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Best Approximation in Uniformity type spaces

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Abstract. Let X be a set, and Γ be a collection of subsets of $X \times X$. The object of this paper, is to define a semi-linear uniform space by assuming certain conditions on Γ. The structure of such spaces turned to be a very rich structure. We define closest elements from a given set to a given element in X. Then we study best approximation in semi-linear uniform spaces.

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1. Introduction

Let X be a set and D_X be a collection of subsets of $X \times X$, such that each element V of D_X contains the diagonal $\Delta = \{(x, x) : x \in X\}$and $V = V^{-1} = \{(y, x) : (x, y) \in V\}$ for all $V \in D_X$ (**symmetric**), D_X is called the family of all entourages of the diagonal. Let Γ be a sub collection of D_X, then

The pair (X, Γ) is called a **uniform space** if

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(i) V_1 and V_2 are in Γ then $V_1 \cap V_2 \in \Gamma$

(ii) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$.

(iii) $\bigcap_{V \in \Gamma} V = \Delta$

(vi) If $V \in \Gamma$ and $V \subseteq W \in D_X$, then $W \in \Gamma$.

Uniform spaces had been studied extensively through years. We refer the reader to [1], and [2], for the basic structure of uniform spaces. The object of this paper is to define uniform type spaces and a set valued map, to be called metric type, on such spaces that enables us to study analytical concepts on uniform type spaces, namely best approximation. Since the problem of best approximation is a problem of nearness between elements and sets, the problem of best approximation is usually discussed in metric and normed spaces [3], [4]. Best approximation never been studied in spaces other than metric and normed spaces We believe that the new structure that we introduced in this paper is very fruitful and will give rise to many problems in approximation theory in uniform spaces.

2. Uniform type spaces

Let (X, Γ) be a uniform space. By a **chain** in $X \times X$ we mean a totally(or linearly) ordered collection of subsets of $X \times X$, where $V_1 \leq V_2$ means $V_1 \subseteq V_2$.

Definition 1.1. We call (X, Γ) a **semi-linear uniform space** if it is a uniform space where Γ is a chain and condition (vi) is replaced by $\bigcup_{V \in \Gamma} V = X \times X$.

An example of a semi-linear uniform space is the following.

Example 2.1. Let $V_t = \{(x, y) : y - t < x < y + t, \text{ and } -\infty < y < \infty\}$. Then $(\mathbb{R}, \Gamma)$, with $\Gamma = \{V_t : 0 < t < \infty\}$ is a semi-linear uniform space.

One can generate semi-linear uniform spaces as follows. Let D_X be a chain in the power set of $X \times X$, such that, each element of D_X is symmetric , contains $\triangle$, $\bigcup_{U \in D_X} U = X \times X$ and $\bigcap_{U \in D_X} U = \triangle$. Then one can easily see that (X, D_X) is a semi-linear uniform space.

We should remark that **the topology in metric and normed spaces can be generated by semi-linear uniformities**.

Throughout the rest of this paper, (X,Γ) will be assumed semi-linear uniform space.

Now we introduce one of the main concepts in this paper.

Let (X,Γ) be a semi-linear uniform space. For $x, y \in X$, let

$C\ (x,y) = \cap\{V \in \Gamma : (x,y) \in V\}$, and $\Sigma = \left\{C\ (x,y) : x,y \in X\ \right\}$.

Clearly $C\ (x,y) = \cap\{V^{-1} \in \Gamma : (x,y) \in V\}$.

Definition 3.1. Let (X,Γ) be a semi-linear uniform space. We define the set valued map: $\rho : X \times X\ \rightarrow \Sigma$, $\rho(x,y) = C\ (x,y)$.The map ρ will be called a **set metric** on (X,Γ).

The proof of the following result is immediate and will be omitted.

Proposition 4.1. For a semi-linear uniform space, we have the followings.

(i) $\rho(x,y) = \Delta$ if and only if $x = y$.

(ii) $\rho(x,y) = \rho(y,x)$.

Now we have the following natural questions.

Question1: Is $\rho(x,y) \subseteq \rho(x,z) \cap \rho(z,y)$?.

In metric spaces, it is known that if $d(x,y) = d(x,z)$ then y need not equal z.. In semi-linear type spaces, the story is different. So we pose the following question.

Question 2. If $\rho(x,z) = \rho(x,w)$,for some $x \in X$. Must $w = z$?.

Using the concept of set metric, we introduce the following concepts.

Definition 5.1. For $x \in X$ and $E \subset X$, we define $\rho(x,E) = \underset{y \in E}{\cap} \rho(x,y)$.

Clearly, if $x \in E$, then $\rho(x,E) = \Delta$.

Definition 6.1. For $x \in X$ and $V \in \Gamma$, we define The open ball of center x and radius V to be $B(x,V) = \{y : (x,y) \in V\}$. Equivalently $B(x,V) = \{y : \rho(x,y) \subseteq V\}$.Clearly if $y \in B(x,V)$, then there is a $W \in \Gamma$ such that $B(y,W) \subseteq B(x,V)$.

Definition 7.1. $B \subseteq X$ is called bounded if $B \subseteq B(x, V)$, for some $V \in \Gamma, x \in X$.

Definition 8.1. Let (x_n) be a sequence in X.We say x_n **converges** to x in X , and we write $x_n \rightarrow x$, if for every $V \in \Gamma$ there exists k such that $(x_n, x) \in V$ for every $n \geq k$.

Clearly if $x_n \rightarrow x$, then for every j, $\bigcap_{n=j}^{\infty} \rho(x, x_n) = \Delta$.Unfortunately the converse is not true. But we have

Lemma 9.1. Let (x_n) be a sequence in X. If $\bigcap_{n=j}^{\infty} \rho(x, x_n) = \Delta$, for every j, then there exist a subsequence $x_{n_k} \rightarrow x$.

proof. We may assume that, for every j there is $n_j \geq j$ such that $\rho(x, x_{n_j}) \neq \Delta$, also we may assume $\rho(x, x_{n_j})$ is a decreasing sequence and $\bigcap_{j=1}^{\infty} \rho(x, x_{n_j}) = \Delta$.Let $V \in \Gamma$,then there exist j_1 such that $\rho(x, x_{n_j}) \subseteq V$ for all $j \geq j_1$, hence $x_{n_j} \rightarrow x$.

Definition 10.1. Let (x_n) be a sequence in X, (x_n) is called **Cauchy** if for every $V \in \Gamma$ there exists k such that $(x_n, x_m) \in V$ for every $n, m \geq k$.

Now it is easy to prove the following Corollary

Corollary 11.1. Let (x_n) be a Cauchy sequence in X. Then $x_n \rightarrow x$, iff for every $j \in \mathbb{N}$, $\bigcap_{n=j}^{\infty} \rho(x, x_n) = \Delta$.

Now, we prove:

Lemma 12.1. Let (x_n) be a sequence in (X, Γ). Then.

(i) Every convergent sequence is Cauchy.

(ii) Every Cauchy sequence is bounded.

Proof. (i) Let (x_n) converges to x in X, and $V \in \Gamma$.Let $U \in \Gamma$ such that $U \circ U \subset V$. From the definition of convergence, there exists k such that $(x, x_n) \in U$ for all $n > k$. Since U is symmetric, $(x_m, x) \in U$ for all $m > k$. Hence $(x_n, x) \circ (x, x_m) = (x_n, x_m) \in U \circ U \subset V$ for all $n, m > k$, and (x_n) is Cauchy.

(ii) Let x_n be Cauchy, and $V \in \Gamma$. Then there exists k such that $(x_n, x_m) \in V$ for every $n, m \geq k$.

Let $U \in \Gamma$ be such that $\{(x_k, x_1), (x_k, x_2), ..., (x_k, x_{k-1})\} \subseteq U$. Then

$\{x_1, x_2, ...\} \subseteq B(x_k, W)$, where $W = U \cup V$.

Lemma 13.1. Let (x_n) be a sequence in X. If (x_n) converges then the limit is unique.

Proof. If possible assume that $x_n \to x$ and $x_n \to y$. Let $V \in \Gamma$ be arbitrary. Condition (iii) of uniform spaces implies the existence of some $W \in \Gamma$ such that $W \circ W \subset V$. From the definition of convergence, there exists $n_\circ$ such that (x, x_n) and (y, x_n) are in W. Hence $(x, y) \in W \circ W \subset V$. Thus, since V was arbitrary, $(x, y) \in \Delta$, and so $x = y$.

Now, a set E will be called **open** if for every point x in E there exists $V \in \Gamma$, such that $B(x, V) \subseteq E$. The set E is called **closed** if E^c is open. A point x is called a **limit point** of E if there is a sequence (x_n) in E such that $x_n \to x$. The set of limit points of the set E will be denoted by E^ℓ. For any set E in X, we let $\bar{E} = E \cup E^\ell$.

The proof of the following lemma is similar to that in metric spaces and will be omitted.

Lemma 14.1. A set E is closed if and only if $E^\ell \sqsubseteq E$.

Question 3. If $\rho(x, E) = \Delta$, must $x \in E^\ell$?

Proposition 15.1. If $x \in E^\ell$, then $\rho(x, E) = \Delta$.

Proof. Let $x \in E^\ell$. Then there exists (x_n) in E such that $x_n \to x$. Hence, $\bigcap_{n=1}^{\infty} \rho(x, x_n) = \Delta$. But $\rho(x, E) = \bigcap_{y \in E} \rho(x, y) \subset \bigcap_{n=1}^{\infty} \rho(x, x_n) = \Delta$. So $\rho(x, E) = \Delta$.

A nice property of semi-linear uniform spaces is:

Theorem 16.1. Open balls separate points in (X, Γ).

Proof. Let x, y be any two elements in (X, Γ) such that $x \neq y$. If possible assume that $B(x, U) \cap B(y, U) \neq \phi$ for all $U \in \Gamma$. Let V be any element in Γ. Since X is a uniform space, then there exists $W \in \Gamma$ such that $W \circ W \subset V$. By assumption $B(x, W) \cap B(y, W) \neq \phi$. Hence, there exists $z \in X$ such that (x, z), $(z, y) \in W$. Consequently, $(x, y) \in W \circ W \subset V$. So $(x, y) \in V$ for all $V \in \Gamma$. But this implies that $(x, y) \in \Delta$, which in turn implies that $x = y$. This contradicts the assumption. So there must exist some $W \in \Gamma$ such that $B(x,W) \cap B(y,W) = \phi$.

Now, let us define a set $E \subset (X,\Gamma)$ to be **compact**, if every sequence in E has a convergent subsequence in E. Clearly, every finite set is compact, and every compact set is closed.

3. Proximinality in Semi-Linear Uniform Spaces

What is nice about semi-linear uniform spaces is that theory of best approximation can be studied in such spaces without tools that metric structure usually offers. In this section we present some results in approximation theory in semi-linear uniform spaces.

Definition 1.2. Let (X,Γ) be semi-linear uniform space, and $E \subset X$. The set E is called **proximinal** if for any $x \in X$, there exists some $e \in E$ such that $\rho(x,E) = \rho(x,e)$.

Proposition 2.2. If $E \subset X$ is proximinal, then E is closed.

Proof. Let $x \in E^{\ell}$. By Proposition 11.1, $\rho(x,E) = \Delta$, then by assumption of proximinality, there exists some $e \in E$ such that $\rho(x,E) = \rho(x,e) = \Delta$. So x must equal e and E is closed.

Compact sets are nice proximinal sets in normed spaces [4]. But what about proximinality of compact sets in semi-linear spaces.

Question 4. If E is compact, must E be proximinal?.

Every finite set is compact, so the following is a partial answer to our question.

Theorem 2.3. Let (X,Γ) be a semi-linear uniform space. Then every finite set is proximinal.

Proof. Since E is finite, then $E = \{e_1, e_2, ...e_n\}$.Let $x \in X$. Then $\rho(x,E) = \bigcap_{i=1}^{n} \rho(x,e_i)$. The chain property of semi-linear uniform spaces implies that any two elements $\rho(x,e_i)$, $\rho(x,e_k)$ one of them must be contained in the other. Thus $\{\rho(x,e_1), ...\rho(x,e_n)\}$

is a finite chain. Consequently, $\overset{n}{\underset{i=1}{\cap}}\rho(x,e_i)=\rho(x,e_k)$ for some k, with $1\leq k\leq n$. Hence E is proximinal.

Corollary 2.4. If $E_1,E_2,...,E_n$ are proximinal in (X,Γ), then $\bigcup\limits_{i=1}^{n}E_i$ is proximinal too.

Proof. Let $x\in X$. Then $\rho(x,\bigcup\limits_{i=1}^{n}E_i)=\underset{y\in\bigcup\limits_{i=1}^{n}E_i}{\cap}\rho(x,y)=\bigcap\limits_{i=1}^{n}(\underset{y\in E_i}{\cap}\rho(x,y)))$. Since E_i all are proximinal, then $\underset{y\in E_i}{\cap}\rho(x,y)=\rho(x,e_i)$ for some $e_i\in E_i$. Hence $\rho(x,\bigcup\limits_{i=1}^{n}E_i)=\bigcap\limits_{i=1}^{n}\rho(x,e_i)=\rho(x,e_k)$ for some $k\in\{1,2,...,n\}$. So $\rho(x,\bigcup\limits_{i=1}^{n}E_i)=\bigcap\limits_{i=1}^{n}\rho(x,e_i)=\rho(x,e_k)$, $e_k\in\bigcup\limits_{i=1}^{n}E_i$.

Also every sequence with it's limit is compact, so we have another partial answer to our question.

Theorem 2.5. Let (X,Γ) be a semi-linear uniform space and (y_n) be a convergent sequence in X. Then $E=\left\{y,y_1,y_2,...\right\}$ is proximinal, where $y=\lim y_n$.

Proof. Let $x\in X\setminus E$ (If $x\in E$ then $\rho(x,E)=\rho(x,x)$). So we may assume $\rho(x,y_n)\neq\Delta$ for all n. Now if there exist n_0 such that $\rho(x,y_{n_0})\subseteq\rho(x,y_n)$. for all n , then $\rho(x,E)=\rho(x,y_{n_0})\cap\rho(x,y)$ and by Theorem 2.3 we are done. If not, then for all n there exist m_n such that $m_n<m_{n+1}$and $\rho(x,y_{m_n})\subsetneqq\rho(x,y_n)\cap\rho(x,y_{m_{n-1}})$.So $\bigcap\limits_{n=1}^{\infty}\rho(x,y_{m_n})=\bigcap\limits_{n=1}^{\infty}\rho(x,y_n)$.Now we want to show that $\rho(x,y)\subseteq\bigcap\limits_{n=1}^{\infty}\rho(x,y_{m_n})$.Let $U\in\Gamma$ be such that $(x,y_{m_n})\in U$, for some $m_n\in\mathbb{N}$, therefor $(x,y_{m_k})\in U$ for all $j\geq n$.

let $W_j\in\Gamma$ be such that $B\left(x,2W_j\right)\times B\left(y_{m_k},2W_j\right)\subseteq U$, also we may assume that $W_j\supseteq W_{j+1}$.If $\bigcap\limits_{j=1}^{\infty}W_j=\Delta$,let $\left(t_j,s_j\right)\in B\left(x,W_j\right)\times B\left(y_{m_k},W_j\right)$, then $t_j\to x$ and $s_j\to y$ $(\lim s_j=\lim y_{m_k})$.

therefor $(x, y) \in \overline{B\left(x, W_j\right)} \times \overline{B\left(y_{m_k}, W_j\right)} \subseteq B\left(x, 2W_j\right) \times B\left(y_{m_k}, 2W_j\right) \subseteq U$.

Now if $\bigcap_{j=1}^{\infty} W_j \neq \Delta$, then there exist $W \in \Gamma$ such that $W \subseteq \bigcap_{j=1}^{\infty} W_j$, so $B(x, W) \times$

$B\left(y_{m_k}, W\right) \subseteq U$ for all j.Since $y_{m_k} \longrightarrow y$,

there exist N such that $\left(y_{m_N}, y\right) \in W$, so $(x, y) \in U$, and the result follows.

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