EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 1, 2016, 114-124
ISSN 1307-5543 – www.ejpam.com

Solution of Burger's Equation in a One-Dimensional Groundwater Recharge by Spreading Using q-Homotopy Analysis Method

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Abstract. The ground water is recharged by spreading of the water in downward direction and the moisture content of soil increases. The mathematical formulation of the phenomena leads to the governing equation, which is a nonlinear partial differential equation in the form of Burger's equation which has been solved by using q-homotopy analysis method with appropriate initial and boundary conditions. The average diffusivity coefficient over the whole range of moisture content is regarded as constant. It is concluded that the moisture content of soil increases with the depth Z and increasing time T . The numerical solutions of the governing equation have been obtained in the form of tables and graphs by using Mathematica coding. The numerical solution represents moisture content distribution in the vertically downward direction at any depth Z for time $T > 0$. This type of problems appears particularly in soil mechanics, hydrology, ceramic engineering and petroleum technology.

2010 Mathematics Subject Classifications: 35Q53, 76S05, 65H20

Key Words and Phrases: Moisture Content, Unsaturated Porous Medium, Burger's Equation, q-homotopy analysis method

1. Introduction

The soil plays one of the most important roles in the hydrological cycle. Moisture content is the quantity of water contained in a material, such as soil (called soil moisture). The saturated zone is one in which the entire void space is occupied by water. In the unsaturated zone only part of the void space is occupied by water. The phreatic surface (or water table), is an imaginary surface that bounds the saturated zone from above. It separates the saturated and unsaturated zones. In the dry soil there is no moisture, so the value of moisture content is 0 in the unsaturated porous medium and its value is 1, when the porous medium is fully saturated by water. The range of moisture content is $[0, 1]$. The region of the unsaturated

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soil is known as Vadose zone (or unsaturated zone), and this is the region where the most interesting nonlinear hysteretic behaviour is observed. Just above the water table the void space is practically fully saturated for a certain distance. This region is called the capillary fringe, or capillary zone. The term moisture content is used in hydrogeology, soil sciences and soil Mechanics. In saturated ground water aquifers, all available pore spaces are filled with water. Above a capillary fringe pore spaces have air in them too. If the porous medium is soil, water content is same as the soil moisture. The role of the unsaturated zone is clearly depicted in Fig. 1 that describes the hydrological cycle. In typical soil profiles some distance separates the earth's surface from the water table, which is the upper limit of completely water-saturated soil. In this inverting zone the water saturation varies between 0 and 1 the rest of the pore space normally being occupied by air. Water flow in this unsaturated zone is complicated by the fact that the soil's permeability to water depends on its water saturation [1].

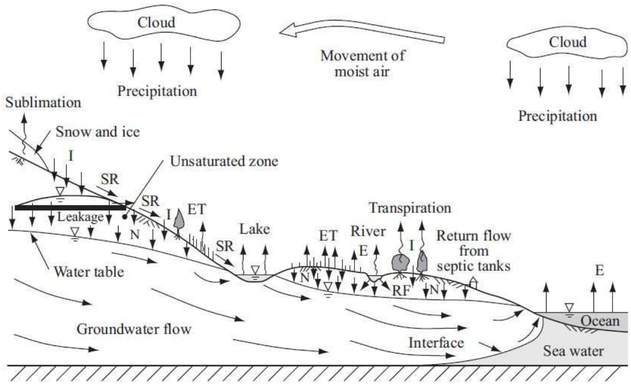


Figure 1: The hydrological cycle [8]

The flow of water through soil in many practical situations is unsteady and slightly saturated. It is unsteady because the moisture content changes as a function of time. It is slightly saturated because all the pore spaces are not completely filed with flowing liquid. Examples of such flows are the infiltration of water through the ground surface, the flow through the capillary fringe of an unconfined aquifer, draining soils, evaporation from an aquifer close to the ground surface, ground water level fluctuations, the inflow of water from irrigation channels and the underground disposal of sewage and waste. Since it is important to know the water content of the soil in these flows, solutions to the equation describing such flows are very useful to many branches of engineering, such as civil, hydrologic, sanitary and irrigation.

The phenomenon of the one dimensional vertical groundwater recharge by spreading is of great importance to hydrologists, agriculturists and for the people related with water resources sciences. The hydrological situation of such problem is confirmed by [21]. The water infiltration system, seepage problem and the underground disposal of wastewater, all harmonize with the flow discussed in the present paper.

This phenomenon has been discussed by many researchers from the different viewpoints; here are some examples. Klute [7], has reduced diffusion equation to an ordinary differential equation and employed a forward integration and iteration method. Other researchers have examined diffusion equation with a linear diffusion coefficient. Graphical illustration of the mathematical solution for horizontal water function described in [15]. Verma [20] has employed Laplace transformation technique to solve this problem; Mehta [12] has obtained an approximate solution considering average diffusivity coefficient of the whole range of moisture content and treated as small constant by the method of singular perturbation technique. Allen [1] had reviewed modification, assumptions and different techniques used by other researchers. Prasad *et al.* [4] had developed a numerical model to simulate water flow through unsaturated zones and study the effect of unsaturated soil parameters on water movement during different processes such as gravity drainage and infiltration. They had developed a numerical model to simulate moisture flow through unsaturated zones using the finite element method. This model is also applied to predict moisture contents during a field internal drainage test. De Vries and Simmers [18] had discussed processes and challenges principally on recharge of unconfined aquifers, then the most readily available and affordable source of water in (semi-unconfined aquifers), (semi-) arid regions. Faybishenko [3] has given review of the theoretical concepts, presented results, and provided perspectives on investigations of flow and transport in unsaturated heterogeneous soils and fractured rock, using the methods of nonlinear dynamics and determine chaos. Mehta and Patel [13] have studied ground water recharge by spreading in vertical downward direction. They constitute governing differential equation as Burgers equation, with permeability as nonlinear function of moisture content. Mishra and Verma [19] obtained a similarity solution of a unidimensional vertical ground water recharges through porous media. Mehta and Yadav [14] have considered aqueous conductivity directly proportional to depth, moisture content and inversely proportional to time. They obtained an approximate solution for the vertical groundwater recharge problem in slightly saturated porous media by using small parameter method.

The moisture in wet porous media migrates generally due to the following causes: the total pressure gradient, the moisture content gradient, and the temperature gradient. Under negligible total pressure gradient or in a medium with poor permeability, the moisture migrations caused by the later two causes are prevailing. [11] and [6] discussed it from different point of view.

In the present paper the mathematical formulation of the moisture content phenomena yields a non-linear partial differential equation in the form of Burger's equation, its analytical solution has been obtained by using q-homotopy analysis method [17]. The diffusivity coefficient is assumed to be constant over a whole range of moisture content and permeability of the porous media is assumed to be varying directly to the square of the moisture content [21].

2. Statement of the Problem

In the investigated model it is considered that the ground water recharge takes place over a large basin (taken as a homogeneous porous medium) of such geological configuration that the sides are limited by rigid boundaries while the bottom is confined by a thick layer of water table. Under these circumstances, water, from the spreading grounds, will flow vertically downwards through the unsaturated porous medium (negligible amount of water may spread in other directions; hence, it is ignored comparing to the large size of basin). That is initial saturation is very small but not-zero [2, 16]. It is assumed that the diffusivity coefficient is equivalent to its average value over the whole range of moisture content; it is small enough constant and, regarded as a perturbation parameter. Further the permeability of the medium is considered to vary directly as square of the moisture content. For the present analysis the following assumptions have been made [4]: The medium is homogenous. There is no air resistance to the flow (i.e. the porous medium contains only the flowing liquid water and empty voids). The air in the void space is approximately at atmospheric pressure i.e. air is stationary. The soil properties are taken to be constant. The flowing liquid (water) is considered continuous at a microscopic level, incompressible and isothermal. The initial moisture content is uniform throughout the soil profile and the moisture content at the soil surface is constant and near saturation also rainfall or irrigation rate is constant. Darcy's law is applicable.

3. Mathematical Formulation

The motion of water in isotropic homogeneous medium is given by Darcy's law as [2],

$$V = -K.\nabla\phi, \tag{1}$$

where V is volume flux of moisture content, K is coefficients of aqueous conductivity and $\nabla\phi$ is gradient of the whole (total) moisture potential.

The motion of water flow through unsaturated porous media is governed by continuity equation

$$\frac{\partial(\rho_S\theta)}{\partial t} = -\nabla M, \tag{2}$$

where ρ_S is the bulk density of medium on dry weight basis, θ is the moisture content at any depth Z on a dry weight basis and M is a mass of flux of moisture at any time $t \geq 0$. Here, moisture content $\theta = \rho_S$ [5], where ρ is the porosity of the medium and S is the saturation of the water.

Using incompressibility of the water from the equation (1) and (2),

$$\frac{\partial(\rho_S\theta)}{\partial t} = -\nabla(\rho V) = \nabla(\rho K \nabla\phi), \tag{3}$$

where ρ is the flux density of the medium.

Since, in the present problem, flow takes place only in the vertical direction, therefore (3) reduces to

$$\rho_S\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(\rho K\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\rho K g, \tag{4}$$

where ψ is the pressure (capillary) potential, g is the gravitation constant and $\phi = \psi - zg$ the positive direction of z—axis is the same as that of gravity.

Considering ψ and ϕ to be connected by a single valued function, equation (4) may be written as [2],

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right) - \frac{\rho}{\rho_S}g.\frac{\partial K}{\partial Z}, \tag{5}$$

where $D = \frac{\rho}{\rho_S} K \frac{\partial \psi}{\partial \theta} = \frac{\rho}{\rho_S}$ is called the diffusivity coefficients.

Replacing D by its average value D_a over the whole range of the moisture content [12] and $K \propto \theta^2$ [13].

i.e. $K = K_0 \theta^2$, where K_0 is constant. Hence equation (5) becomes

$$\frac{\partial \theta}{\partial t} + \frac{\rho}{\rho_S} 2 g K_0 \theta \frac{\partial \theta}{\partial z} = D_a \frac{\partial^2 \theta}{\partial Z^2}. \tag{6}$$

Substituting $\frac{\rho}{\rho_S} 2 g K_0 = K_1$, equation (6) becomes

$$\frac{\partial \theta}{\partial t} + K_1 \theta \frac{\partial \theta}{\partial z} = D_a \frac{\partial^2 \theta}{\partial Z^2} \tag{7}$$

For the sake of simplicity of the problem, the value of the constant $K_1 = 1$ is considered.

We choose new variable as,

$$Z = \frac{z}{L} \ \ \& \ \ T = \frac{t D_a}{L^2}, 0 \leq Z \leq 1, 0 \leq T \leq 1. \tag{8}$$

Hence the equation (7) can be written as

$$\frac{\partial \theta}{\partial T} + \theta \frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial Z^2}, \ 0 \leq Z \leq 1 \tag{9}$$

The equation (9) is the governing non-linear partial differential equation known as Burger's equation for the moisture content distribution phenomenon. Consider θ_0 as the initial moisture content. Since, the moisture content of the soil increases as the depth Z increases [11] considered the initial moisture content as exponential function of Z, so it is appropriate to choose initial condition for this phenomenon as,

$$\theta(Z, 0) = \theta_0(Z) = Z.e^{Z-1}, 0 \leq Z \leq 1. \tag{10}$$

Also the moisture content at the top (i.e. depth $Z = 0$) in the unsaturated porous media is very small and at the bottom the moisture content of soil will be fully saturated. So it is suitable to choose the boundary condition as

$$\begin{aligned} \theta(0, T) &= 0.01 = \theta_0 \\ \theta(1, T) &= 1. \end{aligned} \tag{11}$$

4. Solution of the Problem

In this section, we employ the q-homotopy analysis method [17] to solve the governing equation (9) with the appropriate conditions (10).

Let us consider equation (9) as,

$$N[\theta(Z, T; q)] = 0, \tag{12}$$

where N is a nonlinear operator and $q \in [0, \frac{1}{n}]$ is an embedding parameter.

Let us consider the auxiliary linear operator as $L = \frac{\partial}{\partial T}$ with $L[c_1] = 0$, where c_1 is arbitrary constant.

We construct a so-called zero order deformation equation for the moisture content distribution phenomenon as [10],

$$(1-nq)L[\theta(Z,T;q)-\theta_0(Z,T)] = qhH(Z,T)N[\theta(Z,T;q)], \tag{13}$$

where $h \neq 0$ is an auxiliary parameter, $H(Z,T) \neq 0$ is an auxiliary function & L is an auxiliary linear operator with the property $L[\theta(Z,T;q)] = 0$ when $\theta(Z,T;q) = 0$.

In equation (13), if we take $q = 0$ then we get

$$\theta(Z,T;0) = \theta_0(Z,T), \tag{14}$$

and if we take $q = \frac{1}{n}$, we get,

$$N[\theta(Z,T;\frac{1}{n})] = 0 \Rightarrow \theta(Z,T,\frac{1}{n}) = \theta(Z;T). \tag{15}$$

Thus as q varies from 0 to $\frac{1}{n}$, solution $\theta(Z;T;q)$ varies from initial condition $\theta_0(Z,T)$ to $\theta(Z,T)$.

We can write [10]

$$\theta(Z,T;q) = \theta(Z,T;0) + \sum_{m=1}^{\infty} \theta_m(Z,T) q^m, \tag{16}$$

where

$$\theta_m(Z,T) = \frac{1}{m!}\frac{\partial}{\partial q^m}\theta(Z,T;q)]_{q=0}. \tag{17}$$

As suggested in [9] and [10], if we choose properly the auxiliary linear operator L, initial guess θ_0, the auxiliary parameter h and the auxiliary function $H(Z,T)$ then equation (16) is converges at $q = \frac{1}{n}$.

Hence we have,

$$\theta(Z,T) = \theta_0(Z,T) + \sum_{m=1}^{\infty} \theta_m(Z,T)\left(\frac{1}{n}\right)^m. \tag{18}$$

This is one of the solution of equation (9) for the moisture content distribution phenomenon.

Define the vector

$$\overrightarrow{\theta}_n = \{\theta_0, \theta_1, \ldots, \theta_n\}. \tag{19}$$

Differentiate equation (13) m times with respect to embedding parameter q and then setting $q = 0$ and divide them by $m!$, we get so-called m^{th}-order deformation equation as,

$$L[\theta_m(Z,T) - \chi_m \theta_{m-1}(Z,T)] = qhH(Z,T)R_m[\overrightarrow{\theta}_{m-1}(Z,T)], \tag{20}$$

where

$$R_m[\overrightarrow{\theta}_{m-1}(Z,T)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\theta(Z,T;q)]}{\partial q^{m-1}}]_{q=0}$$
$$= \frac{\partial\theta_{m-1}}{\theta T} + \sum_{K=0}^{m-1}\theta_K\frac{\partial\theta_{m-1-k}}{\partial Z} - \frac{\partial^2\theta_{m-1}}{\partial Z^2},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ n, & \text{otherwise} \end{cases} \tag{21}$$

Choosing $H(Z,T) = 1$ in equation (20), the solution of the equation (9) for $m > 1$ becomes

$$\theta_m(Z,T) = \chi_m\theta_{m-1}(Z,T) + hL^{-1}[R_m(\overrightarrow{\theta}_{m-1}(Z,T))] \tag{22}$$

Using $q-$homotopy analysis method, we obtain components of the solution successively by Mathematica coding as follows,

$$\theta_1(Z,T) = e^{-2+Z}hT(e^Z Z(1+Z) - e(2+Z))$$
$$\theta_2(Z,T) = \frac{1}{2}e^{-3+Z}hT(e^{2Z}hTZ(2+6Z+3Z^2) + e^2(-2n(2+Z) + h(-2(2+Z) + T(4+Z))) + 2e^{1+Z}(nZ(1+Z) + h(Z(1+Z) - T(4+9Z+3Z^2)))$$

Similarly, $\theta_m(Z,T)$ for $m = 3,4,\ldots$ can be obtained.

The series solution expression by $q-$HAM can be written in the form

$$\theta(Z,T) = Ze^{Z-1} + e^{-2+Z}hT(e^Z Z(1+Z) - e(2+Z))(\frac{1}{n}) + \frac{1}{2}e^{-3+Z}hT(e^{2Z}hTZ(2+6Z+3Z^2) + e^2(-2n(2+Z) + h(-2(2+Z) + T(4+Z))) + 2e^{1+Z}(nZ(1+Z) + h(Z(1+Z) - T(4+9Z+3Z^2)))((\frac{1}{n})^2) + \ldots \tag{23}$$

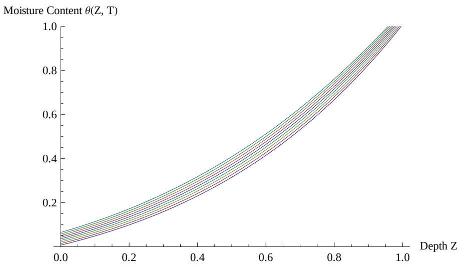
Equation (23) is an approximate solution of (9) in terms of convergence parameter h and n.

5. Numerical and Graphical Representation of the Solution

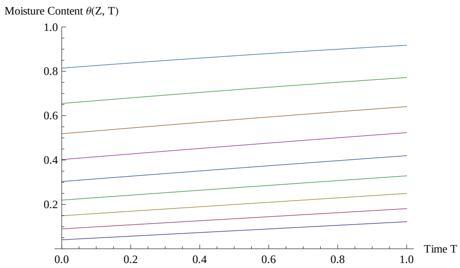
In this section, we have presented graphs of solution to Equation (9) using Mathematica with different values of n and testing the effect of large n. It should be observed from the graphs that for the larger values of n the moisture content $\theta(Z,T)$ at different depth Z and time levels T almost co-linear due to the presence of fraction factor $\left(\frac{1}{n}\right)^m$ in the solution.

Table 1: Moisture content $\theta(Z,T)$ for the depth Z at different time levels T.

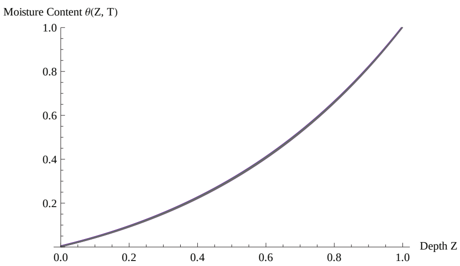
T	Z	Moisture Content $\theta(Z,T)$ HAM ($n=1$)	q-HAM ($n=100$)
0.2	0.1	0.05695	0.04082
	0.2	0.10818	0.09005
	0.3	0.16931	0.14919
	0.4	0.24181	0.21945
	0.5	0.32728	0.30351
	0.6	0.42751	0.40245
	0.7	0.54448	0.51884
	0.8	0.68037	0.65525
	0.9	0.83758	0.81460
0.4	0.1	0.07326	0.04099
	0.2	0.12644	0.09024
	0.3	0.18954	0.14939
	0.4	0.26390	0.21998
	0.5	0.35096	0.30376
	0.6	0.45231	0.40272
	0.7	0.56966	0.51911
	0.8	0.70478	0.65552
	0.9	0.85953	0.81484
0.6	0.1	0.08957	0.04116
	0.2	0.14466	0.09043
	0.3	0.20965	0.14960
	0.4	0.28578	0.22021
	0.5	0.37428	0.30401
	0.6	0.47661	0.40298
	0.7	0.59411	0.51938
	0.8	0.72826	0.65578
	0.9	0.88020	0.81509
0.8	0.1	0.10588	0.04153
	0.2	0.16284	0.09062
	0.3	0.22964	0.14981
	0.4	0.30740	0.22044
	0.5	0.39727	0.30426
	0.6	0.50039	0.40324
	0.7	0.65284	0.51965
	0.8	0.75064	0.65605
	0.9	0.89958	0.81533



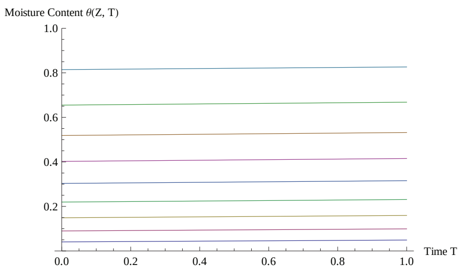
(a) vs Depth ($h = -0.05, n = 1$)



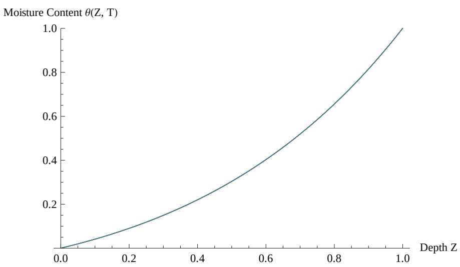
(b) vs Time ($h = -0.05, n = 1$)



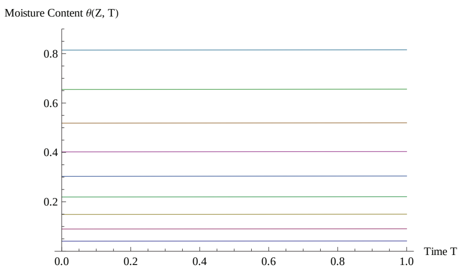
(c) vs Depth ($h = -0.05, n = 10$)



(d) vs Time ($h = -0.05, n = 10$)



(e) vs Depth ($h = -0.05, n = 100$)



(f) vs Time ($h = -0.05, n = 100$)

Figure 2: Moisture Content vs. Depth and Time

6. Conclusion

The solution of the specific one-dimensional vertical groundwater recharge by spreading (moisture content phenomena) in unsaturated homogeneous porous medium, given by the Burger's equation (9) has been obtained by using q-homotopy analysis method, with the appropriate initial and boundary conditions of the expressions (10) and (11). Fig. 2a to Fig. 2f) clearly indicate that the moisture content distribution in the one-dimensional fluid flow through unsaturated homogeneous porous medium is steadily increasing for different time $T > 0$. Figs. 2b, 2d and 2f represents the moisture content $\theta(Z,T)$ versus time T for $0 \leq T \leq 1$ and for the fixed depth $Z = 0.1, 0.2, \ldots, 0.9$. The nature of the graphs reflect that the height of the groundwater recharge by spreading in unsaturated homogeneous porous medium is increasing as per Figs. 2a, 2c and 2e according to the physical phenomenon throughout the domain for the depth $0 \leq Z \leq 1$ and the fixed time levels taken as $T = 0.1, 0.2, 0.3, \ldots, 0.9$.

From Table 1 and Fig. 2, it is evident that the moisture content distribution in unsaturated homogeneous porous medium steadily and uniformly increases due to the groundwater recharge by spreading with the increasing depth (vertical height Z) as well as time T . It is shown by the graphs and table for the different time levels T and the given depth $0 \leq Z \leq 1$. Thus, the solution obtained by the q-homotopy analysis method is converging to 1 as Z tending to 1.

ACKNOWLEDGEMENTS The authors would like to express their deep gratitude to Dr. J. Andrew Howe, Editor of "European Journal of Pure and Applied Mathematics" for kind cooperation, smooth behavior during communication and for their efforts to send the reports of the manuscript timely.

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