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On Gamma Acts over Gamma Semigroups

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Abstract. For a semigroup S, actions of S on non-empty sets, namely S-acts, are of interest to consider for their applications in many branches of science. The well known generalization of a semigroup is the Γ-semigroup. The notion of a Γ-act over a Γ-semigroup is a generalization of actions over semigroups. In this paper, certain intrinsic and basic properties of Γ-acts including cyclic, indecomposable and free are studied as well. Among other results, it is shown that a Γ-act is free only if $|\Gamma| = 1$.

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1. Introduction

Nobusawa [10] introduced the notion of a Γ-ring, which is more general than a ring. Then Barnes [2] studied Γ-rings in a different way than that of Nobusawa. Motivated by these generalizations of rings, Sen [12] defined the concept of a Γ-semigroup, as a generalization of a semigroup. The investigation on Γ-semigroups was done by certain mathematicians which are parallel to the results in semigroup theory, for example, one may see [11, 13, 14]. Recently on this area some new papers appeared, such as [3–5]. The algebraic structure of a module over a ring has also been generalized to the Γ-module over a Γ-ring in [1]. A useful algebraic structure in a variety of applications like algebraic automata theory, theoretical computer science and information theory is the notion of S-act over a semigroup S which is more general than a module over a ring (see, for example, [8]). A generalization of an S-act to the Γ-act over a Γ-semigroup can be found in [14] in connection with the consideration of radicals of Γ-semigroups. Here we study some properties of Γ-acts originating by the basic properties of S-acts. First, we describe a Γ-act in terms of a Γ-representation of a Γ-semigroup by Γ-transformations of a set. Then

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some particular morphisms in the category of Γ-acts are characterized. Finally, some results concerning cyclic, indecomposable and free Γ-acts are presented.

In the sequel we recall the definitions of S-act and Γ-semigroup.

For a semigroup S, a non-empty set A together with a mapping $\mu : S \times A \to A$ where $(s, a) \mapsto sa := \mu(s, a)$ is called a (*left*) *S-act* if for all $s, t \in S$ and $a \in A$, $(st)a = s(ta)$ holds. This is written as ${}_SA$. For a monoid S with an identity 1, we add the condition $1a = a$, for all $a \in A$. The definition of an S-act, in this form, first proposed by Hoehnke in [6, 7], with a different name in connection with the consideration of radicals of semigroups. For more information on this basic concept, see [9].

There are some different definitions for a Γ-semigroup in the literature (see for example [11–14]). Here we consider the one which is introduced in [11] as follows. Let S and Γ be non-empty sets. Then S is said to be a *Γ-semigroup* if there exists a mapping $\lambda : S \times \Gamma \times S \to S$, writing $\lambda(a, \gamma, b)$ as $a\gamma b$ satisfying the identity $(a\gamma b)\beta c = a\gamma(b\beta c)$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$. Let S be a Γ-semigroup. An element e of S is said to be a *left* (*right*) *identity* of S if, $e\gamma s = s$ ($s\gamma e = s$) for all $s \in S$ and $\gamma \in \Gamma$. A left as well as right identify is an *identity* of S. A Γ-semigroup with an identity is called a *Γ-monoid*. A non-empty subset I of S satisfying $S\Gamma I \subseteq I$ is called a *left Γ-ideal* of S. By a *left Γ-congruence* on S we mean an equivalence relation ρ on S for which $s\rho s'$ implies $(t\gamma s)\rho(t\gamma s')$ for $s, s', t \in S$ and $\gamma \in \Gamma$. Let S and T be two Γ-semigroups with left identities e and e', respectively. A map $f : S \to T$ satisfying $f(e) = e'$ and $f(s\gamma s') = f(s)\gamma f(s')$ for all $s, s' \in S, \gamma \in \Gamma$, is called a *Γ-semigroup homomorphism*.

2. The structure of Γ-S-acts

The purpose of this section is to study some basic properties of Γ-S-acts. Let us first give some definitions.

Let S be a Γ-semigroup and A be a non-empty set. Recall from [14] that if there exists a mapping $\lambda : S\times\Gamma\times A \to A$ where $(s, \gamma, a) \mapsto s\gamma a := \lambda(s, \gamma, a)$ such that $(s\gamma t)\beta a = s\gamma(t\beta a)$ for all $a \in A$, $s, t \in S$ and $\gamma, \beta \in \Gamma$, and if S has a left identity e, $e\gamma a = a$ for every $a \in A$ and $\gamma \in \Gamma$, then A is called a (*left*) *Γ-S-act*. If no confusion arises, a Γ-S-act A is simply called a *Γ-act* and is denoted by ${}_\Gamma A$. A non-empty subset A' of A is said to be a *Γ-subact* of A if $S\Gamma A' \subseteq A'$, that is, $s\gamma a' \in A'$ for all $s \in S$, $a' \in A'$ and $\gamma \in \Gamma$. Clearly, S itself is a Γ-S-act with its Γ-operation as the Γ-action. Also any left Γ-ideal of S is a Γ-subact of S. Let A be a Γ-S-act. An element $\theta \in A$ is called a *zero element* of A if $s\gamma\theta = \theta$ for every $s \in S$ and $\gamma \in \Gamma$. Let ${}_\Gamma A, {}_\Gamma B$ be two Γ-S-acts. A mapping $f : {}_\Gamma A \to {}_\Gamma B$ is called a *Γ-S-homomorphism*, or simply *Γ-homomorphism*, if $f(s\gamma a) = s\gamma f(a)$ for every $s \in S, a \in A$ and $\gamma \in \Gamma$. If S is a Γ-monoid with identity 1 and A is a Γ-S-act, then for every $s, t \in S$ and $\gamma, \beta \in \Gamma$, we have $s\gamma t = s\beta t$ and $s\gamma a = s\beta a$. Indeed, $s\gamma t = (s\beta 1)\gamma t = s\beta(1\gamma t) = s\beta t$; and $s\gamma a = (s\beta 1)\gamma a = s\beta(1\gamma a) = s\beta a$. Then it is more interesting to consider Γ-S-acts for a Γ-semigroup S with a left identity (not necessarily an identity). Therefore, from now on, S stands for a Γ-semigroup with a left identity e unless otherwise stated.

The idea of representing something by some other objects which are better known at least in some respects is quite familiar in mathematics. It is well known that every representation of a ring by endomorphisms of an abelian group gives a module over that ring and vice versa. Analogously, representations of semigroups (monoids) by transformations of sets give rise to the notion of acts over semigroups (monoids) (see [9, Proposition I.4.4]). In the same way, we here describe Γ-acts in terms of Γ-representations of Γ-semigroups.

Let A be a non-empty set and $\mathcal{A}$ denote the set of all maps $\varphi : \Gamma \to A^A$, the so called Γ-*transformations* of A, where A^A is the monoid of all transformations of A with the usual composition of mappings as its operation. Now we have

Lemma 1. *The set $\mathcal{A}$ is a Γ-semigroup with a left (not necessarily right) identity under the Γ-operation $\varphi\gamma\varphi' : \Gamma \to A^A$ defined by $\varphi\gamma\varphi'(\gamma') := \varphi(\gamma)\varphi'(\gamma')$ for all $\varphi, \varphi' \in \mathcal{A}$ and $\gamma, \gamma' \in \Gamma$.*

Proof. Let $\varphi, \varphi', \varphi'' \in \mathcal{A}$ and γ, γ'. Then $(\varphi\gamma\varphi')\gamma'\varphi'' = \varphi\gamma(\varphi'\gamma'\varphi'')$. Indeed, for any $\gamma'' \in \Gamma$ we have

$$\begin{aligned}
&[(\varphi\gamma\varphi')\gamma'\varphi''](\gamma'')\\
&= (\varphi\gamma\varphi')(\gamma')\varphi''(\gamma'')\\
&= \varphi(\gamma)\varphi'(\gamma')\varphi''(\gamma'')\\
&= \varphi(\gamma)(\varphi'\gamma'\varphi'')(\gamma'')\\
&= [\varphi\gamma(\varphi'\gamma'\varphi'')](\gamma''),
\end{aligned}$$

as desired. Also the constant mapping $\varepsilon : \Gamma \to A^A$ which maps every element of Γ to id_A is a left identity element of $\mathcal{A}$ which is not necessarily a right identity. □

In the following, the notion of representations of a semigroup by transformations of a set is generalized.

Definition 1. A Γ-*representation* of a Γ-semigroup S by Γ-transformations of a non-empty set A is a Γ-semigroup homomorphism $\Phi : S \to \mathcal{A}$, where $\mathcal{A}$ is the Γ-semigroup described in Lemma 1.

Proposition 1. *Every Γ-representation of a Γ-semigroup S by Γ-transformations of a non-empty set A in $\mathcal{A}$ turns A into a Γ-S-act. Conversely, for every Γ-S-act ${}_\Gamma A$, there is an associated Γ-representation of S by Γ-transformations of A in $\mathcal{A}$.*

Proof. Let A be a non-empty set and S be a Γ-semigroup. If $\Phi : S \to \mathcal{A}$ is a Γ-representation, define $\lambda : S \times \Gamma \times A \to A$ by $s\gamma a = \lambda(s, \gamma, a) := \Phi(s)(\gamma)(a)$, for all $s \in S, \gamma \in \Gamma, a \in A$. Then A is a Γ-S-act. For this, let $s, s' \in S, \gamma, \gamma' \in \Gamma, a \in A$. We have

$$\begin{gathered}
(s\gamma s')\gamma' a = \Phi(s\gamma s')(\gamma')(a) = (\Phi(s)\gamma\Phi(s'))(\gamma')(a) = \\
(\Phi(s)(\gamma))(\Phi(s')(\gamma')(a)) = (\Phi(s)(\gamma))(s'\gamma' a) = s\gamma(s'\gamma' a).
\end{gathered}$$

Also if e and ε are left identities of S and $\mathcal{A}$, respectively, then $e\gamma a = \Phi(e)(\gamma)(a) = \varepsilon(\gamma)(a) = id_A(a) = a$. For the converse, consider any Γ-S-act ${}_\Gamma A$. Define $\Phi : S \to \mathcal{A}$

by $\Phi(s) = \varphi_s : \Gamma \to A^A$, where $\varphi_s(\gamma)(a) := s\gamma a$ for all $s \in S, \gamma \in \Gamma, a \in A$. It must be shown that Φ is a Γ-semigroup homomorphism. Let $s, s' \in S$ and $\gamma \in \Gamma$. Then $\Phi(s\gamma s') = \Phi(s)\gamma\Phi(s')$, or equivalently, $\varphi_{s\gamma s'} = \varphi_s\gamma\varphi_{s'}$. To see this, let $\beta \in \Gamma$ and $a \in A$. We get

$$\begin{aligned}\varphi_{s\gamma s'}(\beta)(a) &= (s\gamma s')\beta a = s\gamma(s'\beta a) = s\gamma(\varphi_{s'}(\beta)(a)) \\ &= \varphi_s(\gamma)\varphi_{s'}(\beta)(a) = (\varphi_s\gamma\varphi_{s'})(\beta)(a).\end{aligned}$$

Therefore, Φ is a Γ-representation. $\square$

Remark 1. (i) Every S-act A over a semigroup S can be generalized to a Γ-S-act over the induced Γ-semigroup S. Indeed, first note that a semigroup S can be made into a Γ-semigroup by setting $s\gamma t := st$ for every $s, t \in S$ and $\gamma \in \Gamma$. Now define a mapping from $S \times \Gamma \times A$ to A by $s\gamma a := sa$ for every $s \in S$, $\gamma \in \Gamma$ and $a \in A$. Then A is a Γ-S-act. Moreover, if A is a Γ-S-act over a Γ-semigroup S and γ is a fixed element of Γ, then S is a semigroup under the operation $st := s\gamma t$ for all $s, t \in S$, and A with the action $sa := s\gamma a$, for every $s \in S$ and $a \in A$, is an S-act.

(ii) A Γ-S-act can have more than one zero, for instance, any non-empty set A becomes a Γ-S-act by definition $s\gamma a = a$ for every $a \in A$, $s \in S$ and $\gamma \in \Gamma$, i.e. all elements of A are zero. If S has a right zero z, i.e. $s\alpha z = z$ for any $s \in S, \alpha \in \Gamma$, then every element $z\gamma a$ for $a \in A$ and $\gamma \in \Gamma$ is a zero element of A. Indeed, for every $s \in S$ and $\alpha \in \Gamma$, $s\alpha(z\gamma a) = (s\alpha z)\gamma a = z\gamma a$. Note that every Γ-S-act A can be extended to a Γ-S-act with a zero θ by taking the disjoint union $A \cup \{\theta\}$.

In the following, we give some examples of Γ-S-acts.

Example 1. (i) Let S, Γ and M be the sets of all $3 \times 2, 2 \times 3$ and 3×3 matrices over $\mathbb{Z}$, respectively. Under the usual matrix products, S is a Γ-semigroup and M is a Γ-S-act but not an S-act.

(ii) Let $S = \{5n + 4 : \ n \in \mathbb{N}\}$, $\Gamma = \{5n + 1 : \ n \in \mathbb{N}\}$ and $A = \{5n : n \in \mathbb{N}\}$. Under the usual addition of natural numbers, S is a Γ-semigroup and A is a Γ-S-act but not an S-act.

(iii) If A is a Γ-S-act, then the power set of A, $P(A)$, is a Γ-S-act under the Γ-action $s\gamma X =: \{s\gamma x \mid x \in X\}$ for $s \in S, X \in P(A)$ and $\gamma \in \Gamma$.

(iv) Let S be a Γ-semigroup. Then the set of all 2×2 matrices over S is a Γ-S-act under the Γ-action:

$$s_1\gamma \begin{pmatrix} s & s' \\ t & t' \end{pmatrix} := \begin{pmatrix} s_1\gamma s & s_1\gamma s' \\ s_1\gamma t & s_1\gamma t' \end{pmatrix}$$

for $s_1, s, s', t, t' \in S$ and $\gamma \in \Gamma$.

(v) Let S and T be Γ-semigroups. Clearly, the cartesian product $S \times T$ is a Γ-semigroup with the Γ-operation $(s_1, t_1)\gamma(s_2, t_2) := (s_1\gamma s_2, t_1\gamma t_2)$ for every $s_1, s_2 \in S$, $t_1, t_2 \in T$ and

$\gamma \in \Gamma$. Now suppose S and T contain right zero elements z, z', respectively. Then the set $A = \{\begin{pmatrix} s & z \\ z' & t \end{pmatrix} : s \in S, t \in T\}$ is a Γ-$S \times T$-act under the Γ-action:

$$(s_1, t_1)\gamma \begin{pmatrix} s_2 & z \\ z' & t_2 \end{pmatrix} := \begin{pmatrix} s_1\gamma s_2 & z \\ z' & t_1\gamma t_2 \end{pmatrix}$$

for $s_1, s_2 \in S, t_1, t_2 \in T$ and $\gamma \in \Gamma$. To see this, let $s_1, s_2, s \in S, t_1, t_2, t \in T$ and $\alpha, \gamma \in \Gamma$. We have:

$$((s_1, t_1)\alpha(s_2, t_2))\gamma \begin{pmatrix} s & z \\ z' & t \end{pmatrix} = (s_1\alpha s_2, t_1\alpha t_2)\gamma \begin{pmatrix} s & z \\ z' & t \end{pmatrix}$$

$$= \begin{pmatrix} (s_1\alpha s_2)\gamma s & z \\ z' & (t_1\alpha t_2)\gamma t \end{pmatrix} = \begin{pmatrix} s_1\alpha(s_2\gamma s) & z \\ z' & t_1\alpha(t_2\gamma t) \end{pmatrix}$$

$$= (s_1, t_1)\alpha((s_2, t_2)\gamma \begin{pmatrix} s & z \\ z' & t \end{pmatrix}).$$

Hence, A is a Γ-$S \times U$-act.

We here generalize some basic properties of S-acts to Γ-S-acts.

Since the composition of two Γ-homomorphisms is a Γ-homomorphism, and the identity map on a Γ-act is a Γ-homomorphism, we conclude that all Γ-acts together with all Γ-homomorphisms between them forms a category which is denoted by **Γ-S-Act**, or simply **Γ-Act** if no confusion arise. The notions of Γ-*monomorphisms*, Γ-*epimorphisms* and Γ-*isomorphisms* in their categorical forms are defined as monomorphisms, epimorphisms and isomorphisms, respectively, in the category **Γ-Act**. Here we characterize these notions in terms of injective, surjective and bijective Γ-homomorphisms. Let us list some preliminaries.

If ${}_\Gamma A$ is a Γ-act, $a \in {}_\Gamma A$ and $\gamma \in \Gamma$, then the map $\lambda_{a,\gamma} : {}_\Gamma S \to {}_\Gamma A$ defined by $\lambda_{a,\gamma}(s) = s\gamma a$ for every $s \in S$ is a Γ-homomorphism. To see this, for every $t \in S$ and $\beta \in \Gamma$ we have $\lambda_{a,\gamma}(t\beta s) = (t\beta s)\gamma a = t\beta(s\gamma a) = t\beta\lambda_{a,\gamma}(s)$.

Let ${}_\Gamma A$ be a Γ-S-act. An equivalence relation ρ on A is called a Γ-S-*congruence*, or simply a Γ-*congruence*, on ${}_\Gamma A$ if $a\rho a'$ implies $(s\gamma a)\rho(s\gamma a')$ for every $a, a' \in {}_\Gamma A$, $s \in S$ and $\gamma \in \Gamma$. The set $\frac{{}_\Gamma A}{\rho} = \{[a]_\rho : a \in {}_\Gamma A\}$ with the Γ-action $s\gamma[a]_\rho = [s\gamma a]_\rho$ for every $s \in S$ and $\gamma \in \Gamma$ is called the *factor* Γ-*act* of ${}_\Gamma A$ by ρ, and the canonical surjection $\pi_\rho : {}_\Gamma A \to \frac{{}_\Gamma A}{\rho}$ where $a \mapsto [a]_\rho$ is called the *canonical* Γ-*epimorphism.* Also for a Γ-homomorphism $f : {}_\Gamma A \to {}_\Gamma B$, the Γ-congruence $\rho = ker f$ on ${}_\Gamma A$ where $a\rho a'$ if and only if $f(a) = f(a')$, for all $a, a' \in A$, is called the *kernel* Γ-*congruence* of f. For each Γ-subact ${}_\Gamma B$ of ${}_\Gamma A$, the *Rees* Γ-*congruence* ρ_B on A is given as $a\rho_B a'$ if and only if $a = a'$ or $a, a' \in B$, for any $a, a' \in A$. The resulting factor Γ-act $\frac{{}_\Gamma A}{\rho_B}$ is simply denoted by $\frac{{}_\Gamma A}{{}_\Gamma B}$.

Proposition 2. *In the category* Γ**-Act**, Γ*-monomorphisms,* Γ*-epimorphisms and* Γ*-isomorphisms are exactly injective, surjective and bijective* Γ*-homomorphisms, respectively.*

Proof. It is easy to see that every injective Γ-homomorphism is a Γ-monomorphism, every surjective Γ-homomorphism is a Γ-epimorphism, and every Γ-isomorphism is bijective. Take any Γ-homomorphism $f : {}_\Gamma A \to {}_\Gamma B$. Suppose f is a Γ-monomorphism and $f(a) = f(a')$ for any $a, a' \in A$. We show that $a = a'$. Let $\gamma \in \Gamma$. Consider the Γ-homomorphisms $\lambda_{a,\gamma}, \lambda_{a',\gamma} : {}_\Gamma S \to {}_\Gamma A$. We claim that $f\lambda_{a,\gamma} = f\lambda_{a',\gamma}$. For every $s \in S$, $f\lambda_{a,\gamma}(s) = f(\lambda_{a,\gamma}(s)) = f(s\gamma a) = s\gamma f(a) = s\gamma f(a') = f(s\gamma a') = f(\lambda_{a',\gamma}(s)) = f\lambda_{a',\gamma}(s)$. Since f is a Γ-monomorphism, $\lambda_{a,\gamma} = \lambda_{a',\gamma}$. Hence, $a = e\gamma a = \lambda_{a,\gamma}(e) = \lambda_{a',\gamma}(e) = e\gamma a' = a'$, where e is a left identity of S. Then f is injective. Now let f be a Γ-epimorphism. Clearly, Imf is a Γ-subact of B. Consider Γ-homomorphisms $g, h : {}_\Gamma B \to \frac{{}_\Gamma B}{Imf}$ defined by $g(b) = Imf$ and $h(b) = [b]_{Imf}$ for every $b \in B$, respectively. Clearly, $gf = hf$ and then $g = h$ because f is a Γ-epimorphism. It follows that for every $b \in B$, $Imf = g(b) = h(b) = [b]_{Imf}$ whence $Imf = B$, i.e. f is surjective. Finally, assume that f is bijective. It suffices to show that f^{-1} is a Γ-homomorphism. Let $s \in S, \gamma \in \Gamma, b \in B$. Then there exists $a \in A$ such that $f(a) = b$ and hence $f^{-1}(s\gamma b) = f^{-1}(s\gamma f(a)) = f^{-1}(f(s\gamma a)) = s\gamma a = s\gamma f^{-1}(b)$. This implies that f is a Γ-isomorphism. $\square$

Remark 2. Let S be a Γ-semigroup, ${}_\Gamma A$ a Γ-S-act and $f : {}_\Gamma A \to {}_\Gamma S$ a Γ-homomorphism. Then A is a Γ-semigroup under the Γ-operation $a\gamma a' := f(a)\gamma a'$ for every $a, a' \in A$ and $\gamma \in \Gamma$. For this, let $a, a', a'' \in A$ and $\alpha, \gamma \in \Gamma$. Then

$$\begin{aligned}(a\alpha a')\gamma a'' &= (f(a)\alpha a')\gamma a'' = f(f(a)\alpha a')\gamma a'' = (f(a)\alpha f(a'))\gamma a'' \\ &= f(a)\alpha(f(a')\gamma a'') = a\alpha(f(a')\gamma a'') = a\alpha(a'\gamma a'').\end{aligned}$$

Theorem 2 (Homomorphism Theorem for Γ-Acts)**.** *Let* $f : {}_\Gamma A \to {}_\Gamma B$ *be a* Γ*-homomorphism and* ρ *be a* Γ*-congruence on* ${}_\Gamma A$ *such that* $a\rho a'$ *implies* $f(a) = f(a')$*, i.e.* $\rho \leq kerf$*. Then* $f' : \frac{{}_\Gamma A}{\rho} \to {}_\Gamma B$ *with* $f'([a]_\rho) := f(a)$*,* $a \in {}_\Gamma A$*, is the unique* Γ*-homomorphism such that* $f'\pi_\rho = f$*. If* ρ=$kerf$*, then* f' *is injective. Also if* f *is surjective, then so is* f'*.*

Proof. The mapping f' is well-defined, because for every $[a]_\rho, [a']_\rho \in \frac{{}_\Gamma A}{\rho}$, $[a]_\rho = [a']_\rho \Leftrightarrow a\rho a' \Rightarrow f(a) = f(a') \Rightarrow f'([a]_\rho) = f'([a']_\rho)$. For every $s \in S$,$\gamma \in \Gamma$ and $a \in A$, $f'(s\gamma[a]_\rho) = f'([s\gamma a]_\rho) = f(s\gamma a) = s\gamma f(a) = s\gamma f'([a]_\rho)$. Hence, f' is a Γ-homomorphism. Also for every $a \in {}_\Gamma A$, $(f'\pi_\rho)(a) = f'(\pi_\rho(a)) = f'([a]_\rho) = f(a)$. Now we show that f' is unique. Suppose there exists $f'' : \frac{{}_\Gamma A}{\rho} \to {}_\Gamma B$ such that $f''\pi_\rho = f$. This implies that $f''\pi_\rho = f'\pi_\rho$. Since π_ρ is an epimorphism, $f'' = f'$. The remainder is an easy verification. $\square$

Corollary 1. *Let* $f : {}_\Gamma A \to {}_\Gamma B$ *be a* Γ*-epimorphism. Then* $\frac{{}_\Gamma A}{kerf} \cong {}_\Gamma B$*.*

3. Cyclic, Indecomposable and Free Γ-S-Acts

In this section we study the notions of cyclic, free and indecomposable Γ-S-acts and investigate their properties. For each Γ-act, a unique decomposition into indecomposable Γ-subacts is obtained. It is also proved that if a Γ-act is free, then Γ is a singleton.

Definition 2. A subset $U \neq \emptyset$ of a Γ-S-act ${}_\Gamma A$ is said to be a *generating set* of ${}_\Gamma A$ if every element $a \in A$ can be presented as $a = s\gamma u$ for some $s \in S, u \in U$ and $\gamma \in \Gamma$. In this case, we write ${}_\Gamma A = \langle U \rangle$ (or $S\Gamma U$), where $S\Gamma U = \{s\gamma u : s \in S, \gamma \in \Gamma, u \in U\}$. For simplicity, we use the notations $S\gamma U$ and $S\Gamma u$ for $S\{\gamma\}U$ and $S\Gamma\{u\}$, respectively. Also A is *finitely generated* if it has a finite generating set of elements. We call ${}_\Gamma A$ a *cyclic* Γ-S-*act* if ${}_\Gamma A = \langle a \rangle$ $(= S\Gamma a)$ for some $a \in {}_\Gamma A$. Not that ${}_\Gamma A = \langle A \rangle$, i.e. ${}_\Gamma A$ is always a generating set of itself.

Lemma 3. *Let U be a non-empty subset of a Γ-act ${}_\Gamma A$ and $a \in {}_\Gamma A$. Then the following assertions hold:*

(i) *$S\Gamma a = S\gamma a$ for every $\gamma \in \Gamma$.*

(ii) *$S\gamma a = S\beta a$ for every $\gamma, \beta \in \Gamma$.*

(iii) *$S\Gamma U = S\gamma U$ for every $\gamma \in \Gamma$.*

Proof. (i) Let $\gamma \in \Gamma$ and $a \in {}_\Gamma A$. Clearly, $S\gamma a \subseteq S\Gamma a$. For the reverse inclusion, take any $\beta \in \Gamma$ and $s \in S$. Then $s\beta a = s\beta(e\gamma a) = (s\beta e)\gamma a \in S\gamma a$ which implies that $S\Gamma a = S\gamma a$.

(ii) Let $\gamma, \beta \in \Gamma$. Using (i), we get $S\gamma a = S\Gamma a$ and $S\beta a = S\Gamma a$. Then $S\gamma a = S\beta a$.

(iii) Let $\gamma \in \Gamma$. It follows from (i) that $S\gamma U = \bigcup_{u \in U} S\gamma u = \bigcup_{u\in U} S\Gamma u = S\Gamma U$. □

The above lemma presents a simple characterization for generating subsets of a Γ-act. In particular, one can consider a cyclic Γ-act ${}_\Gamma A = \langle a \rangle$ as $S\gamma a$ for any $\gamma \in \Gamma$.

In the following, we characterize cyclic Γ-acts in terms of the factor Γ-acts of ${}_\Gamma S$.

Theorem 4. *If a Γ-act ${}_\Gamma A$ is cyclic, then there exists a Γ-congruence ρ on ${}_\Gamma S$ such that ${}_\Gamma A \cong \frac{{}_\Gamma S}{\rho}$. The converse also holds provided S is a Γ-monoid.*

Proof. Let ${}_\Gamma A = S\gamma a$ for some $a \in {}_\Gamma A$ and $\gamma \in \Gamma$. Then the Γ-homomorphism $\lambda_{a,\gamma} : {}_\Gamma S \to {}_\Gamma A$ is obviously a Γ-epimorphism. Using Corollary 1, we get ${}_\Gamma A \cong \frac{{}_\Gamma S}{ker\lambda_{a,\gamma}}$. Then setting $\rho = ker\lambda_{a,\gamma}$ we get the result. Conversely, if ρ is a Γ-congruence on a Γ-monoid ${}_\Gamma S$ with identity 1, then for every $[s]_\rho \in \frac{{}_\Gamma S}{\rho}$ and $\gamma \in \Gamma$, $[s]_\rho = [s\gamma 1]_\rho = s\gamma[1]_\rho$ which shows that $\frac{{}_\Gamma S}{\rho} = \langle [1]_\rho \rangle$. □

A Γ-act is called *simple* if it contains no proper Γ-subacts. It is clear that a simple act must be cyclic. Now we give conditions under which cyclic Γ-acts, principal left Γ-ideals and Rees factor Γ-acts of a Γ-monoid by left Γ-ideals are simple.

Proposition 3. *Let ρ be a left Γ-congruence on a Γ-monoid ${}_\Gamma S$. The cyclic Γ-act $\frac{{}_\Gamma S}{\rho}$ is simple if and only if $[1]_\rho \cap S\gamma t \neq \emptyset$ for any $t \in S$ and $\gamma \in \Gamma$.*

Proof. For a left Γ-congruence ρ on ${}_\Gamma S$, consider the canonical Γ-epimorphism $\pi : {}_\Gamma S \to \frac{{}_\Gamma S}{\rho}$. Let $\frac{{}_\Gamma S}{\rho}$ be simple and $t \in S, \gamma \in \Gamma$. Since $\pi(S\gamma t)$ is a Γ-subact of $\frac{{}_\Gamma S}{\rho}$ and $\frac{{}_\Gamma S}{\rho}$ is simple, $\pi(S\gamma t) = \frac{{}_\Gamma S}{\rho}$. Hence, there exists $u \in S\gamma t$ such that $\pi(u) = [1]_\rho$. Thus $u \in [1]_\rho$ and then $[1]_\rho \cap S\gamma t \neq \emptyset$. Conversely, let ${}_\Gamma A$ be a Γ-subact of $\frac{{}_\Gamma S}{\rho}$. Take any $t \in \pi^{-1}(A)$ and $\gamma \in \Gamma$. Using the assumption, there exists $s \in S$ such that $s\gamma t \in [1]_\rho$. Now $[1]_\rho = \pi(s\gamma t) = s\gamma\pi(t) \in {}_\Gamma A$. This implies that ${}_\Gamma A = \frac{{}_\Gamma S}{\rho}$ and hence $\frac{{}_\Gamma S}{\rho}$ is simple. □

The following two statements are corollaries of the previous proposition. They can also be obtained straightforward from the definition of a simple Γ-act.

Corollary 2. *A principal left* Γ*-ideal* $S\gamma z, z \in S, \gamma \in \Gamma$ *is a simple* Γ*-act if and only if* $z \in S\beta t\gamma z$ *for all* $t \in S, \beta \in \Gamma$.

Corollary 3. *Let* I *be a left* Γ*-ideal of* S. *The Rees factor* Γ*-act* $\frac{{}_\Gamma S}{I}$ *is simple if and only if* $I = S$.

Definition 3. A Γ-act ${}_\Gamma A$ is called *decomposable* if there exist two Γ-subacts ${}_\Gamma B$ and ${}_\Gamma C$ of ${}_\Gamma A$ such that ${}_\Gamma A = {}_\Gamma B \cup {}_\Gamma C$ and ${}_\Gamma B \cap {}_\Gamma C = \emptyset$. In this case, the disjoint union ${}_\Gamma B \,\dot\cup\, {}_\Gamma C$ is called a *decomposition* of ${}_\Gamma A$. Otherwise, ${}_\Gamma A$ is called *indecomposable.* If we consider Γ-S-acts with unique zero θ, then we have to replace $\emptyset$ by $\{\theta\}$ to define decomposable and indecomposable Γ-acts with unique zero.

Recall that every S-act has a unique decomposition into indecomposable subacts (see [9, I.5.10]). In the following, an analogous result is obtained for the decomposition of Γ-acts. To this end, first note the following:

Proposition 4. *Every cyclic* Γ*-act is indecomposable.*

Proof. Suppose ${}_\Gamma A = S\gamma a$, $\gamma \in \Gamma, a \in A$, is cyclic and $A = {}_\Gamma B \,\dot\cup\, {}_\Gamma C$ for some Γ-subacts ${}_\Gamma B$ and ${}_\Gamma C$ of ${}_\Gamma A$. Then $a = e\gamma a \in {}_\Gamma B$, say, and then ${}_\Gamma A = S\gamma a \subseteq {}_\Gamma B$ which is a contradiction. □

Lemma 5. *Let* $A_i \subseteq {}_\Gamma A$, $i \in I$, *be indecomposable* Γ*-subacts of an* Γ*-act* ${}_\Gamma A$ *such that* $\cap_{i\in I} A_i \neq \emptyset$. *Then* $\cup_{i\in I} A_i$ *is an indecomposable* Γ*-subact of* ${}_\Gamma A$.

Proof. First note that $\cup_{i\in I} A_i$ is a Γ-subact of ${}_\Gamma A$. Indeed, $S\Gamma A_i \subseteq A_i$ for every $i \in I$ whence $S\Gamma(\cup_{i\in I} A_i) = \cup_{i\in I}(S\Gamma A_i) \subseteq \cup_{i\in I} A_i$. Suppose there exists a decomposition $\cup_{i\in I} A_i = {}_\Gamma B \,\dot\cup\, {}_\Gamma C$. Take $x \in \cap_{i\in I} A_i$ with $x \in {}_\Gamma B$, say. Then $x \in A_i \cap {}_\Gamma B$ for all $i \in I$. Since $A_i = A_i \cap ({}_\Gamma B \,\dot\cup\, {}_\Gamma C) = (A_i \cap {}_\Gamma B) \,\dot\cup\, (A_i \cap {}_\Gamma C)$ and A_i is indecomposable, $A_i \cap {}_\Gamma C = \emptyset$ for every $i \in I$. Thus $\cup_{i\in I} A_i = {}_\Gamma B$ which is a contradiction. □

Theorem 6. *Every* Γ*-*S*-act* ${}_\Gamma A$ *has a unique decomposition into indecomposable* Γ*-subacts.*

Proof. Take $x \in {}_\Gamma A$. Then $S\gamma x$, $\gamma \in \Gamma$, is indecomposable by Proposition 4. Using Lemma 5, we get $U_x = \bigcup \{{}_\Gamma U \subseteq {}_\Gamma A : {}_\Gamma U \text{ is indecomposable and } x \in {}_\Gamma U\}$ is an indecomposable Γ-subact of ${}_\Gamma A$. For $x, y \in {}_\Gamma A$, $U_x = U_y$ or $U_x \cap U_y = \emptyset$. Indeed, $z \in U_x \cap U_y$ implies $U_x, U_y \subseteq U_z$. Thus $x \in U_x \subseteq U_z$, $y \in U_y \subseteq U_z$, i.e. $U_z \subseteq U_x \cap U_y$. Therefore, $U_x = U_y = U_z$. Denote by A' a representative subset of elements $x \in {}_\Gamma A$ with respect to the equivalence relation $\sim$ defined by $x \sim y$ if and only if $U_x = U_y$. Then ${}_\Gamma A = \dot\bigcup_{x\in A'} U_x$ is the unique decomposition of ${}_\Gamma A$ into indecomposable subacts. □

Definition 4. A set U of generating elements of a Γ-S-act ${}_\Gamma A$ is said to be a *basis* of ${}_\Gamma A$ if every element $a \in {}_\Gamma A$ can be uniquely presented in the from $a = s\gamma u$ for some $s \in S, u \in U$ and $\gamma \in \Gamma$, i.e. if $a = s\gamma u = s'\gamma' u'$ for $s, s' \in S$, $u, u' \in U$ and $\gamma, \gamma' \in \Gamma$, then $s = s', u = u'$ and $\gamma = \gamma'$. If a Γ-act ${}_\Gamma A$ has a basis U, then it is called a *free* Γ*-act*.

Proposition 5. *Let $f : {}_\Gamma A \to {}_\Gamma B$ be a Γ-homomorphism.*

(i) *If ${}_\Gamma A$ if finitely generated then so is $f({}_\Gamma A)$.*

(ii) *If ${}_\Gamma A = \langle U \rangle$ and $g : {}_\Gamma A \to {}_\Gamma B$ is a Γ-homomorphism, then $f(u) = g(u)$ for every $u \in U$ implies $f = g$.*

(iii) *If f is a Γ-epimorphism and ${}_\Gamma A = \langle U \rangle$, then ${}_\Gamma B = \langle f(U) \rangle$.*

(iv) *If is a Γ-isomorphism and ${}_\Gamma A$ is a free Γ-act, then so is ${}_\Gamma B$.*

Proof. It is straightforward. □

The following result shows that there is no free Γ-act whenever $|\Gamma| > 1$.

Theorem 7. *If ${}_\Gamma A$ is a free Γ-act, then $|\Gamma| = 1$.*

Proof. Suppose ${}_\Gamma A$ is a free Γ-act with a basis U. Consider $\gamma, \gamma' \in \Gamma$, $s \in S$ and $u \in U$. Using Lemma 3(ii), $s\gamma u \in S\gamma u = S\gamma' u$ and then $s\gamma u = s'\gamma' u'$ for some $s, s' \in S$ and $u, u' \in U$. Since U is a basis, $\gamma = \gamma'$. □

Remark 3. Let $|\Gamma| = 1$. Using Remark 1(i), the category **Γ-S-Act** is equivalent to the category of all acts over the induced semigroup S (containing a left identity). Therefore, any categorical property of such Γ-acts coincides with the analogous property of their corresponding acts. For a $\Gamma = \{\gamma\}$ and a Γ-semigroup S with a left identity, in view of the constructing free acts over monoids as in [9, Construction I.5.14], a free Γ-S-act with a basis $X \neq \emptyset$ is isomorphic to $S \times \Gamma \times X$ with the action $s\gamma(t, \gamma, x) := (s\gamma t, \gamma, x)$ for all $s, t \in S$ and $x \in X$. Furthermore, any free Γ-act is invariant under the cardinality of its bases and the universal property of freeness holds for free Γ-acts. Hence, every Γ-act is a factor Γ-act of a free Γ-act (see [9, Theorem I.5.15, Proposition I.5.16]).

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