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Weak Separation Axioms via e-$\mathscr{I}$- Sets in Ideal Topological Spaces

Wadei Faris Al-Omeri[1], M.S. Md. Noorani[1], A. AL-Omari[2], T. Noiri[3,*]

[1] *School of Mathematical Sciences, Faculty of Science and Technology Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia*
[2] *Department of Mathematics, Faculty of Science Al AL-Bayat University, P.O.Box 130095, Mafraq25113, Jordan*
[3] *2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken 869-5142, JAPAN.*

Abstract. In this paper, we use the notion of e-$\mathscr{I}$-open sets to introduce and define some new weak separation axioms. Also we study some of their basic properties. Additionally, we investigate the relationship and implications of these axioms among themselves and with other known axioms.

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1. Introduction

The notion of R_0 topological spaces is introduced by Shanin [15] in 1943. Later, Davis [4] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e.g. [6, 10, 13]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of R_1 topological spaces which are independent of both T_0 and T_1 but strictly weaker than T_2. A subset A of a space (X,τ) is said to be regular open (resp. regular closed) [16] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is said to be δ-open [18] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a δ-open set is said to be δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(U)) \cap A \neq \emptyset$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$ [18]. The set δ-interior of A [18] is the union of all regular open sets of X contained in A and is denoted by $Int_\delta(A)$. A is δ-open if $Int_\delta(A) = A$. The collection of all δ-open sets of (X,τ) is denoted by $\delta O(X)$ and forms a topology τ^δ.

*Corresponding author.

Email addresses: wadeimoon1@hotmail.com (W. Al-omeri), msn@ukm.my (M. Noorani), omarimutah1@yahoo.com (A. AL-Omari), t.noiri@nifty.com (T. Noiri)

An ideal $\mathscr{I}$ on a topological space $(X,\mathscr{I})$ is a nonempty collection of subsets of X which satisfies the following conditions: $A \in \mathscr{I}$ and $B \subset A$ implies $B \in \mathscr{I}$; $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. Applications to various fields were further investigated by Jankovic and Hamlett [11]; Dontchev [5]; Mukherjee *et al.* [12]; Arenas *et al.* [3]; Nasef and Mahmoud [14], etc. Given a topological space $(X,\mathscr{I})$ with an ideal $\mathscr{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [11, 17] of A with respect to τ and $\mathscr{I}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathscr{I},\tau) = \{x \in X \mid U \cap A \notin \mathscr{I} \text{ for every } U \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. Furthermore $Cl^*(A) = A \cup A^*(\mathscr{I},\tau)$ defines a Kuratowski closure operator for the topology τ^*. When there is no chance for confusion, we will simply write A^* for $A^*(\mathscr{I},\tau)$. X^* is often a proper subset of X. By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X,τ), respectively. A subset A of a topological space (X,τ) is said to be e-open [9] if $A \subset Int(\delta Cl(A)) \cup Cl(\delta Int(A))$. The notion of e-open sets has been study extensively in recent years by many topologists. In this paper, we use the notion of e-$\mathscr{I}$-open sets to introduce and define some new weak separation axioms. Also we study some of their basic properties. Additionally, we investigate the relationship and implications of these axioms among themselves and with other known axioms.

2. Preliminaries

A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is said to be e-$\mathscr{I}$-open [2] if

$$A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)).$$

The complement of an e-$\mathscr{I}$-open set is called an e-$\mathscr{I}$-closed set [2]. The intersection of all e-$\mathscr{I}$-closed sets containing A is called the e-$\mathscr{I}$-closure of A and is denoted by $Cl_e^*(A)$. The e-$\mathscr{I}$-interior of A is defined by the union of all e-$\mathscr{I}$-open sets contained in A and is denoted by $Int_e^*(A)$. The family of all e-$\mathscr{I}$-open (resp. e-$\mathscr{I}$-closed) sets of $(X,\tau,\mathscr{I})$ containing a point $x \in X$ is denoted by $E\mathscr{I}O(X,x)$ (resp. $E\mathscr{I}C(X,x)$). A subset U of X is called an e-$\mathscr{I}$-neighborhood of a point $x \in X$ if there exists an e-$\mathscr{I}$-open set V of $(X,\tau,\mathscr{I})$ such that $x \in V \subset U$. A function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma)$ is said to be e-$\mathscr{I}$-continuous if $f^{-1}(V) \in E\mathscr{I}O(X)$ for every open set V of Y.

Definition 1. *A topological space (X,τ) is said to be:*

(i) R_0 *[4] if every open set contains the closure of each of its singletons.*

(ii) R_1 *[4] if for x, y in X with $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\}) \subset U$ and $Cl(\{y\} \subset V$.*

Definition 2. *A topological space (X,τ) is said to be:*

(i) e-T_1 *[7, 8] if for each pair of distinct points x and y in X, there exist e-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.*

(ii) *e-T_2 [7, 8] if for each pair of distinct points x and y in X, there exist disjoint e-open sets U and V such that $x \in U$ and $y \in V$.*

Definition 3. *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(i) *e-$\mathscr{I}$-T_1 [1] if for each pair of distinct points x and y in X, there exist e-$\mathscr{I}$-open sets U and V of X, such that $x \in U$ and $y \notin U$, $y \in V$ and $x \notin V$.*

(ii) *e-$\mathscr{I}$-T_2 [1] if for each pair of distinct points x and y in X, there exist disjoint e-$\mathscr{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.*

3. On e-$\mathscr{I}$-R_0 Spaces

Definition 4. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subset X$. Then the e-$\mathscr{I}$-kernel of A, denoted by $\mathscr{I}_e Ker(A)$, is defined to be the set $\mathscr{I}_e Ker(A) = \cap\{G \in E\mathscr{I}O(X) | A \subset G\}$.*

Lemma 1. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $x, y \in X$. Then, $y \in \mathscr{I}_e Ker(\{x\})$ if and only if $x \in Cl_e^*(\{y\})$.*

Proof. Suppose that $y \notin \mathscr{I}_e Ker(\{x\})$. Then there exists $U \in E\mathscr{I}O(X, x)$ such that $y \notin U$. Therefore, we have $x \notin Cl_e^*(\{y\})$. The proof of the converse case can be done similarly. □

Lemma 2. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and S a subset of X. Then, $\mathscr{I}_e Ker(S) = \{x \in X | Cl_e^*(\{x\}) \cap S \neq \emptyset\}$.*

Proof. Let $x \in \mathscr{I}_e Ker(S)$. Suppose that $Cl_e^*(\{x\}) \cap S = \emptyset$. Hence $x \notin X \backslash Cl_e^*(\{x\})$ which is an e-$\mathscr{I}$-open set containing S. Since $x \notin \mathscr{I}_e Ker(S)$, this is a contradiction. Hence $Cl_e^*(\{x\}) \cap S \neq \emptyset$. Conversely, suppose that $Cl_e^*(\{x\}) \cap S \neq \emptyset$. Next, let $x \in X$ such that $Cl_e^*(\{x\}) \cap S \neq \emptyset$ and suppose that $x \notin \mathscr{I}_e Ker(S)$. Then, there exists an e-$\mathscr{I}$-open set U containing S and $x \notin U$. Let $y \in Cl_e^*(\{x\}) \cap S$. Hence, U is an e-$\mathscr{I}$-neighborhood of y which does not contains x. By this contradiction $x \in \mathscr{I}_e Ker(S)$ and hence the claim. □

Definition 5. *An ideal topological space $(X, \tau, \mathscr{I})$ is called an e-$\mathscr{I}$-R_0 space if every e-$\mathscr{I}$-open set contains the e-$\mathscr{I}$-closure of each of its singletons.*

Definition 6. *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be e-$\mathscr{I}$-T_0 if for each pair of distinct points x and y in X, there exists an e-$\mathscr{I}$-open set U such that $x \in U$ and $y \notin U$, or there exists an e-$\mathscr{I}$-open set V such that $y \in V$ and $x \notin V$.*

Theorem 1. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then X is e-$\mathscr{I}$-T_1 if and only if it is e-$\mathscr{I}$-T_0 and e-$\mathscr{I}$-R_0.*

Proof. Let X be an e-$\mathscr{I}$-T_1 space. By the definition of an e-$\mathscr{I}$-T_1 space, it is an e-$\mathscr{I}$-T_0 and e-$\mathscr{I}$-R_0 space.

Conversely, let X be an e-$\mathscr{I}$-T_0 and e-$\mathscr{I}$-R_0 space. Let x, y be any two distinct points of X. Since X is e-$\mathscr{I}$-T_0, then there exists an e-$\mathscr{I}$-open set U such that $x \in U$ and $y \notin U$ or there exists an e-$\mathscr{I}$-open set V such that $y \in V$ and $x \notin V$. Let $x \in U$ and $y \notin U$. Since X is e-$\mathscr{I}$-R_0, then $Cl_e^*(x) \subset U$. We have $y \notin U$ and then $y \notin Cl_e^*(x)$. We obtain $y \in X \backslash Cl_e^*(x)$. Take $S = X \backslash Cl_e^*(x)$. Thus, U and S are e-$\mathscr{I}$-open sets containing x and y, respectively, such that $y \notin U$ and $x \notin S$. Hence, X is e-$\mathscr{I}$-T_1. □

Remark 1. *Since an ideal topological space $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-T_1 if and only if the singletons are e-$\mathscr{I}$-closed, it is clear that every e-$\mathscr{I}$-T_1 space e-$\mathscr{I}$-R_0. But the converse is not true in general.*

Example 1. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\mathscr{I} = \{\varnothing, \{c\}, \{b\}, \{b, c\}\}$. Since e-$\mathscr{I}$-open=$\{\phi, X, \{a\}, \{b, c\}\}$. It is clear that every e-$\mathscr{I}$-open set contains the e-$\mathscr{I}$-closure of each of its singletons so the ideal topological space is e-$\mathscr{I}$-R_0, but none of e-$\mathscr{I}$-T_0 and e-$\mathscr{I}$-T_1.*

Remark 2. *The following example and Example 1 show that the notions e-$\mathscr{I}$-T_0-ness and e-$\mathscr{I}$-R_0-ness are independent.*

Example 2. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}\}$ and $\mathscr{I} = \{\varnothing, \{a\}\}$. Now, we determine e-$\mathscr{I}$-open=$\{\phi, X, \{a\}\}$. Then $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-T_0 but it is not e-$\mathscr{I}$-R_0.*

Lemma 3. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then for any points x and y in X, the following statements are equivalent:*

(i) $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$.

(ii) $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$.

Proof. $(i) \Rightarrow (ii)$: Let $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$, then there exists a point k in X such that $k \in \mathscr{I}_e Ker(\{x\})$ and $k \notin \mathscr{I}_e Ker(\{y\})$. By Lemma 1, $x \in Cl_e^*(\{x\})$ and $y \notin Cl_e^*(\{x\})$. Therefore, $Cl_e^*(\{x\}) \subset Cl_e^*(Cl_e^*(\{k\})) = Cl_e^*(\{k\})$ and hence $y \notin Cl_e^*(\{x\})$. Hence $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. By using $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$, we obtain $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$.
$(ii) \Rightarrow (i)$: Let $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$, then there exists a point k in X such that $k \in Cl_e^*(\{x\})$ and $k \notin Cl_e^*(\{y\})$ and then there exists an e-$\mathscr{I}$-open set containing k and therefore x but not y, namely, $y \notin \mathscr{I}_e Ker(\{x\})$ and thus $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$. □

Proposition 1. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is an e-$\mathscr{I}$-R_0 space,*

(ii) *For any $K \in E\mathscr{I}C(X)$, $x \notin K$ implies $K \subset U$ and $x \notin U$ for some $U \in E\mathscr{I}O(X)$,*

(iii) *For any $K \in E\mathscr{I}C(X)$, $x \notin K$ implies $K \cap Cl_e^*(\{x\}) = \emptyset$,*

(iv) *For any distinct points x and y of X, either $Cl_e^*(\{x\}) = Cl_e^*(\{y\})$ or $Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$.*

Proof. $(i) \Rightarrow (ii)$: Let $K \in E\mathscr{I}C(X)$ and $x \notin K$. Then by (i), $Cl_e^*(\{x\}) \subset X \backslash K$. Set $U = X \backslash Cl_e^*(\{x\})$, then $U \in E\mathscr{I}O(X)$, $K \subset U$ and $x \notin U$.
$(ii) \Rightarrow (iii)$: Let $K \in E\mathscr{I}C(X)$ and $x \notin K$. There exists $U \in E\mathscr{I}O(X)$ such that $K \subset U$ and

$x \notin U$. Since $U \in E\mathscr{I}O(X)$, $U \cap Cl_e^*(\{x\}) = \emptyset$ and $K \cap Cl_e^*(\{x\}) = \emptyset$.
$(iii) \Rightarrow (iv)$: Suppose that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$ for distinct points $x, y \in X$. There exists $k \in Cl_e^*(\{x\})$ such that $k \notin Cl_e^*(\{y\})$ (or $k \in Cl_e^*(\{y\})$ such that $k \notin Cl_e^*(\{x\})$. There exists $V \in E\mathscr{I}O(X)$ such that $y \notin V$ and $k \in V$; hence $x \in V$. Therefore, we have $x \notin Cl_e^*(\{y\})$. By (iii), we obtain $Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$. The proof for otherwise is similar.
$(iv) \Rightarrow (i)$: Let $V \in E\mathscr{I}O(X, x)$. For each $y \notin V$, $x \neq y$ and $x \notin Cl_e^*(\{y\})$. This shows that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. By (iv), $Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$ for each $y \in X \backslash V$ and hence $Cl_e^*(\{x\}) \cap (\cup_{y \in X \backslash V} Cl_e^*(\{y\})) = \emptyset$. On the other hand, since $V \in E\mathscr{I}O(X)$ and $y \in X \backslash V$, we have $Cl_e^*(\{y\}) \subset X \backslash V$ and hence $X \backslash V = \cup_{y \in X \backslash V} Cl_e^*(\{y\})$. Therefore, we obtain
$(X \backslash V) \cap Cl_e^*(\{x\}) = \emptyset$ and $Cl_e^*(\{x\}) \subset V$. This shows that $(X, \tau, \mathscr{I})$ is an *e*-$\mathscr{I}$-R_0 space. □

Theorem 2. *An ideal topological space* $(X, \tau, \mathscr{I})$ *is e-*$\mathscr{I}$*-*R_0 *space if and only if for any* x *and* y *in* X*,* $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$ *implies* $Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$*.*

Proof. Let $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_0. By Proposition 1, we obtain the assertion. Conversely, let $V \in E\mathscr{I}O(X; x)$. We will show that $Cl_e^*(\{x\}) \subset V$. Let $y \in X \backslash V$. Then $x \neq y$ and
$x \notin Cl_e^*(\{y\})$. This shows that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. By assumption, $Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$. Hence $y \notin Cl_e^*(\{x\})$ and therefore $Cl_e^*(\{x\}) \subset V$. □

Theorem 3. *Let* $(X, \tau, \mathscr{I})$ *be an ideal topological space. Then the following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is an e-*$\mathscr{I}$*-*R_0 *space,*

(ii) $x \in Cl_e^*(\{y\})$ *if and only if* $y \in Cl_e^*(\{x\})$ *for any points* x *and* y *in* X*.*

Proof. $(i) \Rightarrow (ii)$: Assume that $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_0. Let $x \in Cl_e^*(\{y\})$ and $A \in E\mathscr{I}O(X, y)$. Now by hypothesis, $x \in Cl_e^*(\{y\}) \subset A$ and $x \in A$. Therefore, every *e*-$\mathscr{I}$-open set containing y contains x. Hence $y \in Cl_e^*(\{x\})$.
$(ii) \Rightarrow (i)$: Let $U \in E\mathscr{I}O(X, x)$. If $y \notin U$, then $x \notin Cl_e^*(\{y\})$ and hence $y \notin Cl_e^*(\{x\})$. This implies that $Cl_e^*(\{x\}) \subset U$. Hence $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_0 □

Theorem 4. *For an ideal topological space* $(X, \tau, \mathscr{I})$*, the following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is an e-*$\mathscr{I}$*-*R_0 *space;*

(ii) For any nonempty set S *of* X *and any* $G \in E\mathscr{I}O(X)$ *such that* $S \cap G \neq \emptyset$*, there exists* $K \in E\mathscr{I}C(X)$ *such that* $S \cap K \neq \emptyset$ *and* $K \subset G$*;*

(iii) For any $G \in E\mathscr{I}O(X)$*,* $G = \cup\{K \in E\mathscr{I}C(X) | K \subset G\}$*;*

(iv) For any $K \in E\mathscr{I}C(X)$*,* $K = \cap\{G \in E\mathscr{I}O(X) | K \subset G\}$*;*

(v) For any $x \in X$*,* $Cl_e^*(\{x\}) \subset \mathscr{I}_e Ker(\{x\})$*.*

Proof. $(i) \Rightarrow (ii)$:Let S be a nonempty set of X and $G \in E\mathscr{I}O(X)$ such that $S \cap G \neq \emptyset$. There exists $x \in S \cap G$. Since $x \in G \in E\mathscr{I}O(X)$, it follows that $Cl_e^*(\{x\}) \subset G$. Take $K = Cl_e^*(\{x\})$, then $K \in E\mathscr{I}C(X)$, $K \subset G$ and $S \cap K \neq \emptyset$.
$(ii) \Rightarrow (iii)$: Let $G \in E\mathscr{I}O(X)$. We have $G \supset \cup\{K \in E\mathscr{I}C(X) | K \subset G\}$. Let x be any point of G. By (ii) there exists $K \in E\mathscr{I}C(X)$ such that $x \in K$ and $K \subset G$. Thus, we have
$x \in K \subset \cup\{K \in E\mathscr{I}C(X) | K \subset G\}$ and hence $G = \cup\{K \in E\mathscr{I}C(X) | K \subset G\}$.
$(iii) \Rightarrow (iv)$: This is obvious.
$(iv) \Rightarrow (v)$: Let x be any point of X and $y \notin \mathscr{I}_e Ker(\{x\})$. There exists $V \in E\mathscr{I}O(X)$ such that $x \in V$ and $y \notin V$; hence $Cl_e^*(\{y\}) \cap V = \emptyset$. By (iv), $[\cap\{G \in E\mathscr{I}O(X) | Cl_e^*(\{y\}) \subset G\}] \cap V = \emptyset$ and there exists $G \in E\mathscr{I}O(X)$ such that $x \notin G$ and $Cl_e^*(\{y\}) \subset G$. Hence, $Cl_e^*(\{x\}) \cap G = \emptyset$ and $y \notin Cl_e^*(\{x\})$. Thus, $Cl_e^*(\{x\}) \subset \mathscr{I}_e Ker(\{x\})$.
$(v) \Rightarrow (i)$: Let $G \in E\mathscr{I}O(X)$ and $x \in G$. Let $y \in \mathscr{I}_e Ker(\{x\})$. We have $x \in Cl_e^*(\{y\})$ and $y \in G$. It follows that $\mathscr{I}_e Ker(\{x\}) \subset G$. Thus, we obtain $x \in Cl_e^*(\{x\}) \subset \mathscr{I}_e Ker(\{x\}) \subset G$. This shows that $(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space. □

Theorem 5. *An ideal topological space $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_0 if and only if for any pair of points x and y in X, $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$ implies $\mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{y\}) = \emptyset$.*

Proof. Suppose that $(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space. Thus by Lemma 3, for any points x and y in X if $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$, then $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. Now we prove that $\mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{y\}) = \emptyset$. Assume that $z \in \mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{y\})$. By $z \in \mathscr{I}_e Ker(\{x\})$ and Lemma 1, it follows that $x \in Cl_e^*(\{z\})$. Since $x \in Cl_e^*(\{x\})$, by Theorem 2,
$Cl_e^*(\{x\}) = Cl_e^*(\{z\})$. Similarly, we have $Cl_e^*(\{x\}) = Cl_e^*(\{z\}) = Cl_e^*(\{y\})$. This is a contradiction. Therefore, we have $\mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{y\}) = \emptyset$. Conversely, let $(X, \tau, \mathscr{I})$ be an ideal topological space such that for any points x and y in X, $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$ implies $\mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{y\}) = \emptyset$. If $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$, then by Lemma 3,
$\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$. Hence, $\mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{y\}) = \emptyset$ which implies
$Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$. Because $z \in Cl_e^*(\{x\})$ implies that $x \in \mathscr{I}_e Ker(\{z\})$ and therefore $\mathscr{I}_e Ker(\{x\}) \cap \mathscr{I}_e Ker(\{z\}) \neq \emptyset$. By hypothesis, we have $\mathscr{I}_e Ker(\{x\}) = \mathscr{I}_e Ker(\{z\})$. Then $z \in Cl_e^*(\{x\}) \cap Cl_e^*(\{y\})$ implies that $\mathscr{I}_e Ker(\{x\}) = \mathscr{I}_e Ker(\{z\}) = \mathscr{I}_e Ker(\{y\})$. This is a contradiction. Therefore, $Cl_e^*(\{x\}) \cap Cl_e^*(\{y\}) = \emptyset$ and by Theorem 2 $(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space. □

Theorem 6. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(i) *$(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space,*

(ii) *If F is an e-$\mathscr{I}$-closed subset of X, then $F = \mathscr{I}_e Ker(F)$,*

(iii) *If F is an e-$\mathscr{I}$-closed subset of X and $x \in F$, then $\mathscr{I}_e Ker(\{x\}) \subset F$,*

(iv) *If $x \in X$, then $\mathscr{I}_e Ker(\{x\}) \subset Cl_e^*(\{x\})$.*

Proof. $(i) \Rightarrow (ii)$: Let F be an e-$\mathscr{I}$-closed subset of X and $x \notin F$. Thus $X \backslash F \in E\mathscr{I}O(X\,x)$. Since $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_0, $Cl_e^*(\{x\}) \subset X \backslash F$. Since $F \subset X \setminus Cl_e^*(\{x\})$, $\mathscr{I}_e Ker(F) \subset X - Cl_e^*(\{x\})$

and $x \notin \mathscr{I}_e Ker(F)$. Therefore, $\mathscr{I}_e Ker(F) = F$.
$(ii) \Rightarrow (iii)$: In general, $A \subset B$ implies $\mathscr{I}_e Ker(A) \subset \mathscr{I}_e Ker(B)$. Therefore, it follows from (ii) that $\mathscr{I}_e Ker(\{x\}) \subset \mathscr{I}_e Ker(F) = F$.
$(iii) \Rightarrow (iv)$: Since $x \in Cl_e^*(\{x\})$ and $Cl_e^*(\{x\})$ is e-$\mathscr{I}$-closed, by (iii) $\mathscr{I}_e Ker(\{x\}) \subset Cl_e^*(\{x\})$.
$(iv) \Rightarrow (i)$: We show the implication by using Theorem 3. Let $x \in Cl_e^*(\{y\})$. Then by Lemma 1 $y \in \mathscr{I}_e Ker(\{x\})$. By (iv), we obtain $y \in \mathscr{I}_e Ker(\{x\}) \subset Cl_e^*(\{x\})$. Therefore, $x \in Cl_e^*(\{y\})$ implies $y \in Cl_e^*(\{x\})$. The converse is obvious and $(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space. □

Corollary 1. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is an e-*$\mathscr{I}$*-*R_0 *space,*

(ii) $Cl_e^*(\{x\}) = \mathscr{I}_e Ker(\{x\})$ *for all* $x \in X$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space. By Theorem 4, $Cl_e^*(\{x\}) \subset \mathscr{I}_e Ker(\{x\})$ for each $x \in X$. By Theorem 6, $\mathscr{I}_e Ker(\{x\}) \subset Cl_e^*(\{x\})$. This shows that $Cl_e^*(\{x\}) = \mathscr{I}_e Ker(\{x\})$.
$(ii) \Rightarrow (i)$: This is obvious by Theorem 6. □

Corollary 2. *Let $(X, \tau, \mathscr{I})$ be e-$\mathscr{I}$-R_0 and $x \in X$. If $Cl_e^*(\{x\}) \cap \mathscr{I}_e Ker(\{x\}) = \{x\}$, then $\mathscr{I}_e Ker(\{x\}) = \{x\}$.*

Proof. The proof follows from Theorem 6 (iv). □

Definition 7. *A net $\{x_\lambda\}_{\lambda \in \wedge}$ is said to be e-$\mathscr{I}$-convergent to a point x in X, if for any $U \in E\mathscr{I}O(X, x)$, there exists $\lambda_0 \in \wedge$ such that $x_\lambda \in U$ for any $\lambda \in \wedge$ such that $\lambda \geq \lambda_o$.*

Lemma 4. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and let x and y be any two points in X such that every net in X e-$\mathscr{I}$-converging to y e-$\mathscr{I}$-converges to x. Then $x \in Cl_e^*(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $Cl_e^*(\{y\})$. Since $\{x_n\}_{n \in N}$ e-$\mathscr{I}$-converges to y, then $\{x_n\}_{n \in N}$ e-$\mathscr{I}$-converges to x and this implies that $x \in Cl_e^*(\{y\})$. □

Theorem 7. *For an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is an e-*$\mathscr{I}$*-*R_0 *space,*

(ii) If $x, y \in X$, then $y \in Cl_e^*(\{x\})$ if and only if every net in X e-$\mathscr{I}$-converging to y e-$\mathscr{I}$-converges to x.

Proof. $(i) \Rightarrow (ii)$: Let $x, y \in X$ such that $y \in Cl_e^*(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha \in N}$ be a net in X such that $\{x_\alpha\}_{\alpha \in N}$ e-$\mathscr{I}$-converges to y. Since $y \in Cl_e^*(\{x\})$, by Theorem 2 we have $Cl_e^*(\{x\}) = Cl_e^*(\{y\})$. Therefore $x \in Cl_e^*(\{y\})$. This means that $\{x_\alpha\}_{\alpha \in N}$ e-$\mathscr{I}$-converges to x. Conversely, let $x, y \in X$ such that every net in X e-$\mathscr{I}$-converging to y e-$\mathscr{I}$ converges to x. Then $x \in Cl_e^*(\{y\})$ by Lemma 4. By Theorem 2, we have $Cl_e^*(\{x\}) = Cl_e^*(\{y\})$. Therefore

$y \in Cl^*_e(\{x\})$.

$(ii) \Rightarrow (i)$: Assume that x and y are any two points of X such that $Cl^*_e(\{x\}) \cap Cl^*_e(\{y\}) \neq \emptyset$. Let $z \in Cl^*_e(\{x\}) \cap Cl^*_e(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha \in N}$ in $Cl^*_e(\{x\})$ such that $\{x_\alpha\}_{\alpha \in N}$ e-$\mathscr{I}$-converges to z. Since $z \in Cl^*_e(\{y\})$, then $\{x_\alpha\}_{\alpha \in N}$ e-$\mathscr{I}$-converges to y. It follows that $y \in Cl^*_e(\{x\})$. By the same token we obtain $x \in Cl^*_e(\{y\})$. Therefore $Cl^*_e(\{x\}) = Cl^*_e(\{y\})$ and by Theorem 2 $(X, \tau, \mathscr{I})$ is an e-$\mathscr{I}$-R_0 space. □

4. On e-$\mathscr{I}$-R_1 Spaces

Definition 8. *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be e-$\mathscr{I}$-R_1 if for x, y in X with $Cl^*_e(\{x\}) \neq Cl^*_e(\{y\})$, there exist disjoint e-$\mathscr{I}$-open sets U and V such that $Cl^*_e(\{x\})$ is a subset of U and $Cl^*_e(\{y\})$ is a subset of V.*

Proposition 2. *If $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_1, then it is e-$\mathscr{I}$-R_0.*

Proof. Let $U \in E\mathscr{I}O(X, x)$. If $y \notin U$, since $x \notin Cl^*_e(\{y\})$, we have $Cl^*_e(\{x\}) \neq Cl^*_e(\{y\})$. So, there exists an e-$\mathscr{I}$-open set V_y such that $Cl^*_e(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin Cl^*_e(\{x\})$. Thus $Cl^*_e(\{x\}) \subset U$. Therefore $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_0. □

Theorem 8. *An ideal topological space $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_1 if and only if for $x, y \in X$, $\mathscr{I}_e Ker(\{x\}) \neq \mathscr{I}_e Ker(\{y\})$, there exist disjoint e-$\mathscr{I}$-open sets U and V such that $Cl^*_e(\{x\}) \subset U$ and $Cl^*_e(\{y\}) \subset V$.*

Proof. It follows from Lemma 3. □

Remark 3. *In the following diagram we denote by arrows the implications between the separation axioms which we have introduced and discussed in this paper and examples show that no other implications hold between them:*

$$\begin{array}{ccc} R_1 & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ e\text{-}\mathscr{I}\text{-}R_1 & \longrightarrow & e\text{-}\mathscr{I}\text{-}R_0 \\ \downarrow & & \downarrow \\ e\text{-}R_1 & \longrightarrow & e\text{-}R_0 \end{array}$$

Example 3. *Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathscr{I} = \{\phi, \{b\}\}$. $E\mathscr{I}O = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_0 but not R_0 and e-$\mathscr{I}$-R_1.*

Example 4. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathscr{I} = \{\phi, \{a\}\}$. Since $E\mathscr{I}O = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_0 but not R_0.*

Example 5. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\phi, X, \{a, b\}\}$ and $\mathscr{I} = \{\phi, \{c\}\}$. Since $E\mathscr{I}O = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_1 but not R_1.*

Example 6. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\mathscr{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Since $E\mathscr{I}O = \{\phi, X, \{a\}\{a, b\}, \{a, c\}\}$ and e-open sets is $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $(X, \tau, \mathscr{I})$ is e-R_0 but not e-$\mathscr{I}$-R_o.*

Theorem 9. *The following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is e-$\mathscr{I}$-R_1,*

(ii) for each x, $y \in X$ one of the following holds:

- *If U is e-$\mathscr{I}$-open, then $x \in U$ if and only if $y \in U$,*
- *there exist disjoint e-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$.*

(iii) If $x, y \in X$ such that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$, then there exist e-$\mathscr{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.

Proof. $(i) \Rightarrow (ii)$: Let $x, y \in X$. Then $Cl_e^*(\{x\}) = Cl_e^*(\{y\})$ or $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. If $Cl_e^*(\{x\}) = Cl_e^*(\{y\})$ and U is *e*-$\mathscr{I}$-open, then $x \in U$ implies $y \in Cl_e^*(\{x\}) \subset U$ and $y \in U$ implies $x \in Cl_e^*(\{y\}) \subset U$. Thus consider the case that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. Then there exist disjoint *e*-$\mathscr{I}$-open sets U and V such that $x \in Cl_e^*(\{x\}) \subset U$ and $y \in Cl_e^*(\{y\}) \subset V$.
$(ii) \Rightarrow (iii)$: Let $x, y \in X$ such that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$. Then $x \notin Cl_e^*(\{y\})$ or $y \notin Cl_e^*(\{x\})$, say $x \notin Cl_e^*(\{y\})$. Then there exists an *e*-$\mathscr{I}$-open set A such that $x \in A$ and $y \notin A$, which implies there exist disjoint *e*-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$. Then $F_1 = X \setminus V$ and $F_2 = X \setminus U$ are *e*-$\mathscr{I}$-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.
$(iii) \Rightarrow (i)$: First, we show that $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_0. Let U be *e*-$\mathscr{I}$-open and let $x \in U$. Suppose that $Cl_e^*(\{x\}) \not\subset U$. Let $y \in Cl_e^*(\{x\}) \cap (X \setminus U)$. Then $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$ and there exist $F_1, F_2 \in E\mathscr{I}C(X)$ such that $x \in F_1$, $y \in F_2$, $y \notin F_1$, $x \notin F_2$, and $X = F_1 \cup F_2$. Then $y \in F_2 \setminus F_1 = X \setminus F_1$, which is *e*-$\mathscr{I}$-open, and $x \notin X \setminus F_1$, which is a contradiction. Hence, $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_0. To show X to be *e*-$\mathscr{I}$-R_1 assume that $a, b \in X$ such that $Cl_e^*(\{a\}) \neq Cl_e^*(\{b\})$. Then there exist $P_1, P_2 \in E\mathscr{I}C(X)$ such that $a \in P_1$, $b \notin P_1$, $a \notin P_2$, $b \in P_2$ and $X = P_1 \cup P_2$. Thus $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$, which are *e*-$\mathscr{I}$-open. This implies $Cl_e^*(\{a\}) \subset P_1 \setminus P_2 = X - P_2 \in E\mathscr{I}O(X)$ and $Cl_e^*(\{b\}) \subset P_2 \setminus P_1$. Thus, $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_1. □

Theorem 10. *The following properties are equivalent:*

(i) $(X, \tau, \mathscr{I})$ *is e-$\mathscr{I}$-T_2,*

(ii) $(X, \tau, \mathscr{I})$ *is e-$\mathscr{I}$-R_1 and e-$\mathscr{I}$-T_1,*

(iii) $(X, \tau, \mathscr{I})$ *is e-$\mathscr{I}$-R_1 and e-$\mathscr{I}$-T_0.*

Proof. $(i) \Rightarrow (ii)$: Since $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-T_1, then it is *e*-$\mathscr{I}$-T_1. If $x, y \in X$ such that $Cl_e^*(\{x\}) \neq Cl_e^*(\{y\})$, then $x \neq y$ and there exist disjoint *e*-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$. Therefore, $Cl_e^*(\{x\}) = \{x\} \subset U$ and $Cl_e^*(\{y\}) = \{y\} \subset V$. Hence $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-R_1.
$(ii) \Rightarrow (iii)$: Since $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-T_1, then $(X, \tau, \mathscr{I})$ is *e*-$\mathscr{I}$-T_0.

$(iii) \Rightarrow (i)$: Since $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-R_1, then $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-R_0 and e-$\mathscr{I}$-T_0 and hence by Theorem 1 $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-T_1. Let $x, y \in X$ such that $x \neq y$. Since $Cl^*_e(\{x\}) = \{x\} \neq \{y\} = Cl^*_e(\{y\})$, then there exist disjoint e-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$. Hence, $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-T_2. □

In view of Definition 3, it follows that

Theorem 11. *An ideal topological space $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-T_2 if and only if for $x, y \in X$ such that $x \neq y$, there exist e-$\mathscr{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.*

Remark 4. *Let $\{x_\lambda\}_{\lambda\in\lambda}$ be a net in $(X,\tau,\mathscr{I})$ and $e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in\lambda})$ denote $\{x \in X : \mathscr{I}-$ converges to $x\}$.*

Theorem 12. *The following properties are equivalent:*

(i) $(X,\tau,\mathscr{I})$ *is* e-$\mathscr{I}$-R_1,

(ii) *for* $x, y \in X$ $Cl^*_e(\{x\}) = Cl^*_e(\{y\})$, *whenever there exists a net* $\{x_\lambda\}_{\lambda\in A}$ *such that* $x, y \in e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A})$,

(iii) $(X,\tau,\mathscr{I})$ *is* e-$\mathscr{I}$-R_0, *and for every* e-$\mathscr{I}$-*convergent net* $\{x_\lambda\}_{\lambda\in A}$ *in* X, $e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A}) = Cl^*_e(\{x\})$ *for some* $x \in X$.

Proof. $(i) \Rightarrow (ii)$: Let $x, y \in X$ such that there exists a net $\{x_\lambda\}_{\lambda\in A}$ in X such that $x, y \in e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A})$. Then, by Theorem 9, (a) if U is e-$\mathscr{I}$-open, then $x \in U$ if and only if $y \in U$ or (b) there exist disjoint e-$\mathscr{I}$-open sets U and V such that $x \in U$ and $y \in V$. Since $x, y \in e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A})$, then (a) is satisfied, and we obtain $Cl^*_e(\{x\}) = Cl^*_e(\{y\})$.
$(ii) \Rightarrow (iii)$: Let $U \in E\mathscr{I}O(X,x)$. Let $y \notin U$. For each $n \in N$ let $x_n = x$. Then $\{x_n\}_{n\in N}$ e-$\mathscr{I}$-converges to x and since $Cl^*_e(\{x\}) \neq Cl^*_e(\{y\})$, by (ii) $\{x_n\}$ does not e-$\mathscr{I}$-converge to y and there exists $A \in E\mathscr{I}O(X)$ such that $y \in A$ and $x \notin A$. Thus, $y \notin Cl^*_e(\{x\})$ and $Cl^*_e(\{x\}) \subset U$. Hence $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-R_0. Let $\{x_\lambda\}_{\lambda\in A}$ be an e-$\mathscr{I}$-convergent net in X. Let $x \in X$ such that $\{x_\lambda\}_{\lambda\in A}$ e-$\mathscr{I}$-converges to x. If $y \in Cl^*_e(\{x\})$, then $\{x_\lambda\}_{\lambda\in A}$ e-$\mathscr{I}$-converges to y, which implies $Cl^*_e(\{x\}) \subset e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A})$. Let $y \in e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A})$, then $x, y \in e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A})$, which implies $y \in Cl^*_e(\{y\}) = Cl^*_e(\{x\})$. Hence $e\mathscr{I}lim(\{x_\lambda\}_{\lambda\in A}) = Cl^*_e(\{x\})$.
$(iii) \Rightarrow (i)$: Assume that $(X,\tau,\mathscr{I})$ is not e-$\mathscr{I}$-R_1. Then there exist $x, y \in X$ such that $Cl^*_e(\{x\}) \neq Cl^*_e(\{y\})$ and every e-$\mathscr{I}$-open set containing $Cl^*_e(\{x\})$ intersects every e-$\mathscr{I}$-open set containing $Cl^*_e(\{y\})$. Since $(X,\tau,\mathscr{I})$ is e-$\mathscr{I}$-R_0, then every e-$\mathscr{I}$-open set containing x contains $Cl^*_e(\{x\})$ and every e-$\mathscr{I}$-open set containing y contains $Cl^*_e(\{y\})$, which implies that every e-$\mathscr{I}$-open set containing x intersects every e-$\mathscr{I}$-open set containing y. Let $D_x = \{U \subset X | U \in E\mathscr{I}O(X,x)\}$. Let $\geq_x$ be the binary relation on D_x defined by $U_1 \geq_x U_2$ if and only if $U_1 \subset U_2$. Then, clearly $(D_x, \geq_x)$ is a directed set. Let $D_y = \{U \subset X | U \in E\mathscr{I}O(X,y)\}$ and let $\geq_y$ be the binary relation on Dy defined by $U_1 \geq_y U_2$ if and only if $U_1 \subset U_2$. Then, $(D_x, \geq_y)$ is also a directed set. Let $D = \{(U_1,U_2) | U_1 \in D_x$ and $U_2 \in D_y\}$ and let $\geq$ be the binary relation on D defined by $(U_1,U_2) \geq (V_1,V_2)$ if and only if $U_1 \geq_x V_1$ and $U_2 \geq_y V_2$. Then, $(D, \geq)$ is a directed set. For each $(U_1,U_2) \in D$, let $x_{(U_1,U_2)} \in (U_1,U_2)$.

Then $\{x_{(U_1,U_2)}\}_{(U_1,U_2)} \in D$ is a net in X that e-$\mathscr{I}$-converges to both x and y. Thus, there exists $z \in X$ such that $e\mathscr{I}lim(\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}) = Cl_e^*(\{z\})$, which implies $x, y \in Cl_e^*(\{z\})$. Since $\{Cl_e^*(\{w\}) : w \in X\}$ is a decomposition of X, then $Cl_e^*(\{x\}) = Cl_e^*(\{z\}) = Cl_e^*(\{y\})$, which is a contradiction. Hence $(X, \tau, \mathscr{I})$ is e-$\mathscr{I}$-R_1. □

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