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Modification of Laplace Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations based on Newton Raphson formula

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Abstract. In this paper, we establish a modified Laplace transform Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations. This technique is different from the standard Laplace Adomian decomposition method because of the terms involved in Adomian polynomials. Here, we have used Newton Raphson formula in place of the term u_i in Adomian polynomials. The proposed scheme is investigated with some illustrative examples and has given reliable results.

2010 Mathematics Subject Classifications: 44A10, 45D05, 49M27, 65R10, 65R20.

Key Words and Phrases: Numerical Laplace Transform Method, Volterra Integral Equations, Volterra Integro-differential Equations, Adomian Decomposition Method, Newton Raphson Formula.

1. Introduction

For solving nonlinear functional equations, Adomian decomposition method was introduced by George Adomian in 1980 [4, 19, 23]. Basically, the technique provides an infinite series solution of the equation and the nonlinear term is decomposed into an infinite series of Adomian polynomials [1, 2, 5, 6, 8, 10, 14–17, 20, 22–24, 26–29]. Several linear and nonlinear ordinary, partial, deterministic and stochastic differential equations are solved easily and adequately by Adomian decomposition method [4, 13, 14, 19, 23]. In this work, Laplace transform technique in combination with Adomian decomposition method is presented and modified, which was first studied by Khuri in [14] to solve nonlinear differential equations. In [13], the authors investigated the method for solving coupled nonlinear partial differential equations. Laplace decomposition method was

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employed to logistic differential equations to find the numerical solutions in [10]. Chanquing and Jianhua studied the Adomian decomposition method to solve the nonlinear fractional differential equations in [27]. In [7], the technique was applied on delay differential equations. A comparison was made between Adomian decomposition and tau methods in [4] for finding the solution of Volterra integro-differential equations. Magdy and Mohamed [20] practiced Laplace decomposition method and Pade approximation to get the numerical solution of nonlinear system of partial differential equations. Further, a modified Laplace decomposition method was adopted for Lane-Emden type differential equations in [28]. Hence, there are numerous applications where Laplace Adomian decomposition method is used by many researchers.

In the present paper, we focus to solve nonlinear Volterra integral and integro-differential equations. Nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics and semi-conductor devices [25]. Volterra integro-differential equations also emanated in many physical applications such as biological species coexisting together with increasing and decreasing rates of generating and in engineering applications such as heat transfer, diffusion process in general [3, 4, 21, 24]. Recently, many researchers investigated the solution of these problems. Extant methods are presented to solve these kinds of equations. In [18], quasilinearization technique was employed to solve Volterra integral equations. Kamyad et al. proposed the discretisation and interpolation method for Volterra integral equations [12]. In [3], a comparison was made between Laplace decomposition method, homotopy perturbation method and wavelet-Galerkin method for solving nonlinear Volterra integro-differential equations. Laplace transform combined with Adomian decomposition method is pertained already to solve nonlinear Volterra integral and integro-differential equations [9, 19, 23]. Our work is inspired from these. In this paper, we have followed the combined Laplace transform and Adomian decomposition method but while decomposing the nonlinear term using Adomian polynomials, we have substituted the term u_i with Newton Raphson formula. As we know that Newton Raphson formula is used for finding the better approximate solution of real valued function. By adapting this change, we have achieved the approximate solutions which are in good agreement with the exact one.

The paper is organized as follows: In Section 2, the modified Laplace Adomian decomposition method is presented and discussed for Volterra integral equations. Section 3 summarizes the application of technique to nonlinear Volterra integro-differential equations. In Section 2 and 3, some numerical results are also given to clarify the method. The conclusions are drawn in last section.

2. Nonlinear Volterra integral equations of the second kind

Consider the following nonlinear Volterra integral equation with difference kernel i.e. $k(x,t) = k(x-t)$ defined as

$$u(x) = f(x) + \int_0^x k(x-t)F(u(t))dt, \tag{1}$$

where $f(x)$ is known real valued function and $F(u(x))$ is the nonlinear function of $u(x)$. Apply Laplace transform on both sides of (1). After that using the linear property and convolution theorem of Laplace transform, we have

$$L[u(x)] = L[f(x)] + L[k(x-t)]L[F(u(x))]. \tag{2}$$

The methodology consists of approximating the solution of (1) as an infinite series given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{3}$$

However, the nonlinear term $F(u(x))$ is decomposed as

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{4}$$

where $A_n's$ are modified Adomian polynomials which are based on Newton Raphson formula given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n}\lambda^i\left(u_i - \frac{F(u_i)}{F'(u_i)}\right)\right)\right]_{\lambda=0}, n \geq 0. \tag{5}$$

Substituting (3) and (4) into (2), we get

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = L[f(x)] + L[k(x-t)]L\left[\sum_{n=0}^{\infty} A_n(x)\right].$$

Using the linearity property of Laplace transform, we get

$$\sum_{n=0}^{\infty} L[u_n(x)] = L[f(x)] + L[k(x-t)]\sum_{n=0}^{\infty} L[A_n(x)]. \tag{6}$$

To determine the terms $u_0(x)$,$u_1(x)$,$u_2(x)$,$u_3(x)\ldots$ of infinite series, comparing both sides of (6), we have the following iterative scheme

$$L[u_0(x)] = L[f(x)], \tag{7}$$

In general, the relation is given by

$$L[u_{n+1}(x)] = L[k(x-t)]L[A_n(x)]. \tag{8}$$

Employing the inverse Laplace transform to (7) and (8), we get

$$u_0(x) = L^{-1}\left[L[f(x)]\right], \tag{9}$$

$$u_{n+1}(x) = L^{-1}\left[L[k(x-t)]L[A_n(x)]\right]. \tag{10}$$

Adapting the value of $u_0(x)$ into (5) gives the value of A_0 and then using the general iterative relation (10), we get the values of $u_1(x)$, $u_2(x)$, $u_3(x)$ and so on, which finally gives the solution (3) to the given Volterra integral equation.

The effectiveness of modified technique for solving Volterra integral equations is shown by following numerical examples. Here, we have also found the maximum absolute error estimation to show the adequacy of technique given as:

$$e_j = Max|u_{ex} - u_{app}|$$

where e_j denotes the maximum absolute error at some x_j in the given interval.

Example 1. Consider the following Volterra integral equation [11, 25]

$$u(x) = x + \int_0^x u^2(t)dt, \tag{11}$$

which has the exact solution as $u(x) = \tan x$.

Solution.Taking Laplace transform on both sides of (11) and using the linearity property of Laplace transform, we have

$$L[u(x)] = L[x] + L\left[\int_0^x u^2(t)dt\right],$$

that is

$$L[u(x)] = \frac{1}{s^2} + \frac{1}{s}L[u^2(x)],$$

Using above technique, we have

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = \frac{1}{s^2} + \frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n(x)\right], \tag{12}$$

where the nonlinear term $F(u(x)) = u^2(x)$ is decomposed using the formula given by (5). Certain terms of modified Adomian polynomials are as follows:

$$A_0 = \left(\frac{1}{2}\right)^2 u_0^2,$$

$$A_1 = \left(\frac{1}{2}\right)^2 (u_0 u_1),$$

$$A_2 = \left(\frac{1}{2}\right)^2 (2u_0 u_2 + u_1^2),$$

$$A_3 = \left(\frac{1}{2}\right)^2 (2u_0 u_3 + 2u_1 u_2).$$

Pairing both sides of (12), gives

$$L[u_0(x)] = \frac{1}{s^2}, \tag{13}$$

In general

$$L[u_{n+1}(x)] = \frac{1}{s}L[A_n(x)]. \tag{14}$$

Applying inverse Laplace transform on both sides of (13), gives

$$u_0(x) = x, \tag{15}$$

Using general relation, we have

$$u_1(x) = \frac{x^3}{12},$$

Continuing in this manner, we get

$$u_2(x) = \frac{x^5}{120},$$

$$u_3(x) = \frac{x^7}{20160},$$

$$u_4(x) = \frac{31x^9}{362880},$$

Subsequently, the approximate solution becomes

$$u(x) = x + \frac{x^3}{12} + \frac{x^5}{120} + \frac{x^7}{20160} + \frac{31x^9}{362880} \cdots .$$

The exact solution and the one obtained by our technique corresponding to distinct values of x are presented in Table 1 and demonstrated through figure 1. The absolute error laid out in the table admit that the solutions are very much close to the exact solution and the maximum absolute error is 0.0002.

Example 2. Solve the following Volterra integral equation [12]

$$u(x) = 2x - \frac{x^4}{12} + 0.25\int_0^x (x-t)u^2(t)dt, \tag{16}$$

having exact solution $u(x) = 2x$.

Solution. Applying the modified decomposition method, we have

$$L[u(x)] = L\left[2x - \frac{x^4}{12}\right] + 0.25L[x]L[u^2(x)],$$

The method assumes the series solution of function $u(x)$

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = L\left[2x - \frac{x^4}{12}\right] + \frac{1}{4s^2}L\left[\sum_{n=0}^{\infty} A_n(x)\right], \tag{17}$$

Table 1: Numerical Results for Example 1.

x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0.0000E+00
0.01	0.010000333	0.010000083	2.5001E-07
0.02	0.020002667	0.020000667	2.0004E-06
0.03	0.030009003	0.030002250	6.7530E-06
0.04	0.040021347	0.040005334	1.6013E-05
0.05	0.050041708	0.050010419	3.1289E-05
0.06	0.060072104	0.060018006	5.4097E-05
0.07	0.070114558	0.070028597	8.5961E-05
0.08	0.080171105	0.080042694	1.2841E-04
0.09	0.090243790	0.090060799	1.8299E-04
0.1	0.100334672	0.100083417	2.5126E-04

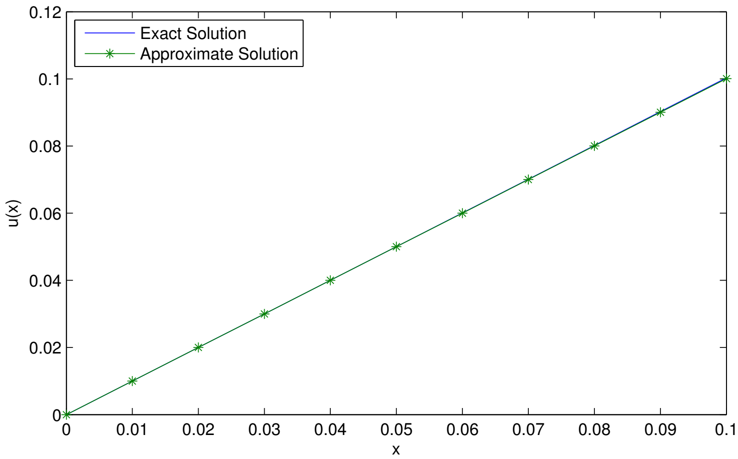

Figure 1: Comparison of Exact Solution and Approximate Solution.

Comparing both sides of (17), gives the continual algorithm

$$L[u_0(x)] = L\left[2x - \frac{x^4}{12}\right], \tag{18}$$

In general

$$L[u_{n+1}(x)] = \frac{1}{4s^2} L[A_n(x)]. \tag{19}$$

Taking inverse Laplace transform on above iterative steps, implies

$$u_0(x) = 2x - \frac{x^4}{12}, \tag{20}$$

$$u_1(x) = \frac{x^{10}}{207360} - \frac{x^7}{2016} + \frac{x^4}{48},$$

$$u_2(x) = -\frac{x^{16}}{4777574400} + \frac{37x^{13}}{905748480} - \frac{11x^{10}}{2903040} + \frac{x^7}{8064},$$

and so on.
Thus, the solution takes the form

$$u(x) = 2x - \frac{x^4}{16} - \frac{x^7}{2688} + \frac{x^{10}}{967680} + \frac{37x^{13}}{905748480} - \frac{x^{16}}{4777574400} + \dots.$$

The numerical results shown in Table 2 and Figure 2 illustrate the performance of proposed

Table 2: Comparison of Approximate Solution with Exact solution for Example 2.

x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0.0000E+00
0.05	0.1	0.099999609	3.9063E-07
0.1	0.2	0.19999375	6.2500E-06
0.15	0.3	0.299968359	3.1641E-05
0.2	0.4	0.399899995	1.0000E-04
0.25	0.5	0.499755837	2.4416E-04
0.3	0.6	0.599493669	5.0633E-04
0.35	0.7	0.69906187	9.3813E-04
0.4	0.8	0.798399391	1.6006E-03
0.45	0.9	0.89743572	2.5643E-03
0.5	1	0.996090845	3.9092E-03

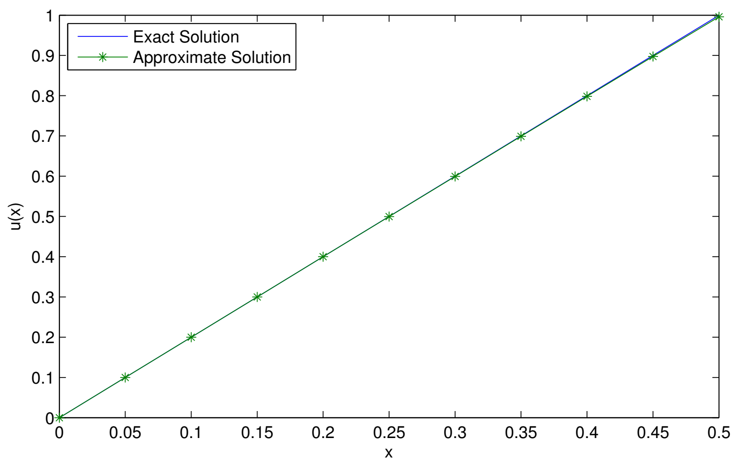

Figure 2: Comparison of Presented Approximate Solution with Exact solution.

method and maximum absolute error is 0.003.

3. Nonlinear Volterra integro-differential equations of the second kind

The nonlinear Volterra integro-differential equation of the second kind with difference kernel $k(x,t) = k(x-t)$ is defined as

$$u^{(i)}(x) = f(x) + \int_0^x k(x-t)F(u(t))dt, \tag{21}$$

where $u^{(i)}(x)$ denotes the ith derivative of $u(x)$ w.r.to x, $f(x)$ is known as source term and $F(u(x))$ is the nonlinear function of $u(x)$.
The derivative property of Laplace transform is defined by

$$L[u^{(i)}(x)] = s^i L[u(x)] - s^{i-1}u(0) - s^{i-2}u'(0) - \ldots - u^{(i-1)}(0). \tag{22}$$

Taking Laplace transform on both sides of (21) and using the properties of Laplace transform, we get

$$s^i L[u(x)] - s^{i-1}u(0) - s^{i-2}u'(0) - \ldots - u^{(i-1)}(0) = L[f(x)] + L[k(x-t)]L[F(u(x))]. \tag{23}$$

which implies

$$L[u(x)] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \ldots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}L[f(x)] + \frac{1}{s^i}L[k(x-t)]L[F(u(x))].$$

Adopting the same process as described in Section 2, we obtain

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = \frac{1}{s}u(0)+\frac{1}{s^2}u'(0)+\ldots+\frac{1}{s^i}u^{(i-1)}(0)+\frac{1}{s^i}L[f(x)]+\frac{1}{s^i}L[k(x-t)]L\left[\sum_{n=0}^{\infty} A_n(x)\right]. \tag{24}$$

The linearity property of Laplace transform gives

$$\sum_{n=0}^{\infty} L[u_n(x)] = \frac{1}{s}u(0)+\frac{1}{s^2}u'(0)+\ldots+\frac{1}{s^i}u^{(i-1)}(0)+\frac{1}{s^i}L[f(x)]+\frac{1}{s^i}L[k(x-t)]\sum_{n=0}^{\infty} L[A_n(x)]. \tag{25}$$

Matching both sides, we have the following recurrence relation

$$L[u_0(x)] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \ldots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}L[f(x)], \tag{26}$$

In general, the relation is given by

$$L[u_{n+1}] = \frac{1}{s^i}L[k(x-t)]L[A_n(x)]. \tag{27}$$

Applying the inverse Laplace transform to (26), we get the value of $u_0(x)$, that will define the value of A_0. Using the value of $A_0(x)$, $u_1(x)$ is obtained. Continuing in this manner we will find $u_n(x)$ from the general relation given by (27). After finding the components of infinite series, the series solution (3) follows. The proposed method will be illustrated by using the following example.

Example 3. Consider the nonlinear Volterra integro-differential equation [3, 19, 21]

$$u'(x) = -1 + \int_0^x u^2(t)dt, u(0) = 0. \tag{28}$$

Solution.Taking Laplace transform on both sides of (28) and employing the initial condition, we get

$$L[u'(x)] = L\left[-1 + \int_0^x u^2(t)dt\right],$$

$$L[u(x)] = -\frac{1}{s^2} + \frac{1}{s^2}L[u^2(x)],$$

Substituting the series form of $u(x)$ gives

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = -\frac{1}{s^2} + \frac{1}{s^2}L\left[\sum_{n=0}^{\infty} A_n(x)\right], \tag{29}$$

Matching both sides of (29) gives the iterative algorithm

$$L[u_0(x)] = -\frac{1}{s^2}, \tag{30}$$

$$L[u_{n+1}(x)] = \frac{1}{s^2}L[A_n(x)]. \tag{31}$$

Taking inverse Laplace transform on both sides of (30) and using the recursive relation (31), we get

$$u_0(x) = -x,$$

$$u_1(x) = \frac{x^4}{48},$$

$$u_2(x) = -\frac{x^7}{4032},$$

$$u_3(x) = \frac{x^{10}}{387072},$$

$$u_4(x) = -\frac{x^{13}}{40255488},$$

The series solution is therefore given by

$$u(x) = -x + \frac{x^4}{48} - \frac{x^7}{4032} + \frac{x^{10}}{387072} - \frac{x^{13}}{40255488} \cdots.$$

Table 3 and Figure 3 show that the approximate numerical solution compared with exact [21] is very superior having maximum absolute error 0.003.

Table 3: Computed Exact and Approximation Solution for Example 3.

x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0.0000E+00
0.0625	-0.0625	-0.062499682	3.1789E-07
0.125	-0.12498	-0.124994914	1.4914E-05
0.1875	-0.1874	-0.187474253	7.4253E-05
0.25	-0.24967	-0.249918635	2.4863E-04
0.3125	-0.31171	-0.31230139	5.9139E-04
0.375	-0.37336	-0.374588271	1.2283E-03
0.4375	-0.43446	-0.436737503	2.2775E-03
0.5	-0.49482	-0.498699852	3.8799E-03

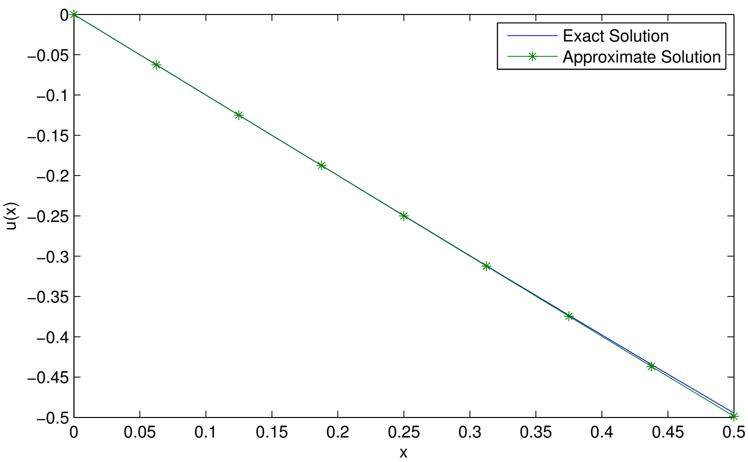

Figure 3: Comparison of Presented Approximate Solution with Exact solution.

4. Concluding Remarks

Newton Raphson formula using as a term in Adomian polynomials exhibits the tenability of combining Laplace transform technique and Adomian decomposition method to solve the nonlinear Volterra integral and integro-differential equations. This is the first time that the Adomian polynomials are modified using Newton Raphson formula. The solution given in tables and demonstrated through figures reveals that the approximate solution using the modified technique is very close to exact solution. Thus, the proposed technique is easy to implement and manifest the accuracy of solution.

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