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Analyzing Group Theoretical Properties of Groups with Order 24 by GAP

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Abstract. We will present our implementation `fgroup.gi` and present subgroup lattice of some groups of order 24. We also will construct tables containing those groups basic properties and generators elements of Frattini and Fitting subgroups.

2010 Mathematics Subject Classifications: 20D05, 20D25, 20D30

Key Words and Phrases: Subgroup lattice, frattini subgroup, fitting subgroup.

1. Introduction

The link between group theory and GAP is constructed by Cayley's theorem which states that every group is isomorphic to a subgroup of a permutation group. If G is a group with order n, then we can define an isomorphism from G into the symmetric group S_n by Cayley's theorem. GAP generates all finite order groups using the symmetric group operations, and gives concrete examples to the abstract notions of algebra, in other words, it visualizes these abstract notions. Some applications of GAP to abstract algebra can be found in [1, 3].

Since most of the cohomological invariants are (directly) related to nilpotency. It is well known that the Frattini and Fitting subgroups are related to nilpotency. In the sense of this relation we compute the generator elements of these special subgroups. It is one of the fundamental property of algebraic structures such as groups, algebras, algebraic models etc.

The goals of this paper are:

* Presenting our implementation `fgroup.gi`
* Constructing tables consisting of some basic properties and generator elements of Fitting, Frattini subgroups of a group of order 24.
* Constructing the subgroup lattices of these groups.

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2. Preliminaries

In GAP programming, the following functions are frequently used for constructing groups and obtain some of their properties.

```
Group
GeneratorsOfGroup
ConjugateGroup
IsGroup
IsCyclic
IsAbelian
Elements
```

Example 1. *The cyclic group C_4 and its properties can be obtained by the following GAP commands,*

```
gap> G:=Group((1,2,3,4));
Group([ (1,2,3,4) ])
gap> Size(G);
4
gap> IsCyclic(G);
true
gap> IsAbelian(G);
true
```

The function `Subgroup()` is used for obtain subgroups. Below we give the useful GAP commands related with subgroups and cosets.

```
Subgroup
Index
IsSubgroup
IsNormal
ConjugateSubgroup
RightCoset
IsRightCoset
ConjugacyClasses
```

Example 2. *The following coding can be used for constructing subgroups.*

```
gap> G:=Group((1,2,3,4),(1,2,3));
Group([ (1,2,3,4), (1,2,3) ])
gap> Subgroup(G,[(1,2,3,4)]);
Group([ (1,2,3,4) ])
gap> H:=Subgroup(G,[(1,2,3,4)]);
Group([ (1,2,3,4) ])
```

Also the conjugacy classes obtained by the function `ConjugacyClasses()` and the function `NrConjugacyClasses()` is used for finding the number of conjugacy classes of the group.

```
gap> G:=Group((1,2,3,4,5));
Group([ (1,2,3,4,5) ])
gap> ConjugacyClasses(G);
[()^G,(1,2,3,4,5)^G,(1,3,5,2,4)^G,(1,4,2,5,3)^G,
(1,5,4,3,2)^G ]
gap> NrConjugacyClasses(G);
5
```

The functions `FactorGroup()` and `FactorGroupNC()` are used for constructing quotient groups and their enumerations. Here we will construct all quotient groups of a group by using second author's implementation in [3] which we called as `fgroup.gi`.

```
gap> Read("fgroup.gi");
gap> G:=Group((1,2,3),(1,3));
Group([ (1,2,3), (1,3) ])
gap> IsAbelian(G);
false
gap> IsCyclic(G);
false
gap> Fgroup(G);
Group( [ (1,2,3), (1,3) ] ) has 3 factor groups.
These :
[Group([  ]), Group([ f1 ]), Group([ (1,2,3), (1,3) ])]
```

Finally, in GAP, a homomorphism between groups is defined as follows

```
gap> G:=Group((1,2,3)(4,5));
Group([ (1,2,3)(4,5) ])
gap> genG:=GeneratorsOfGroup(G);
[ (1,2,3)(4,5) ]
gap> H:=Group((1,2)(3,4));
Group([ (1,2)(3,4) ])
gap> hom:=GroupHomomorphismByImages(G,H,genG,[(1,2)]);
[ (1,2,3)(4,5) ] -> [ (1,2) ]
gap> IsGroupHomomorphism(hom);
true
```

It is well known that for a given group G and its normal subgroup H, we have G/H is abelian if and only if $[G,G] \subseteq H$, where $[G,G]$ is the commutator subgroup of G. So, $[G,G]$ is the smallest normal subgroup which makes the quotient abelian. Consequently, we have the following example.

Example 3. *We will find the commutator subgroup of S_8 and look its normality and show that $S_8/[S_8,S_8]$ is the largest quotient group in the abelian qoutients of S_8.*

```
gap> G:=SymmetricGroup(8);
Sym( [ 1 .. 8 ] )
gap> Size(G);
40320
gap> K:=DerivedSubgroup(G);
Group([ (1,3,2), (1,4,3), (1,4,5), (1,5,6),
(1,4,3,6,7), (1,6,8)(3,4)(5,7) ])
gap> IsNormal(G,K);
true
gap> F:=FactorGroup(G,K);
Group([ f1 ])
gap> IsAbelian(F);
true
gap> Fgroup(G);
SymmetricGroup( [ 1 .. 8 ] ) has 3 factor groups.
These :
[ Sym( [ 1 .. 8 ] ), Group([ f1 ]), Group([  ]) ]
```

Definition 1. *The largest nilpotent normal subgroup of a group G, is called the Fitting subgroup, denoted $\mathbf{F}(G)$. The Frattini subgroup of a group G, denoted $\Phi(G)$, is the intersection of all maximal subgroups of G. Of course, $\Phi(G)$ is characteristic, and hence normal in G, and it is nilpotent. It follows that for any finite group G, we have $\Phi(G) \leq \mathbf{F}(G)$.*

We refer [4] and [2] for details about the Fitting and Frattini subgroups.

Theorem 1 (Frattini Argument)**.** *Let $N_G(P)$ is normaliser of P in G, $N \lhd G$ and suppose that $P \in Syl_p(N)$. Then $G = N_G(P)N$.*

Theorem 2. *Let $\Phi(G) \leq N \lhd G$ and suppose that $N/\Phi(G)$ is nilpotent. Then N is nilpotent.*

Proof. To show that N is nilpotent, we prove that each of its Sylow subgroups is normal. For this purpose, we let $P \in Syl_p(N)$ and we note that $P\Phi(G)/\Phi(G)$ is a Sylow p-sugroup of $N/\Phi(G)$. But $N/\Phi(G)$ is assumed to be nilpotent, and thus its Sylow subgroups are normal and we have $P\Phi(G)/\Phi(G) \lhd N/\Phi(G)$. Thus, in fact, $P\Phi(G)/\Phi(G)$ is actually characteristic in $N/\Phi(G)$ and since $N/\Phi(G) \lhd G/\Phi(G)$, we deduce that $P\Phi(G)/\Phi(G) \lhd G/\Phi(G)$. This gives $P\Phi(G) \lhd G$.

Since P is Sylow in N, it is also Sylow in $P\Phi(G)$. Since the latter subgroup is normal in G, we can apply the Frattini argument to deduce that $N_G(P)P\Phi(G) = G$. Since $P \leq N_G(P)$, however, this yields $N_G(P)\Phi(G) = G$. It follows from this that $N_G(P) = G$ (Otherwise, $N_G(P)$ would be contained in some maximal subgroup of G, which also contains $\Phi(G)$, and this would contradict the fact that $N_G(P)\Phi(G) = G$). We now have $N_G(P) = G$, and thus $P \lhd G$. In particular, $P \lhd N$ as desired. □

Example 4. *The Fitting and Frattini subgroups of the Dihedral group with order 28 can be obtained by GAP as follows:*

```
gap> D:=DihedralGroup(28);
<pc group of size 28 with 3 generators>
gap> FittingSubgroup(D);
Group([ f2*f3^3, f3^5 ])
gap> F1:=FittingSubgroup(D);
Group([ f2*f3^3, f3^5 ])
gap> Size(F);
14
gap> F2:=FrattiniSubgroup(D);
Group([  ])
gap> G:=DihedralGroup(28);
gap> Ns := NormalSubgroups(D);
[ Group([  ]), Group([ f2*f3^3 ]), Group([ f3 ]),
  Group([ f1*f2, f3 ]), Group([ f1, f3 ]), Group([ f2, f3 ]),
  <pc group of size 28 with 3 generators> ]
gap> N := Ns[2];;
gap> FF:=FactorGroup(N,F);
Group([ f2*f3^3 ])
gap> IsNilpotentGroup(FF);
true
gap> IsNilpotentGroup(N);
true
```

2.1. Constructing Subgroup Lattices by GAP

Finding subgroups and classification of group properties of a group has significant importance for analyzing its relations to compare with other groups.

In this section we will look how GAP is used for classifying subgroups. For example subgroup lattice of Klein 4-group is as shown in Figure 1 [5].

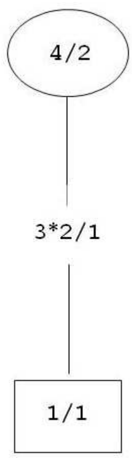

Figure 1: Subgroup Lattice of Klein 4-group.

This means that the group has five subgroups where three of them are trivial and others are 2/1 type, i.e., cyclic group of order 2. These subgroups can be find by using GAP as follows:

```
gap> k4:=Group((1,2),(3,4));
Group([ (1,2), (3,4) ])
gap> l:=LatticeSubgroups(k4);
<subgroup lattice of Group([ (1,2), (3,4) ]),
5 classes, 5 subgroups>
gap> IsAbelian(k4);
true
gap> ConjugacyClassesSubgroups(l);
[ Group( () )^G, Group( [ (3,4) ] )^G, Group( [ (1,2) ] )^G,
  Group( [ (1,2)(3,4) ] )^G, Group( [ (3,4), (1,2) ] )^G ]
gap> M:=MaximalSubgroupsLattice(l);
[ [  ], [ [ 1, 1 ] ], [ [ 1, 1 ] ], [ [ 1, 1 ] ],
  [ [ 4, 1 ], [ 3, 1 ], [ 2, 1 ] ] ]
gap> M[5];
[ [ 4, 1 ], [ 3, 1 ], [ 2, 1 ] ]
gap> u1:=Representative(ConjugacyClassesSubgroups(l)[5]);
Group([ (3,4), (1,2) ])
gap> u2:=ClassElementLattice(ConjugacyClassesSubgroups(l)[4],1);;
gap> u3:=ClassElementLattice(ConjugacyClassesSubgroups(l)[3],1);;
gap> u4:=ClassElementLattice(ConjugacyClassesSubgroups(l)[2],1);;
gap> IsSubgroup(u1,u2);IsSubgroup(u1,u3);IsSubgroup(u1,u4);
true
true
true
```

2.2. Examples

In this section, up to GAP order, we will give tables containing generators of (Frattini, Fitting) subgroups, abelianess property of 14 different groups of order 24. In all cases, the table headings are as defined in Table 1. Also, we give the subgroup lattices near the tables.

Table 1: Group & Generator Properties

Group T	Type of Group	NS	Number of Subgroups
Group N	Name of Group	NNS	Number of Normal Subgroups
GAP T	Type of GAP	FiG	Generators of Fitting Subgroup
GAP N	GAP Name of Group	FrG	Generators of Frattini Subgroup
Deg.	Degree of Group	Gen.	Generators of Group
Ab.	Group is Abelian		

Group T	24/2
Group N	$C_2 \times C_{12}$
GAP T	24/2
GAP N	$c12c2$
Deg.	24
Ab.	Abelian
NS	16
NNS	16
FiG	$(7,9,8),(3,5)(4,6),(3,6,5,4),(1,2)$
FrG	$(3,5)(4,6)$
Gen.	$(1,2),(3,4,5,6)(7,8,9)$

Table 2: Table of $C_2 \times C_{12}$

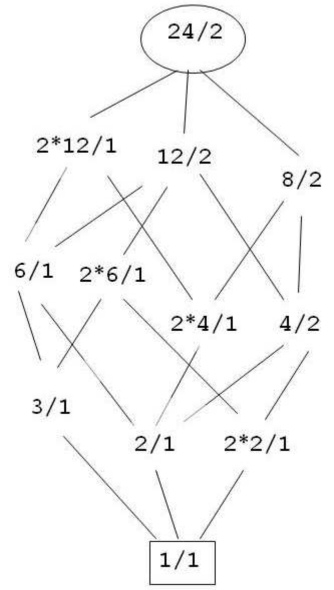

Figure 2: Subgroup lattice of $C_2 \times C_{12}$

Group T	24/3
Group N	$C_6 \times C_2^2$
GAP T	24/1
GAP N	$c6k4$
Deg.	24
Ab.	Abelian
NS	32
NNS	32
FiG	$(8,9),(6,7),(4,5),(1,2,3)$
FrG	$()$
Gen.	$(1,2,3)(4,5),(6,7),(8,9)$

Table 3: Table of $C_6 \times C_2^2$

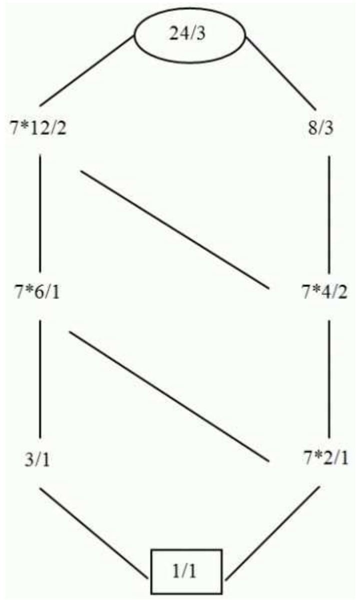

Figure 3: Subgroup lattice of $C_6 \times C_2^2$

Group T	24/4
Group N	$D_6 \times C_2$
GAP T	24/6
GAP N	$d12c2$
Deg.	24
Ab.	Not Abelian
NS	54
NNS	21
FiG	$(6,7),(4,5),(1,3,2)$
FrG	$()$
Gen.	$(1,2,3)(4,5),(2,3),(6,7)$

Table 4: Table of $D_6 \times C_2$

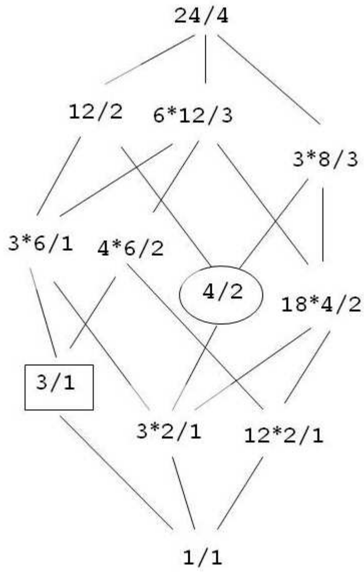

Figure 4: Subgroup lattice of $D_6 \times C_2$

Group T	24/5
Group N	$A_4 \times C_2$
GAP T	24/10
GAP N	$a4c2$
Deg.	24
Ab.	Not Abelian
NS	26
NNS	6
FiG	$(5,6),(1,2)(3,4),(1,3)(2,4)$
FrG	$()$
Gen.	$(1,2,3),(2,3,4),(5,6)$

Table 5: Table of $A_4 \times C_2$

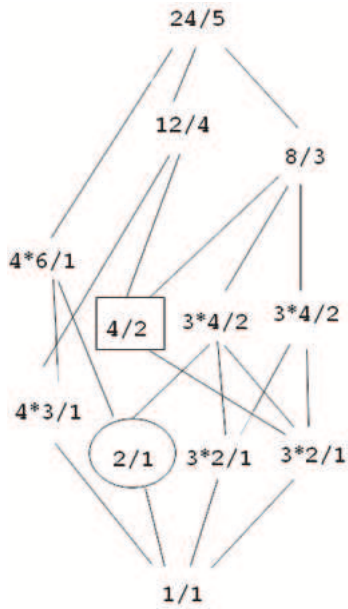

Figure 5: Subgroup lattice of $A_4 \times C_2$

Group T	24/6
Group N	$Q_6 \times C_2$
GAP T	24/8
GAP N	$q12c2$
Deg.	24
Ab.	Not Abelian
NS	22
NNS	13
FiG	$(8,9),(5,7,6),(1,2)(3,4)$
FrG	$(1,2)(3,4)$
Gen.	$(1,2)(3,4)(5,6,7),$ $(1,3,2,4)(6,7),(8,9)$

Table 6: Table of $Q_6 \times C_2$

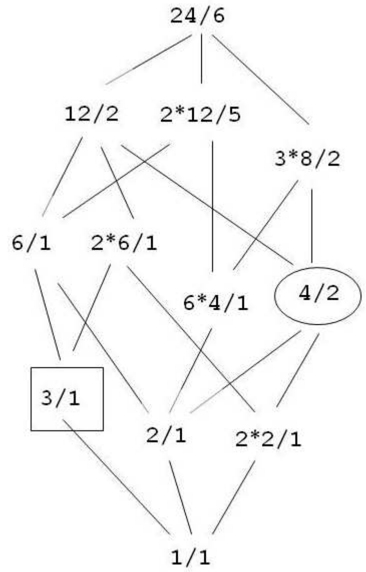

Figure 6: Subgroup lattice of $Q_6 \times C_2$

Group T	24/7
Group N	$D_4 \times C_3$
GAP T	24/4
GAP N	$d8c3$
Deg.	24
Ab.	Not Abelian
NS	20
NNS	12
FiG	$(5,7,6),(2,4),(1,3)(2,4),(1,43,2)$
FrG	$(1,3)(2,4)$
Gen.	$(1,2,3,4)(5,6,7),(2,4)$

Table 7: Table of $D_4 \times C_3$

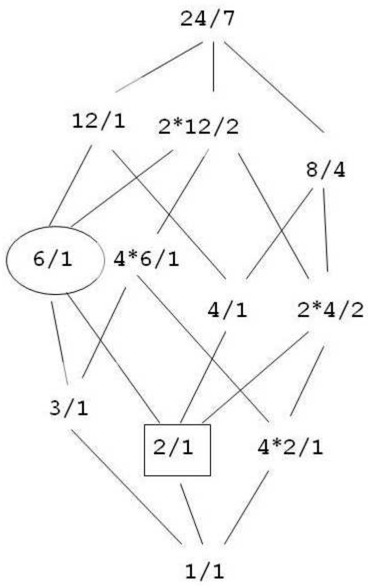

Figure 7: Subgroup lattice of $D_4 \times C_3$

Group T	24/8
Group N	$Q \times C_3$
GAP T	24/5
GAP N	$q8c3$
Deg.	24
Ab.	Not Abelian
NS	12
NNS	12
FiG	$(9,11,10),(1,2)(3,4)(5,6)(7,8)$ $(1,6,2,5)(3,8,4,7)$ $(1,8,2,7)(3,5,4,6)$
FrG	$(1,2)(3,4)(5,6)(7,8)$
Gen.	$(1,5,2,6)(3,7,4,8),$ $(1,7,2,8)(3,6,4,5),(9,10,11)$

Table 8: Table of $Q \times C_3$

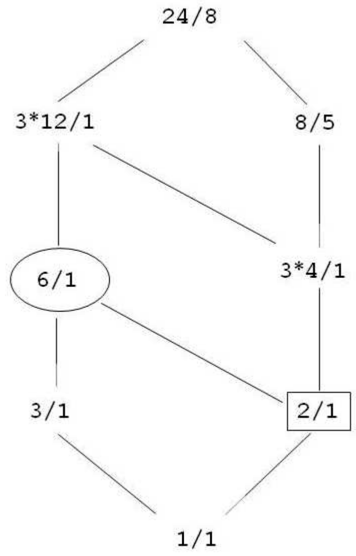

Figure 8: Subgroup lattice of $Q \times C_3$

Group T	24/9
Group N	$S_3 \times C_4$
GAP T	24/7
GAP N	$s3c4$
Deg.	24
Ab.	Not Abelian
NS	26
NNS	11
FiG	$(4,6)(5,7),(4,7,6,5),(1,2,3)$
FrG	$(4,6)(5,7)$
Gen.	$(1,2),(2,3),(4,5,6,7)$

Table 9: Table of $S_3 \times C_4$

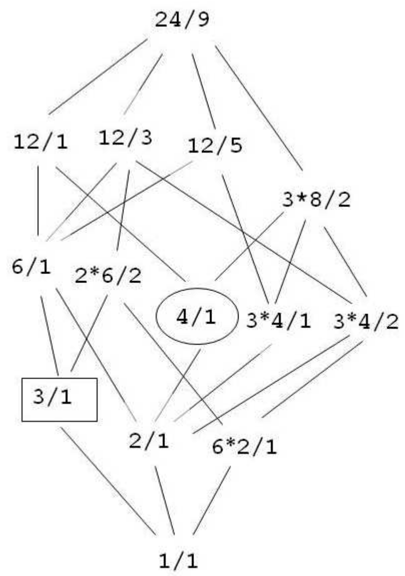

Figure 9: Subgroup lattice of $S_3 \times C_4$

Group T	24/10
Group N	D_{12}
GAP T	24/12
GAP N	$d24$
Deg.	24
Ab.	Not Abelian
NS	34
NNS	9
FiG	$(4,6)(5,7),(4,7,6,5),(1,2,3)$
FrG	$(4,6)(5,7)$
Gen.	$(1,2,3)(4,5,6,7),(2,3)(4,7)(5,6)$

Table 10: Table of D_{12}

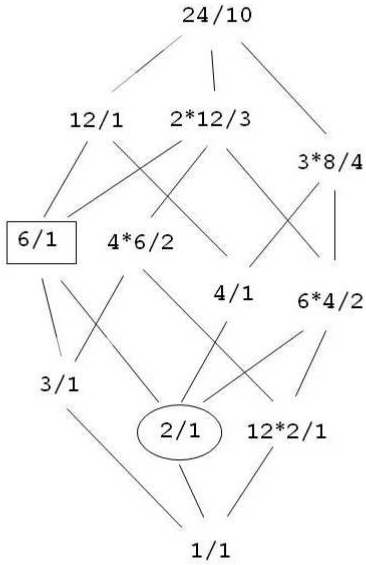

Figure 10: Subgroup lattice of D_{12}

Group T	24/11
Group N	Q_{12}
GAP T	24/13
GAP N	$q24$
Deg.	24
Ab.	Not Abelian
NS	18
NNS	9
FiG	$(9,10,11),(1,3)(2,4)(5,7)(6,8),$ $(1,4,3,2)(5,8,7,6)$
FrG	$(1,3)(2,4)(5,7)(6,8)$
Gen.	$(1,2,3,4)(5,6,7,8)(9,10,11),$ $(1,5,3,7)(2,8,4,6)(10,11)$

Table 11: Table of Q_{12}

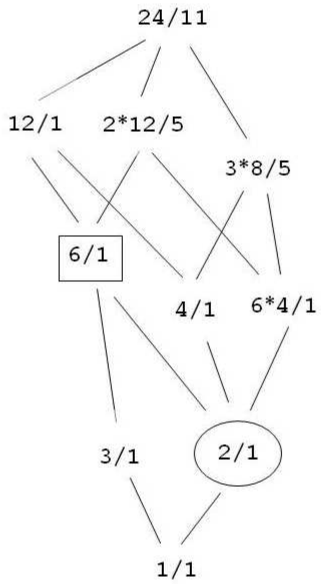

Figure 11: Subgroup lattice of Q_{12}

Group T	24/12
Group N	S_4
GAP T	24/15
GAP N	$s24$
Deg.	24
Ab.	Not Abelian
NS	34
NNS	9
FiG	$(4,6)(5,7),(4,7,6,5),(1,2,3)$
FrG	$(4,6)(5,7)$
Gen.	$(1,2,3)(4,5,6,7),(2,3)(4,7)(5,6)$

Table 12: Table of S_4

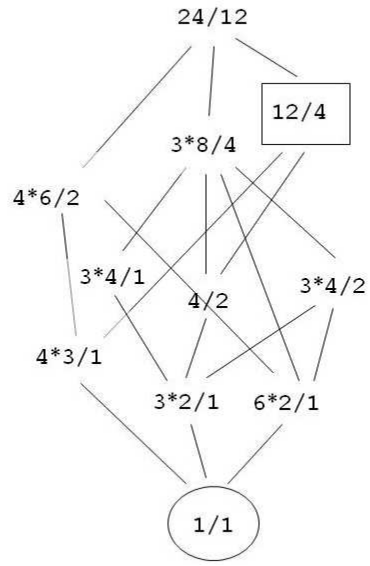

Figure 12: Subgroup lattice of S_4

Group T	24/13
Group N	$SL_2(F_3)$
GAP T	24/14
GAP N	$sl(2,3)$
Deg.	24
Ab.	Not Abelian
NS	15
NNS	4
FiG	$(1,3)(2,4)(5,7)(6,8),(1,4,3,2)$ $(5,6,7,8),(1,8,3,6)(2,5,4,7)$
FrG	$(1,3)(2,4)(5,7)(6,8)$
Gen.	$(1,2,3,4)(5,8,7,6),$ $(1,5,3,7)(2,6,4,8),$ $(2,5,6)(4,7,8)(9,10,11)$

Table 13: Table of $SL_2(F_3)$

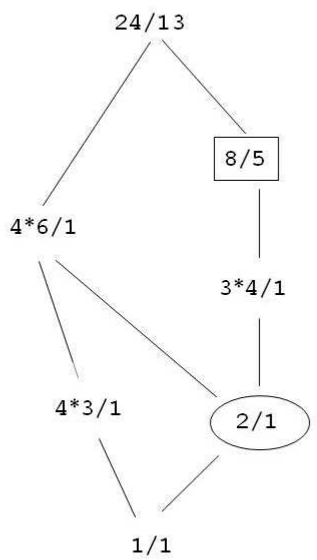

Figure 13: Subgroup lattice of $SL_2(F_3)$

Group T	24/14
Group N	$C_3 \times C_8$
GAP T	24/9
GAP N	$c3 \times c8$
Deg.	24
Ab.	Not Abelian
NS	10
NNS	7
FiG	$(4,6,8,10)(5,7,9,11),(4,8)(5,9)(6,10)(7,11),(1,2,3)$
FrG	$(4,6,8,10)(5,7,9,11),(4,8)(5,9)(6,10)(7,11)$
Gen.	$(1,2,3),(2,3)(4,5,6,7,8,9,10,11)$

Table 14: Table of $C_3 \times C_8$

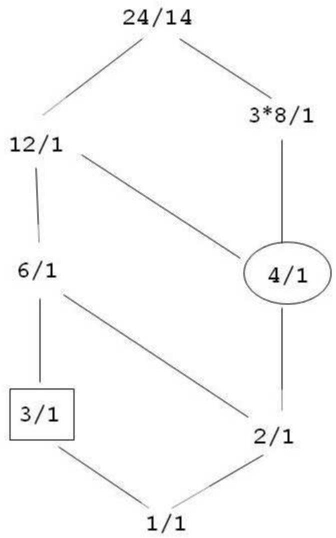

Figure 14: Subgroup lattice of $C_3 \times C_8$

Group T	24/15
Group N	$D_8 \times C_3$
GAP T	24/11
GAP N	$d8 \times c3$
Deg.	24
Ab.	Not Abelian
NS	30
NNS	9
FiG	$(5,6,7),(2,4),(1,3)(2,4)$
FrG	$(1,3)(2,4)$
Gen.	$(5,6,7),(1,2,3,4)(6,7),(2,4)$

Table 15: Table of $D_8 \times C_3$

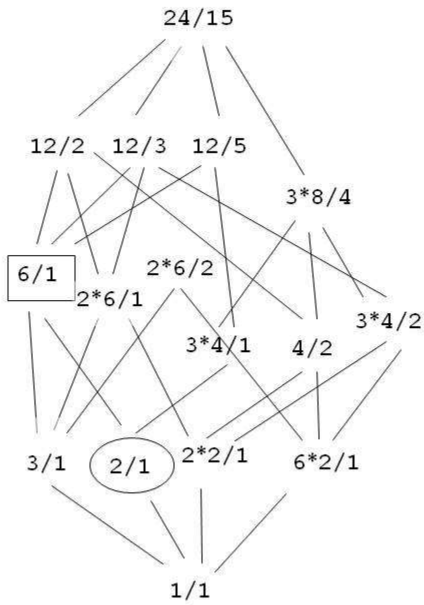

Figure 15: Subgroup lattice of $D_8 \times C_3$

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