{39}{4}p^4-\frac{175}{4}p\\&^5-\frac{33}{4}p^6+\frac{375}{4}p^7+\frac{219}{4}p^8-\frac{399}{4}p^9-\frac{145}{2}p^{10}+54p^{11}+40p^{12}-12p^{13}-8p^{14})k^8,\\f_6:=&\sum_{n=0}^{\infty}f_6(n)=\frac{\eta^{33}(2z)\eta^{5}(12z)}{\eta^{15}(4z)\eta^{7}(6z)}=(\frac{1}{2}p+\frac{15}{4}p^2+\frac{25}{4}p^3-\frac{65}{4}p^4-\frac{207}{4}p^5\\&+\frac{77}{4}p^6+\frac{615}{4}p^7+\frac{45}{4}p^8-\frac{975}{4}p^9-38p^{10}+219p^{11}+20p^{12}-100p^{13}+16p^{15})k^8,\\f_7:=&\sum_{n=0}^{\infty}f_7(n)=\frac{\eta^{6}(4z)\eta^{26}(6z)}{\eta^6(2z)\eta^{10}(12z)}=(\frac{1}{4}p^2+\frac{7}{4}p^3+\frac{77}{16}p^4\end{aligned}$$

$$+\frac{49}{8}p^5+\frac{33}{16}p^6-\frac{33}{8}p^7-\frac{97}{16}p^8-\frac{29}{8}p^9-\frac{17}{16}p^{10}-\frac{1}{8}p^{11})k^8,$$

$$f_8:=\sum_{n=0}^{\infty}f_8(n)=\frac{\eta^2(2z)\eta^2(4z)\eta^{18}(6z)}{\eta^6(12z)}=(\frac{1}{4}p^2+\frac{7}{4}p^3+\frac{69}{16}p^4+\frac{25}{8}p^5$$
$$-\frac{73}{16}p^6-\frac{39}{4}p^7-\frac{73}{16}p^8+\frac{25}{8}p^9+\frac{69}{16}p^{10}+\frac{7}{4}p^{11}+\frac{1}{4}p^{12})k^8,$$

$$f_9:=\sum_{n=0}^{\infty}f_9(n)=\frac{\eta^{10}(2z)\eta^{10}(6z)}{\eta^2(4z)\eta^2(12z)}=(\frac{1}{4}p^2+\frac{7}{4}p^3+\frac{53}{16}p^4-\frac{23}{8}p^5-\frac{237}{16}p$$
$$^6-6p^7+\frac{339}{16}p^8+\frac{141}{8}p^9-\frac{183}{16}p^{10}-\frac{29}{8}p^{11}+\frac{1}{2}p^{12}+4p^{13}+p^{14})k^8,$$

$$f_{10}:=\sum_{n=0}^{\infty}f_{10}(n)=\frac{\eta^{18}(2z)\eta^2(6z)\eta^2(12z)}{\eta^6(4z)}=(\frac{1}{4}p^2+\frac{7}{4}p^3+\frac{61}{16}p^4+\frac{1}{8}p^5$$
$$-\frac{163}{16}p^6-\frac{83}{8}p^7+\frac{83}{16}p^8+\frac{103}{8}p^9+\frac{59}{16}p^{10}-\frac{31}{8}p^{11}-\frac{11}{4}p^{12}-\frac{1}{2}p^{13})k^8,$$

$$f_{11}:=\sum_{n=0}^{\infty}f_{11}(n)=\frac{\eta^{26}(2z)\eta^6(12z)}{\eta^{10}(4z)\eta^6(6z)}=(\frac{1}{4}p^2+\frac{7}{4}p^3+\frac{45}{16}p^4-\frac{47}{8}p^5-\frac{295}{16}p^6$$
$$+\frac{27}{8}p^7+\frac{663}{16}p^8+\frac{75}{8}p^9-\frac{729}{16}p^{10}-\frac{125}{8}p^{11}+\frac{49}{2}p^{12}+9p^{13}-5p^{14}-2p^{15})k^8.$$

We immediately verify that $f_1,\ldots,f_{11}\in S_8\left(\Gamma_0(12)\right)$. Now

$$a_1:=-b_1+2b_2-2b_3-4b_4-b_5+16, \tag{26}$$
$$a_2:=3b_1+b_2+3b_3+10b_4+b_5-40, \tag{27}$$
$$a_3:=3b_1+2b_2+6b_3+4b_4+3b_5-48, \tag{28}$$
$$a_4:=-2b_1-b_2-b_3-4b_4+2b_5+16, \tag{29}$$
$$a_6:=-9b_1-7b_2-9b_3-10b_4-7b_5+120, \tag{30}$$
$$a_{12}:=6b_1+3b_2+3b_3+4b_4+2b_5-48. \tag{31}$$

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z)$$
$$=q^{b_1}\prod_{n=1}^{\infty}\left(1-q^n\right)^{a_1}\left(1-q^{2n}\right)^{a_2}\left(1-q^{3n}\right)^{a_3}\left(1-q^{4n}\right)^{a_4}\left(1-q^{6n}\right)^{a_6}\left(1-q^{12n}\right)^{a_{12}}$$
$$=2^{-\frac{a_1}{6}-\frac{a_2}{3}-\frac{a_3}{6}-2\frac{a_4}{3}-\frac{a_6}{3}-2\frac{a_{12}}{3}}p^{\frac{a_1}{24}+\frac{a_2}{12}+\frac{a_3}{8}+\frac{a_4}{6}+\frac{a_6}{4}+\frac{a_{12}}{2}}(1-p)^{\frac{a_1}{2}+\frac{a_2}{4}+\frac{a_3}{6}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}$$
$$(1+p)^{\frac{a_1}{6}+\frac{a_2}{12}+\frac{a_3}{2}+\frac{a_4}{24}+\frac{a_6}{4}+\frac{a_{12}}{8}}(1+2p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}(2+p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{2}+\frac{a_6}{12}+\frac{a_{12}}{6}}$$
$$k^{\frac{a_1+a_2+a_3+a_4+a_6+a_{12}}{2}}=\frac{k^8}{2^{b_1+b_5}}p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}$$
$$=k^8(k_0+k_1p+k_2p^2+k_3p^3+k_4p^4+k_5p^5+k_6p^6$$
$$+k_7p^7+k_8p^8+k_9p^9+k_{10}p^{10}+k_{11}p^{11}+k_{12}p^{12})$$
$$=\frac{c_1}{480}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)+\frac{c_2}{480}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}\right)$$

$$
\begin{aligned}
&+\frac{c_3}{480}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{3n}\right)+\frac{c_4}{480}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\right)\\
&+\frac{c_6}{480}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{6n}\right)+\frac{c_{12}}{480}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{12n}\right)\\
&+r_1q\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{10}\left(1-q^{6n}\right)^{28}}{\left(1-q^{2n}\right)^{8}\left(1-q^{12n}\right)^{14}}\\
&+r_2q\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{5}\left(1-q^{6n}\right)^{33}}{\left(1-q^{2n}\right)^{7}\left(1-q^{12n}\right)^{15}}\\
&+r_3q^4\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{9}\left(1-q^{6n}\right)^{17}}{\left(1-q^{4n}\right)^{3}\left(1-q^{12n}\right)^{7}}\\
&+r_4q\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{17}\left(1-q^{6n}\right)^{9}}{\left(1-q^{4n}\right)^{7}\left(1-q^{12n}\right)^{3}}\\
&+r_5.q\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{25}\left(1-q^{6n}\right)\left(1-q^{12n}\right)}{\left(1-q^{4n}\right)^{11}}\\
&+r_6q\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{33}\left(1-q^{12n}\right)^{5}}{\left(1-q^{4n}\right)^{15}\left(1-q^{6n}\right)^{7}}\\
&+r_7q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)^{6}\left(1-q^{6n}\right)^{26}}{\left(1-q^{2n}\right)^{6}\left(1-q^{12n}\right)^{10}}\\
&+r_8q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{2}\left(1-q^{4n}\right)^{2}\left(1-q^{6n}\right)^{18}}{\left(1-q^{12n}\right)^{6}}\\
&+r_9q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{10}\left(1-q^{6n}\right)^{10}}{\left(1-q^{4n}\right)^{2}\left(1-q^{12n}\right)^{2}}\\
&+r_{10}q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{18}\left(1-q^{6n}\right)^{2}\left(1-q^{12n}\right)^{2}}{\left(1-q^{4n}\right)^{6}}\\
&+r_{11}q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{26}\left(1-q^{12n}\right)^{6}}{\left(1-q^{4n}\right)^{10}\left(1-q^{6n}\right)^{6}}\\
=&\delta(b_1)+\sum_{n=1}^{\infty}(c_1\sigma_7(n)+c_2\sigma_7\left(\frac{n}{2}\right)+c_3\sigma_7\left(\frac{n}{3}\right)+c_4\sigma_7\left(\frac{n}{4}\right)\\
&+c_6\sigma_7\left(\frac{n}{6}\right)+c_{12}\sigma_7\left(\frac{n}{12}\right))+r_1f_1(n)+\ldots+r_{11}f_{11}(n),
\end{aligned}
$$

where

$$\delta(b_1) = \begin{cases} 0 & \text{if } b_1 \neq 0 \\ 1 & \text{if } b_1 = 0 \end{cases}$$

So

$$\begin{aligned} c(n) =& (c_1\sigma_7(n) + c_2\sigma_7\left(\frac{n}{2}\right) + c_3\sigma_7\left(\frac{n}{3}\right) + c_4\sigma_7\left(\frac{n}{4}\right) \\ &+ c_6\sigma_7\left(\frac{n}{6}\right) + c_{12}\sigma_7\left(\frac{n}{12}\right)) + r_1 f_1(n) + \ldots + r_{11} f_{11}(n). \end{aligned}$$

Therefore, since for $n = 1, 2, \ldots,$

$$f_1(2n) = f_2(2n) = \ldots = f_6(2n) = 0,$$

$$f_7(2n-1) = f_8(2n-1) = f_9(2n-1) = \ldots = f_{11}(2n-1) = 0,$$

we have

$$\begin{aligned} c(2n) =& c_1\sigma_7(2n) + c_2\sigma_7(n) + c_4\sigma_7\left(\frac{n}{2}\right) + \left(129c_3 + c_6\right)\sigma_7\left(\frac{n}{3}\right) \\ &+ (c_{12} - 128c_3)\sigma_7\left(\frac{n}{6}\right) + r_7 f_7(2n) + r_8 f_8(2n) + \ldots + r_{11} f_{11}(2n), \\ c(2n-1) =& c_1\sigma_7(2n-1) + c_3\sigma_7\left(\frac{2n-1}{3}\right) \\ &+ r_1 f_1(2n-1) + r_2 f_2(2n-1) + r_3 f_3(2n-1) + \ldots + r_6 f_6(2n-1), \end{aligned}$$

by the following Lemma. □

Lemma 1.

$$\sigma_k\left(\frac{2n}{3}\right) = \left(2^k + 1\right)\sigma_k\left(\frac{n}{3}\right) - 2^k\sigma_k\left(\frac{n}{6}\right).$$

Proof. If 3 doesn't divide n, both sides are 0. So it is enough to prove that

$$\sigma_k(2n) = \left(2^k + 1\right)\sigma_k(n) - 2^k\sigma_k\left(\frac{n}{2}\right).$$

If 2 doesn't divide n, it is obvious. So assume that $n = 2^l m$, $l \geq 1$, 2 doesn't divide m. Then

$$\begin{aligned} \left(2^k+1\right)\sigma_k(n) - 2^k\sigma_k\left(\frac{n}{2}\right) &= \left(2^k + 1\right)\sigma_k\left(2^l\right)\sigma_k(m) - 2^k\sigma_k\left(2^{l-1}\right)\sigma_k(m) \\ &= \left(\left(2^k+1\right)\sigma_k\left(2^l\right) - 2^k\sigma_k\left(2^{l-1}\right)\right)\sigma_k(m) \\ &= \left(\left(2^k+1\right)\left(1 + 2^k + \left(2^k\right)^2 + \ldots + \left(2^k\right)^l\right) - 2^k\left(1 + 2^k + \left(2^k\right)^2 + \ldots + \left(2^k\right)^{l-1}\right)\right)\sigma_k(m) \\ &= \left(2^k\left(2^k\right)^l + \left(1 + 2^k + \left(2^k\right)^2 + \ldots + \left(2^k\right)^l\right)\right)\sigma_k(m) = \sigma_k\left(2^{l+1}m\right). \end{aligned}$$

□

These formulas are valid for 17,346 nontrivial eta quotients. Among them, we have found 64 nontrivial eta quotients, see Table 1, such that

$$\begin{aligned}c(2n) =& c_1\sigma_7(2n) + c_2\sigma_7(n) + c_4\sigma_7\left(\frac{n}{2}\right) + \left(129c_3 + c_6\right)\sigma_7\left(\frac{n}{3}\right)\\ &+ (c_{12} - 128c_3)\sigma_7\left(\frac{n}{6}\right),\\ c(2n-1) =& c_1\sigma_7(2n-1) + c_3\sigma_7\left(\frac{2n-1}{3}\right)\\ &+ r_1f_1(2n-1) + r_2f_2(2n-1) + r_3f_3(2n-1) + \ldots + r_6f_6(2n-1),\end{aligned}$$

and 130 eta quotients, see Table 2 (appendix), such that

$$\begin{aligned}c(2n-1) =& c_1\sigma_7(2n-1) + c_3\sigma_7\left(\frac{2n-1}{3}\right) = 0,\\ c(2n) =& c_2\sigma_7(n) + c_4\sigma_7\left(\frac{n}{2}\right) + c_6\sigma_7\left(\frac{n}{3}\right)\\ &+ c_{12}\sigma_7\left(\frac{n}{6}\right) + f_7(2n) + f_8(2n) + \ldots + f_{11}(2n).\end{aligned}$$

Remark 1. *$S_8(\Gamma_0(12))$ is 11 dimensional, see [3, Chapter 3, pg.87 and Chapter 5, pg.197], and generated by $\Delta_{2,8}$, $\Delta_{2,8}(2z)$, $\Delta_{2,8}(3z)$, $\Delta_{2,8}(6z)$, $\Delta_{3,8}$, $\Delta_{3,8}(2z)$, $\Delta_{3,8}(4z)$, $\Delta_{6,8}$, $\Delta_{6,8}(2z)$, $\Delta_{12,8,1}(z)$, $\Delta_{12,8,2}(z)$, where $\Delta_{2,8}$ is the unique newform in $S_8(\Gamma_0(2))$, $\Delta_{3,8}$ is the unique newform in $S_8(\Gamma_0(3))$, $\Delta_{6,8}$ is the unique newform in $S_8(\Gamma_0(6))$. $\Delta_{12,8,1}$, $\Delta_{12,8,2}$ are the newforms in $S_8(\Gamma_0(12))$. By simple calculation, we see that*

$$\begin{aligned}f_1 =& \frac{11}{45}\Delta_{2,8} + \frac{88}{45}\Delta_{2,8}(2z) + \frac{19}{5}\Delta_{2,8}(3z) + \frac{152}{5}\Delta_{2,8}(6z)\\ &+ \frac{32}{135}\Delta_{3,8} - \frac{64}{45}\Delta_{3,8}(2z) + \frac{4096}{135}\Delta_{3,8}(4z) + \frac{5}{27}\Delta_{6,8}\\ &- \frac{40}{27}\Delta_{6,8}(2z) + \frac{35}{162}\Delta_{12,8,1}(z) + \frac{19}{162}\Delta_{12,8,2}(z),\\ f_2 =& \frac{11}{45}\Delta_{2,8} + \frac{88}{45}\Delta_{2,8}(2z) + \frac{47}{15}\Delta_{2,8}(3z) + \frac{376}{15}\Delta_{2,8}(6z)\\ &+ \frac{32}{135}\Delta_{3,8} - \frac{64}{45}\Delta_{3,8}(2z) + \frac{4096}{135}\Delta_{3,8}(4z) + \frac{5}{27}\Delta_{6,8}\\ &- \frac{40}{27}\Delta_{6,8}(2z) + \frac{17}{81}\Delta_{12,8,1}(z) + \frac{10}{81}\Delta_{12,8,2}(z),\\ f_3 =& \frac{1}{5}\Delta_{2,8} + \frac{8}{5}\Delta_{2,8}(2z) - \frac{9}{5}\Delta_{2,8}(3z) - \frac{72}{5}\Delta_{2,8}(6z)\\ &+ \frac{16}{45}\Delta_{3,8} - \frac{32}{15}\Delta_{3,8}(2z) + \frac{2048}{45}\Delta_{3,8}(4z) + \frac{1}{9}\Delta_{6,8}\\ &- \frac{8}{9}\Delta_{6,8}(2z) + \frac{1}{9}\Delta_{12,8,1}(z) + \frac{2}{9}\Delta_{12,8,2}(z),\\ f_4 =& -\frac{1}{15}\Delta_{2,8} - \frac{8}{15}\Delta_{2,8}(2z) + \frac{243}{5}\Delta_{2,8}(3z) + \frac{1944}{5}\Delta_{2,8}(6z)\end{aligned}$$

$$
\begin{aligned}
&+\frac{16}{15}\Delta_{3,8}-\frac{32}{5}\Delta_{3,8}(2z)+\frac{2048}{15}\Delta_{3,8}(4z)-\frac{1}{3}\Delta_{6,8}\\
&+\frac{8}{3}\Delta_{6,8}(2z)-\frac{1}{3}\Delta_{12,8,1}(z)+\frac{2}{3}\Delta_{12,8,2}(z),\\
f_5=&-\frac{11}{15}\Delta_{2,8}-\frac{88}{15}\Delta_{2,8}(2z)+\frac{2673}{5}\Delta_{2,8}(3z)+\frac{21384}{5}\Delta_{2,8}(6z)\\
&+\frac{32}{5}\Delta_{3,8}-\frac{192}{5}\Delta_{3,8}(2z)+\frac{4096}{5}\Delta_{3,8}(4z)-5\Delta_{6,8}+40\Delta_{6,8}(2z)\\
&-\frac{13}{3}\Delta_{12,8,1}(z)+\frac{14}{3}\Delta_{12,8,2}(z),\\
f_6=&\frac{47}{5}\Delta_{2,8}+\frac{376}{5}\Delta_{2,8}(2z)+\frac{24057}{5}\Delta_{2,8}(3z)+\frac{192456}{5}\Delta_{2,8}(6z)\\
&+\frac{288}{5}\Delta_{3,8}-\frac{1728}{5}\Delta_{3,8}(2z)+\frac{36864}{5}\Delta_{3,8}(4z)-45\Delta_{6,8}\\
&+360\Delta_{6,8}(2z)-51\Delta_{12,8,1}(z)+30\Delta_{12,8,2}(z),\\
f_7=&\frac{4}{15}\Delta_{2,8}(2z)+\frac{68}{5}\Delta_{2,8}(6z)+\frac{13}{45}\Delta_{3,8}(2z)+\frac{128}{45}\Delta_{3,8}(4z)+\frac{4}{9}\Delta_{6,8}(2z),\\
f_8=&\frac{2}{5}\Delta_{2,8}(2z)-\frac{18}{5}\Delta_{2,8}(6z)-\frac{17}{45}\Delta_{3,8}(2z)-\frac{128}{45}\Delta_{3,8}(4z)+\frac{2}{9}\Delta_{6,8}(2z),\\
f_9=&\frac{8}{15}\Delta_{2,8}(2z)+\frac{216}{5}\Delta_{2,8}(6z)+\frac{7}{15}\Delta_{3,8}(2z)-\frac{128}{15}\Delta_{3,8}(4z),\\
f_{10}=&-\frac{2}{5}\Delta_{2,8}(2z)+\frac{1458}{5}\Delta_{2,8}(6z)+\frac{17}{5}\Delta_{3,8}(2z)-\frac{128}{5}\Delta_{3,8}(4z)-2\Delta_{6,8}(2z),\\
f_{11}=&\frac{68}{5}\Delta_{2,8}(2z)+\frac{8748}{5}\Delta_{2,8}(6z)+\frac{117}{5}\Delta_{3,8}(2z)+\frac{1152}{5}\Delta_{3,8}(4z)-36\Delta_{6,8}(2z).
\end{aligned}
$$

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Appendix: Table 1

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
1	1	0	0	0	15	−22	44	6	−12	$\frac{44287}{28549120}$	$-\frac{5713023}{28549120}$	$-\frac{45927}{28549120}$	$\frac{44287}{223040}$	$\frac{5924583}{28549120}$	$-\frac{45927}{223040}$
2	1	0	1	0	13	−21	39	11	−13	$\frac{19683}{14274560}$	$-\frac{2539107}{14274560}$	$-\frac{19683}{14274560}$	$\frac{19683}{111520}$	$\frac{2539107}{14274560}$	$\frac{19683}{111520}$
3	1	0	2	0	11	−20	34	16	−14	$\frac{2187}{1784320}$	$-\frac{282123}{1784320}$	$-\frac{2187}{1784320}$	$\frac{2187}{13940}$	$\frac{282123}{1784320}$	$-\frac{2187}{13940}$
4	1	0	3	0	9	−19	29	21	−15	$\frac{243}{223040}$	$-\frac{31347}{223040}$	$-\frac{243}{223040}$	$\frac{486}{3485}$	$\frac{31347}{223040}$	$-\frac{486}{3485}$
5	1	0	4	0	7	−18	24	26	−16	$\frac{27}{27880}$	$-\frac{3483}{27880}$	$-\frac{27}{27880}$	$\frac{432}{3485}$	$\frac{3483}{27880}$	$-\frac{432}{3485}$
6	1	0	5	0	5	−17	19	31	−17	$\frac{3}{3485}$	$-\frac{387}{3485}$	$-\frac{3}{3485}$	$\frac{384}{3485}$	$\frac{387}{3485}$	$\frac{101101}{6970}$
7	1	0	6	0	3	−16	14	36	−18	$\frac{8}{10455}$	$-\frac{344}{3485}$	$-\frac{8}{10455}$	$\frac{1024}{10455}$	$\frac{344}{3485}$	$-\frac{1024}{10455}$
8	1	0	7	0	1	−15	9	41	−19	$\frac{64}{94095}$	$-\frac{2752}{31365}$	$-\frac{64}{94095}$	$\frac{8192}{94095}$	$\frac{2752}{31365}$	$-\frac{8192}{94095}$
9	1	1	0	1	13	−13	35	3	−9	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
10	1	2	0	2	11	−4	26	0	−6	$\frac{1}{1784320}$	$-\frac{129}{1784320}$	$-\frac{6561}{1784320}$	$\frac{1}{13940}$	$\frac{846369}{1784320}$	$-\frac{6561}{13940}$
11	1	3	0	3	9	5	17	−3	−3	$-\frac{1}{223040}$	$\frac{129}{223040}$	$\frac{6561}{223040}$	$-\frac{2}{3485}$	$-\frac{846369}{223040}$	$\frac{13122}{3485}$
12	1	4	0	4	7	14	8	−6	0	$\frac{1}{27880}$	$-\frac{129}{27880}$	$-\frac{6561}{27880}$	$\frac{16}{3485}$	$\frac{846369}{27880}$	$-\frac{104976}{3485}$
13	1	5	0	5	5	23	−1	−9	3	$-\frac{1}{3485}$	$\frac{129}{3485}$	$\frac{6561}{3485}$	$-\frac{128}{3485}$	$-\frac{846369}{3485}$	$\frac{839808}{3485}$
14	1	6	0	6	3	32	−10	−12	6	$\frac{8}{3485}$	$-\frac{1032}{3485}$	$-\frac{52488}{3485}$	$\frac{1024}{3485}$	$\frac{6770952}{3485}$	$-\frac{6718464}{3485}$
15	1	7	0	7	1	41	−19	−15	9	$-\frac{64}{3485}$	$\frac{8256}{3485}$	$\frac{419904}{3485}$	$-\frac{8192}{3485}$	$-\frac{54167616}{3485}$	$-\frac{407889}{3485}$
16	3	0	0	0	13	−18	36	2	−4	$\frac{4921}{28549120}$	$-\frac{634809}{28549120}$	$-\frac{6561}{28549120}$	$\frac{4921}{223040}$	$\frac{846369}{28549120}$	$-\frac{6561}{223040}$
17	3	0	1	0	11	−17	31	7	−5	$\frac{2187}{14274560}$	$-\frac{282123}{14274560}$	$-\frac{2187}{14274560}$	$\frac{2187}{111520}$	$\frac{282123}{14274560}$	$-\frac{2187}{111520}$
18	3	0	2	0	9	−16	26	12	−6	$\frac{243}{1784320}$	$-\frac{31347}{1784320}$	$-\frac{243}{1784320}$	$\frac{243}{13940}$	$\frac{31347}{1784320}$	$-\frac{243}{13940}$
19	3	0	3	0	7	−15	21	17	−7	$\frac{27}{223040}$	$-\frac{3483}{223040}$	$-\frac{27}{223040}$	$\frac{54}{385}$	$\frac{3483}{223040}$	$-\frac{54}{3485}$
20	3	0	4	0	5	−14	16	22	−8	$\frac{3}{27880}$	$-\frac{387}{27880}$	$-\frac{3}{27880}$	$\frac{48}{3485}$	$\frac{387}{27880}$	$-\frac{48}{3485}$
21	3	0	5	0	3	−13	11	27	−9	$\frac{1}{10455}$	$-\frac{43}{3485}$	$-\frac{1}{10455}$	$\frac{128}{10455}$	$\frac{43}{3485}$	$-\frac{128}{10455}$
22	3	0	6	0	1	−12	6	32	−10	$\frac{8}{94095}$	$-\frac{344}{31365}$	$-\frac{8}{94095}$	$\frac{1024}{94095}$	$\frac{344}{31365}$	$-\frac{1024}{94095}$
23	3	1	0	1	11	−9	27	−1	−1	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
24	3	2	0	2	9	0	18	−4	2	$\frac{1}{1784320}$	$-\frac{129}{1784320}$	$\frac{-6561}{1784320}$	$\frac{1}{13940}$	$\frac{846369}{1784320}$	$-\frac{6561}{13940}$
25	3	3	0	3	7	9	9	−7	5	$-\frac{1}{223040}$	$\frac{129}{223040}$	$\frac{6561}{223040}$	$-\frac{2}{3485}$	$-\frac{846369}{223040}$	$\frac{13122}{3485}$
26	3	4	0	4	5	18	0	−10	8	$\frac{1}{27880}$	$-\frac{129}{27880}$	$-\frac{6561}{27880}$	$\frac{16}{3485}$	$\frac{846369}{27880}$	$-\frac{104976}{3485}$

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
27	3	5	0	5	3	27	−9	−13	11	$-\frac{1}{3485}$	$\frac{129}{3485}$	$\frac{6561}{3485}$	$-\frac{128}{3485}$	$-\frac{846369}{3485}$	$\frac{839808}{3485}$
28	3	6	0	6	1	36	−18	−16	14	$\frac{8}{3485}$	$-\frac{1032}{3485}$	$-\frac{52488}{3485}$	$\frac{1024}{3485}$	$\frac{6770952}{3485}$	$-\frac{6718464}{3485}$
29	5	0	0	0	11	−14	28	−2	4	$\frac{547}{28549120}$	$-\frac{70563}{28549120}$	$-\frac{2187}{28549120}$	$\frac{547}{223040}$	$\frac{282123}{28549120}$	$-\frac{2187}{223040}$
30	5	0	1	0	9	−13	23	3	3	$\frac{243}{14274560}$	$-\frac{31347}{14274560}$	$-\frac{243}{14274560}$	$\frac{243}{14274560}$	$\frac{31347}{14274560}$	$-\frac{243}{111520}$
31	5	0	2	0	7	−12	18	8	2	$\frac{27}{1784320}$	$-\frac{3483}{1784320}$	$-\frac{27}{1784320}$	$\frac{27}{13940}$	$\frac{3483}{1784320}$	$-\frac{27}{13940}$
32	5	0	4	0	3	−10	8	18	0	$\frac{1}{83640}$	$-\frac{43}{27880}$	$-\frac{1}{83640}$	$\frac{16}{10455}$	$\frac{43}{27880}$	$-\frac{16}{10455}$
33	5	0	5	0	1	−9	3	23	−1	$\frac{1}{94095}$	$-\frac{43}{31365}$	$-\frac{1}{94095}$	$\frac{128}{94095}$	$\frac{43}{31365}$	$-\frac{128}{94095}$
34	5	1	0	1	9	−5	19	−5	7	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
35	5	2	0	2	7	4	10	−8	10	$\frac{1}{1784320}$	$-\frac{129}{1784320}$	$-\frac{6561}{1784320}$	$\frac{1}{13940}$	$\frac{846369}{1784320}$	$-\frac{6561}{13940}$
36	5	3	0	3	5	13	1	−11	13	$-\frac{1}{223040}$	$\frac{129}{223040}$	$\frac{6561}{223040}$	$-\frac{2}{3485}$	$-\frac{846369}{223040}$	$\frac{13122}{3485}$
37	5	4	0	4	3	22	−8	−14	16	$\frac{1}{27880}$	$-\frac{129}{27880}$	$-\frac{6561}{27880}$	$\frac{16}{3485}$	$\frac{846369}{27780}$	$-\frac{104976}{3485}$
38	5	5	0	5	1	31	−17	−17	19	$-\frac{1}{3485}$	$\frac{129}{3485}$	$\frac{6561}{3485}$	$-\frac{128}{3485}$	$-\frac{846369}{3485}$	$\frac{839808}{3485}$
39	7	0	0	0	9	−10	20	−6	12	$\frac{61}{28549120}$	$-\frac{7869}{28549120}$	$-\frac{1701}{28549120}$	$\frac{61}{223040}$	$\frac{219429}{28549120}$	$-\frac{1701}{223040}$
40	7	0	1	0	7	−9	15	−1	11	$\frac{27}{14274560}$	$-\frac{3483}{14274560}$	$-\frac{27}{14274560}$	$\frac{27}{111520}$	$\frac{3483}{14274560}$	$-\frac{27}{111520}$
41	7	0	2	0	5	−8	10	4	10	$\frac{3}{1784320}$	$-\frac{387}{1784320}$	$-\frac{3}{1784320}$	$\frac{3}{13940}$	$\frac{387}{1784320}$	$-\frac{3}{13940}$
42	7	0	3	0	3	−7	5	9	9	$\frac{1}{669120}$	$-\frac{43}{223040}$	$-\frac{1}{669120}$	$\frac{2}{10455}$	$\frac{43}{223040}$	$-\frac{2}{10455}$
43	7	0	4	0	1	−6	0	14	8	$\frac{1}{752760}$	$-\frac{43}{250920}$	$-\frac{1}{752760}$	$\frac{16}{94095}$	$\frac{43}{250920}$	$-\frac{16}{94095}$
44	7	1	0	1	7	−1	11	−9	15	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
45	7	2	0	2	5	8	2	−12	18	$\frac{1}{1784320}$	$-\frac{129}{1784320}$	$-\frac{6561}{1784320}$	$\frac{1}{13940}$	$\frac{846369}{1784320}$	$-\frac{6561}{13940}$
46	7	3	0	3	3	17	−7	−15	21	$-\frac{1}{223040}$	$\frac{129}{223040}$	$\frac{6561}{223040}$	$-\frac{2}{3485}$	$-\frac{846369}{223040}$	$-\frac{132122}{3485}$
47	7	4	0	4	1	26	−16	−18	24	$\frac{1}{27880}$	$-\frac{129}{27880}$	$-\frac{6561}{27880}$	$\frac{16}{3485}$	$\frac{846369}{27880}$	$-\frac{104976}{3485}$
48	9	0	0	0	7	−6	12	−10	20	$\frac{7}{28549120}$	$-\frac{903}{28549120}$	$-\frac{1647}{28549120}$	$\frac{7}{223040}$	$\frac{212463}{28549120}$	$-\frac{1647}{223040}$
49	9	0	1	0	5	−5	7	−5	19	$\frac{3}{14274560}$	$-\frac{387}{14274560}$	$-\frac{3}{14274560}$	$\frac{3}{111520}$	$\frac{387}{14274560}$	$-\frac{3}{111520}$
50	9	0	2	0	3	−4	2	0	18	$\frac{1}{5352960}$	$-\frac{43}{1784320}$	$-\frac{1}{5352960}$	$\frac{1}{41820}$	$\frac{43}{1784320}$	$-\frac{1}{41820}$
51	9	0	3	0	1	−3	−3	5	17	$\frac{1}{6022080}$	$-\frac{43}{2007360}$	$-\frac{1}{6022080}$	$\frac{2}{94095}$	$\frac{43}{2007360}$	$-\frac{2}{94095}$
52	9	1	0	1	5	3	3	−13	23	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
53	9	2	0	2	3	12	−6	−16	26	$\frac{1}{1784320}$	$-\frac{129}{1784320}$	$\frac{-6561}{1784320}$	$\frac{1}{13940}$	$\frac{846369}{1784320}$	$-\frac{6561}{13940}$

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
54	9	3	0	3	1	21	−15	−19	29	$-\frac{1}{223040}$	$\frac{129}{223040}$	$\frac{6561}{223040}$	$-\frac{2}{3485}$	$-\frac{846369}{223040}$	$\frac{13122}{3485}$
55	11	0	0	0	5	−2	4	−14	28	$\frac{1}{28549120}$	$-\frac{129}{28549120}$	$-\frac{1641}{28549120}$	$\frac{1}{223040}$	$\frac{211689}{28549120}$	$-\frac{1641}{223040}$
56	11	0	1	0	3	−1	−1	−9	27	$\frac{1}{42823680}$	$-\frac{43}{14274560}$	$-\frac{1}{42823680}$	$\frac{1}{334560}$	$\frac{43}{14274560}$	$-\frac{1}{334560}$
57	11	0	2	0	1	0	−6	−4	26	$\frac{1}{48176640}$	$\frac{-43}{16058880}$	$-\frac{1}{48176640}$	$\frac{1}{376380}$	$\frac{43}{16058880}$	$-\frac{1}{376380}$
58	11	2	0	2	1	16	−14	−20	34	$\frac{1}{1784320}$	$-\frac{129}{1784320}$	$\frac{-6561}{1784320}$	$\frac{1}{13940}$	$\frac{846369}{1784320}$	$-\frac{6561}{13940}$
59	13	0	0	0	3	2	−4	−18	36	$\frac{1}{85647360}$	$-\frac{43}{28549120}$	$-\frac{4921}{85647360}$	$\frac{1}{669120}$	$\frac{211603}{28549120}$	$-\frac{4921}{669120}$
60	5	0	3	0	5	−11	13	13	1	$\frac{1}{223040}$	$-\frac{387}{223040}$	$-\frac{3}{223040}$	$\frac{6}{3485}$	$\frac{387}{223040}$	$-\frac{6}{3485}$
61	11	1	0	1	3	7	−5	−17	31	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
62	13	0	1	0	1	3	−9	−13	35	$\frac{1}{385413120}$	$-\frac{43}{128471040}$	$-\frac{1}{385413120}$	$\frac{1}{3011040}$	$\frac{43}{128471040}$	$-\frac{1}{30110940}$
63	13	1	0	1	1	11	−13	−21	39	$-\frac{1}{14274560}$	$\frac{129}{14274560}$	$\frac{6561}{14274560}$	$-\frac{1}{111520}$	$-\frac{846369}{14274560}$	$\frac{6561}{111520}$
64	15	0	0	0	1	6	−12	−22	44	$\frac{7}{770826240}$	$-\frac{301}{2569942080}$	$-\frac{44287}{770826240}$	$\frac{7}{6022080}$	$\frac{1904341}{2569942080}$	$-\frac{44287}{6022080}$

Appendix: Table 2

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
1	0	0	0	0	16	−24	48	8	−16	0	$\frac{33231}{55760}$	0	$-\frac{37311}{55760}$	$-\frac{137781}{55760}$	$\frac{26906661}{55760}$
2	0	0	1	0	14	−23	43	13	−17	0	$\frac{1441}{2720}$	0	$-\frac{67241}{2720}$	$\frac{53806761}{111520}$	$-\frac{433647}{71372800}$
3	0	0	2	0	12	−22	38	18	−18	0	$\frac{1313}{2788}$	0	$\frac{-6561}{2788}$	$\frac{32805}{68}$	$\frac{1974861}{69632}$
4	0	0	3	0	10	−21	33	23	−19	0	$\frac{1459}{3485}$	0	$-\frac{1714}{3485}$	$-\frac{8019}{3485}$	$\frac{1681074}{3485}$
5	0	0	4	0	8	−20	28	28	−20	0	$\frac{1297}{3485}$	0	$-\frac{1552}{3485}$	$-\frac{7857}{3485}$	$\frac{1680912}{3485}$
6	0	0	5	0	6	−19	23	33	−21	0	$\frac{1153}{3485}$	0	$-\frac{1408}{3485}$	$-\frac{7713}{3485}$	$\frac{1680768}{3485}$
7	0	0	6	0	4	18	18	38	−22	0	$\frac{5}{17}$	0	$-\frac{256}{697}$	$-\frac{37}{17}$	$\frac{336128}{697}$
8	0	0	7	0	2	−17	13	43	−23	0	$\frac{8201}{31365}$	0	$-\frac{256}{765}$	$-\frac{67241}{31365}$	$\frac{368896}{765}$
9	0	0	8	0	0	−16	8	48	−24	0	$\frac{21872}{94095}$	0	$-\frac{28672}{94095}$	$-\frac{198992}{94095}$	$\frac{45371392}{94095}$
10	0	1	0	1	14	−15	39	5	−13	0	$\frac{33}{111520}$	0	$-\frac{8193}{111520}$	$-\frac{216513}{111520}$	$\frac{53754273}{111520}$
11	0	1	1	1	12	−14	34	10	−14	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
12	0	1	2	1	10	−13	29	15	−15	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
13	0	1	3	1	8	−12	24	20	−16	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
14	0	1	4	1	6	−11	19	25	−17	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
15	0	1	5	1	4	−10	14	30	−18	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
16	0	1	6	1	2	−9	9	35	−19	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
17	0	1	7	1	0	−8	4	40	−20	0	$\frac{8}{31365}$	0	$-\frac{2048}{31365}$	$-\frac{59048}{31365}$	$\frac{15116288}{31365}$
18	0	2	0	2	12	−6	30	2	−10	0	$\frac{3}{13940}$	0	$-\frac{1023}{13940}$	$-\frac{19683}{13940}$	$\frac{6711903}{13940}$
19	0	2	1	2	10	−5	25	7	−11	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
20	0	2	2	2	8	−4	20	12	−12	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
21	0	2	3	2	6	−3	15	17	−13	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
22	0	2	4	2	4	−2	10	22	−14	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
23	0	2	5	2	2	−1	5	27	−15	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
24	0	2	6	2	0	0	0	32	−16	0	0	0	0	$-\frac{32}{17}$	$\frac{8192}{17}$
25	0	3	0	3	10	3	21	−1	−7	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
26	0	3	1	3	8	4	16	4	−8	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
27	0	3	2	3	6	5	11	9	-9	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
28	0	3	3	3	4	6	6	14	-10	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
29	0	3	4	3	2	7	1	19	-11	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
30	0	3	5	3	0	8	-4	24	-12	0	$-\frac{8}{3485}$	0	$-\frac{2048}{3485}$	$-\frac{6552}{3485}$	$\frac{1677312}{3485}$
31	0	4	0	4	8	12	12	-4	-4	0	$-\frac{3}{697}$	0	$-\frac{48}{697}$	$\frac{19683}{697}$	$\frac{19683}{6970}$
32	0	4	1	4	6	13	7	1	-5	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
33	0	4	2	4	4	14	2	6	-6	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
34	0	4	3	4	2	15	-3	11	-7	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
35	0	4	4	4	0	16	-8	16	-8	0	$-\frac{16}{697}$	0	$\frac{4096}{697}$	$-\frac{1296}{697}$	$\frac{331776}{697}$
36	0	5	0	5	6	21	3	-7	-1	0	$\frac{129}{3485}$	0	$-\frac{384}{3485}$	$-\frac{846369}{3485}$	$\frac{2519424}{3485}$
37	0	5	1	5	4	22	-2	-2	-2	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
38	0	5	2	5	2	23	-7	3	-3	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
39	0	5	3	5	0	24	-12	8	-4	0	$-\frac{728}{3485}$	0	$\frac{186368}{3485}$	$-\frac{5832}{3485}$	$\frac{1492992}{3485}$
40	0	6	0	6	4	30	-6	-10	2	0	$-\frac{1023}{3485}$	0	$\frac{768}{3485}$	$\frac{6711903}{3485}$	$-\frac{5038848}{3485}$
41	0	6	1	6	2	31	-11	-5	1	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
42	0	6	2	6	0	32	-16	0	0	0	$-\frac{32}{7}$	0	$\frac{8192}{17}$	0	0
43	0	7	0	7	2	39	-15	-13	5	0	$\frac{8193}{3485}$	0	$-\frac{8448}{3485}$	$-\frac{53754273}{3485}$	$\frac{55427328}{3485}$
44	0	7	1	7	0	40	-20	-8	4	0	$-\frac{59048}{3485}$	0	$\frac{15116288}{3485}$	$\frac{52488}{3485}$	$-\frac{13436928}{3485}$
45	0	8	0	8	0	48	-24	-16	8	0	$-\frac{596976}{3485}$	0	$\frac{136114176}{3485}$	$\frac{430506576}{3485}$	$-\frac{564350976}{3485}$
46	2	0	0	0	14	-20	40	4	-8	0	$\frac{7381}{111520}$	0	$-\frac{7381}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
47	2	0	1	0	12	-19	35	9	-9	0	$\frac{6561}{111520}$	0	$-\frac{6561}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
48	2	0	2	0	10	-18	30	14	-10	0	$\frac{729}{13940}$	0	$-\frac{729}{13940}$	$-\frac{729}{13940}$	$\frac{729}{13940}$
49	2	0	3	0	8	-17	25	19	-11	0	$\frac{162}{3485}$	0	$-\frac{162}{3485}$	$-\frac{162}{3485}$	$\frac{162}{3485}$
50	2	0	4	0	6	-16	20	24	-12	0	$\frac{144}{3485}$	0	$-\frac{144}{3485}$	$-\frac{144}{3485}$	$\frac{144}{3485}$
51	2	0	5	0	4	-15	15	29	-13	0	$\frac{128}{3485}$	0	$-\frac{128}{3485}$	$-\frac{128}{3485}$	$\frac{128}{3485}$
52	2	0	6	0	2	-14	10	34	-14	0	$\frac{1024}{31365}$	0	$-\frac{1024}{31365}$	$-\frac{1024}{31365}$	$\frac{1024}{31365}$
53	2	0	7	0	0	-13	5	39	-15	0	$\frac{2731}{94095}$	0	$-\frac{2816}{94095}$	$-\frac{2731}{94095}$	$\frac{2816}{94095}$

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
54	2	1	0	1	12	−11	31	1	−5	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
55	2	1	6	1	0	−5	1	31	−11	0	$\frac{1}{31365}$	0	$-\frac{256}{31365}$	$-\frac{1}{31365}$	$\frac{256}{31365}$
56	2	2	0	2	10	−2	22	−2	−2	0	$-\frac{1}{13940}$	0	$\frac{1}{13940}$	$\frac{6561}{13940}$	$-\frac{6561}{13940}$
57	2	2	5	2	0	3	−3	23	−7	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{1}{3485}$	$\frac{256}{3485}$
58	2	3	0	3	8	7	13	−5	1	0	$\frac{2}{3485}$	0	$-\frac{2}{3485}$	$-\frac{13122}{3485}$	$\frac{13122}{3485}$
59	2	3	4	3	0	11	−7	15	−3	0	$\frac{9}{3485}$	0	$-\frac{2304}{3485}$	$-\frac{9}{3485}$	$\frac{2304}{3485}$
60	2	4	0	4	6	16	4	−8	4	0	$-\frac{16}{3485}$	0	$\frac{16}{3485}$	$\frac{104976}{3485}$	$-\frac{104976}{3485}$
61	2	4	3	4	0	19	−11	7	1	0	$\frac{81}{3485}$	0	$-\frac{20736}{3485}$	$-\frac{81}{3485}$	$\frac{20736}{3485}$
62	2	5	0	5	4	25	−5	−11	7	0	$\frac{128}{3485}$	0	$-\frac{128}{3485}$	$-\frac{839808}{3485}$	$\frac{839808}{3485}$
63	2	5	2	5	0	27	−15	−1	5	0	$\frac{729}{3485}$	0	$-\frac{186624}{3485}$	$-\frac{729}{3485}$	$\frac{186624}{3485}$
64	2	6	0	6	2	34	−14	−14	10	0	$-\frac{1024}{3485}$	0	$\frac{1024}{3485}$	$\frac{6718464}{3485}$	$-\frac{6718464}{3485}$
65	2	6	1	6	0	35	−19	−9	9	0	$\frac{6561}{3485}$	0	$-\frac{1679616}{3485}$	$-\frac{6561}{3485}$	$\frac{1679616}{3485}$
66	2	7	0	7	0	43	−23	−17	13	0	$\frac{67241}{3485}$	0	$-\frac{368896}{85}$	$-\frac{53806761}{3485}$	$\frac{1679616}{3485}$
67	4	0	0	0	12	−16	32	0	0	0	$\frac{1}{136}$	0	$\frac{1}{136}$	0	0
68	4	0	1	0	10	−15	27	5	−1	0	$\frac{729}{111520}$	0	$-\frac{729}{111520}$	$-\frac{729}{111520}$	$\frac{729}{111520}$
69	4	0	2	0	8	−14	22	10	−2	0	$\frac{81}{13940}$	0	$-\frac{81}{13940}$	$-\frac{81}{13940}$	$\frac{81}{13940}$
70	4	0	3	0	6	−13	17	15	−3	0	$\frac{18}{3485}$	0	$-\frac{18}{3485}$	$-\frac{18}{3485}$	$\frac{18}{3485}$
71	4	0	4	0	4	−12	12	20	−4	0	$\frac{16}{3485}$	0	$-\frac{16}{3485}$	$-\frac{16}{3485}$	$\frac{16}{3485}$
72	4	0	5	0	2	−11	7	25	−5	0	$\frac{128}{31365}$	0	$-\frac{128}{31365}$	$-\frac{128}{31365}$	$\frac{128}{31365}$
73	4	0	6	0	0	−10	2	30	−6	0	$\frac{341}{94095}$	0	$-\frac{256}{94095}$	$-\frac{341}{94095}$	$\frac{256}{94095}$
74	4	1	0	1	10	−7	23	−3	3	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
75	4	1	5	1	0	−2	−2	22	−2	0	$-\frac{1}{31365}$	0	$\frac{256}{31365}$	$\frac{1}{31365}$	$-\frac{256}{31365}$
76	4	2	0	2	8	2	14	−6	6	0	$-\frac{1}{13940}$	0	$\frac{1}{13940}$	$\frac{6561}{13940}$	$-\frac{6561}{13940}$
77	4	2	4	2	0	6	−6	14	2	0	$-\frac{1}{3485}$	0	$\frac{256}{3485}$	$\frac{1}{3485}$	$-\frac{256}{3485}$
78	4	3	0	3	6	11	5	−9	9	0	$\frac{2}{3485}$	0	$-\frac{2}{3485}$	$-\frac{13122}{3485}$	$\frac{13122}{3485}$
79	4	3	3	3	0	14	−10	6	6	0	$-\frac{9}{3485}$	0	$\frac{2304}{3485}$	$\frac{9}{3485}$	$-\frac{2304}{3485}$
80	4	4	0	4	4	20	−4	−12	12	0	$-\frac{16}{3485}$	0	$\frac{16}{3485}$	$\frac{104976}{3485}$	$-\frac{104976}{3485}$

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
81	4	4	2	4	0	22	−14	−2	10	0	$-\frac{81}{3485}$	0	$\frac{20736}{3485}$	$\frac{81}{3485}$	$-\frac{20736}{3485}$
82	4	5	0	5	2	29	−13	−15	15	0	$\frac{128}{3485}$	0	$-\frac{128}{3485}$	$-\frac{839808}{3485}$	$\frac{839808}{3485}$
83	4	5	1	5	0	30	−18	−10	14	0	$-\frac{729}{3485}$	0	$\frac{186624}{3485}$	$\frac{729}{3485}$	$-\frac{186624}{3485}$
84	4	6	0	6	0	38	−22	−18	18	0	$-\frac{37}{17}$	0	$\frac{336128}{697}$	$\frac{32805}{17}$	$-\frac{1679616}{697}$
85	6	0	0	0	10	−12	24	−4	8	0	$\frac{91}{111520}$	0	$-\frac{91}{111520}$	$\frac{729}{111520}$	$-\frac{729}{111520}$
86	6	0	1	0	8	−11	19	1	7	0	$\frac{81}{111520}$	0	$-\frac{81}{111520}$	$-\frac{81}{111520}$	$\frac{81}{111520}$
87	6	0	2	0	6	−10	14	6	6	0	$\frac{9}{13940}$	0	$-\frac{9}{13940}$	$-\frac{9}{13940}$	$\frac{9}{13940}$
88	6	0	3	0	4	−9	9	11	5	0	$\frac{2}{3485}$	0	$-\frac{2}{3485}$	$-\frac{2}{3485}$	$\frac{2}{3485}$
89	6	0	4	0	2	−8	4	16	4	0	$\frac{16}{31365}$	0	$-\frac{16}{31365}$	$-\frac{16}{31365}$	$\frac{16}{31365}$
90	6	0	5	0	0	−7	−1	21	3	0	$\frac{43}{94095}$	0	$-\frac{128}{94095}$	$-\frac{43}{94095}$	$\frac{128}{94095}$
91	6	1	0	1	8	−3	15	−7	11	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
92	6	1	4	1	0	1	−5	13	7	0	$\frac{1}{31365}$	0	$-\frac{256}{31365}$	$-\frac{1}{31365}$	$\frac{256}{31365}$
93	6	2	0	2	6	6	6	−10	14	0	$-\frac{1}{13940}$	0	$\frac{1}{13940}$	$\frac{6561}{13940}$	$-\frac{6561}{13940}$
94	6	2	3	2	0	9	−9	5	11	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{1}{3485}$	$\frac{256}{3485}$
95	6	3	0	3	4	15	−3	−13	17	0	$\frac{2}{3485}$	0	$-\frac{2}{3485}$	$-\frac{13122}{3485}$	$\frac{13122}{3485}$
96	6	3	2	3	0	17	−13	−3	15	0	$\frac{9}{3485}$	0	$-\frac{2304}{3485}$	$-\frac{9}{3485}$	$\frac{2304}{3485}$
97	6	4	0	4	2	24	−12	−16	20	0	$-\frac{16}{3485}$	0	$\frac{16}{3485}$	$\frac{104976}{3485}$	$\frac{104976}{3485}$
98	6	4	1	4	0	25	−17	−11	19	0	$\frac{81}{3485}$	0	$-\frac{20736}{3485}$	$-\frac{81}{3485}$	$\frac{20736}{3485}$
99	8	0	0	0	8	−8	16	−8	16	0	$-\frac{1}{11152}$	0	$\frac{1}{11152}$	$\frac{81}{11152}$	$-\frac{81}{11152}$
100	6	5	0	5	0	33	−21	−19	23	0	$\frac{857}{3485}$	0	$-\frac{186752}{3485}$	$-\frac{840537}{3485}$	$\frac{1026432}{3485}$
101	8	0	1	0	6	−7	11	−3	15	0	$\frac{9}{111520}$	0	$-\frac{9}{111520}$	$-\frac{9}{111520}$	$\frac{9}{111520}$
102	8	0	2	0	4	−6	6	2	14	0	$\frac{1}{13940}$	0	$-\frac{1}{13940}$	$-\frac{1}{13940}$	$\frac{1}{13940}$
103	8	0	3	0	2	−5	1	7	13	0	$\frac{2}{31365}$	0	$-\frac{2}{31365}$	$-\frac{2}{31365}$	$\frac{2}{31365}$
104	8	0	4	0	0	−4	−4	12	12	0	$\frac{1}{18819}$	0	$\frac{16}{18819}$	$-\frac{1}{18819}$	$-\frac{16}{18819}$
105	8	1	0	1	6	1	7	−11	19	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
106	8	1	3	1	0	4	−8	4	16	0	$-\frac{1}{31365}$	0	$\frac{256}{31365}$	$\frac{1}{31365}$	$-\frac{256}{31365}$
107	8	2	0	2	4	10	−2	−14	22	0	$-\frac{1}{13940}$	0	$\frac{1}{13940}$	$\frac{6561}{13940}$	$-\frac{6561}{13940}$

No	b_1	b_2	b_3	b_4	b_5	a_2	a_4	a_6	a_{12}	c_1	c_2	c_3	c_4	c_6	c_{12}
108	8	2	2	2	0	12	−12	−4	20	0	$-\frac{1}{3485}$	0	$\frac{256}{3485}$	$\frac{1}{3485}$	$-\frac{256}{3485}$
109	8	3	0	3	2	19	−11	−17	25	0	$\frac{2}{3485}$	0	$-\frac{2}{3485}$	$-\frac{13122}{3485}$	$\frac{13122}{3485}$
110	8	3	1	3	0	20	−16	−12	24	0	$-\frac{9}{3485}$	0	$\frac{2304}{3485}$	$\frac{9}{3485}$	$-\frac{2304}{3485}$
111	8	4	0	4	0	28	−20	−20	28	0	$-\frac{97}{3485}$	0	$\frac{20752}{3485}$	$\frac{105057}{3485}$	$-\frac{125712}{3485}$
112	10	0	1	0	4	−3	3	−7	23	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{1}{111520}$	$\frac{1}{111520}$
113	10	0	0	0	6	−4	8	−12	24	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$\frac{819}{111520}$	$-\frac{819}{111520}$
114	10	0	2	0	2	−2	−2	−2	22	0	$\frac{1}{125460}$	0	$-\frac{1}{125460}$	$-\frac{1}{125460}$	$\frac{1}{125460}$
115	10	0	3	0	0	−1	−7	3	21	0	$\frac{1}{94095}$	0	$-\frac{86}{94095}$	$-\frac{1}{94095}$	$\frac{86}{94095}$
116	10	1	0	1	4	5	−1	−15	22	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
117	10	1	2	1	0	7	−11	−5	25	0	$\frac{1}{31365}$	0	$-\frac{256}{31365}$	$-\frac{1}{31365}$	$\frac{256}{31365}$
118	10	2	0	2	2	14	−10	−18	30	0	$-\frac{1}{13940}$	0	$\frac{1}{13940}$	$\frac{6561}{13940}$	$-\frac{6561}{13940}$
119	10	2	1	2	0	15	−15	−13	29	0	$\frac{1}{3485}$	0	$-\frac{256}{3485}$	$-\frac{1}{3485}$	$\frac{256}{3485}$
120	10	3	0	3	0	23	−19	−21	33	0	$\frac{11}{3485}$	0	$-\frac{2306}{3485}$	$-\frac{13131}{3485}$	$\frac{15426}{3485}$
121	12	0	0	0	4	0	0	−16	32	0	0	0	0	$\frac{1}{136}$	$-\frac{1}{136}$
122	12	0	1	0	2	1	−5	−11	31	0	$\frac{1}{1003680}$	0	$-\frac{1}{1003680}$	$-\frac{1}{1003680}$	$\frac{1}{1003680}$
123	12	0	2	0	0	2	−10	−6	30	0	$-\frac{1}{376380}$	0	$\frac{341}{376380}$	$\frac{1}{376380}$	$-\frac{341}{376380}$
124	12	1	0	1	2	9	−9	−19	35	0	$\frac{1}{111520}$	0	$-\frac{1}{111520}$	$-\frac{6561}{111520}$	$\frac{6561}{111520}$
125	12	1	1	1	0	10	−14	−14	34	0	$-\frac{1}{31365}$	0	$\frac{256}{31365}$	$\frac{1}{31365}$	$-\frac{256}{31365}$
126	12	2	0	2	0	18	−18	−22	38	0	$-\frac{1}{2788}$	0	$\frac{5}{68}$	$\frac{1313}{2788}$	$-\frac{37}{68}$
127	14	0	0	0	2	4	−8	−20	40	0	$-\frac{1}{1003680}$	0	$\frac{1}{1003680}$	$\frac{7381}{1003680}$	$-\frac{7381}{1003680}$
128	14	0	1	0	0	5	−13	−15	39	0	$\frac{11}{3011040}$	0	$-\frac{2731}{3011040}$	$-\frac{11}{3011040}$	$\frac{2731}{3011040}$
129	14	1	0	1	0	13	−17	−23	43	0	$\frac{1}{24480}$	0	$-\frac{8201}{24480}$	$-\frac{1441}{24480}$	$\frac{67241}{24480}$
130	16	0	0	0	0	8	−16	−24	48	0	$-\frac{7}{1505520}$	0	$\frac{1367}{1505520}$	$\frac{11077}{1505520}$	$-\frac{12437}{1505520}$