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On identities for sequences of binomial sums with the terms of sequences $\{u_{kn}\}$ and $\{v_{kn}\}$

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Abstract. In this paper, considering technique used in [4], and the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$, we derive the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$. Also with the aid of generating matrix for the terms of these sequences for a positive integer k, we derive some combinatorial identities for the sequence $\{g_{kn}\}$.

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1. Introduction

Matrix methods are very convenient for deriving certain of linear recurrence sequences. Some authors have used matrix methods of other methods to derive some identities, combinatorial representations of linear recurrence relations etc[3, 6, 10, 13, 14, 15, 16, 8, 9].

In [13], the author gives a new formula for the nth power of an arbitrary 2×2 matrix and derive various matrix identities and formulae for the nth power of particular matrices to obtain various combinatorial identities. The generalized second order sequences $\{u_n\}$ and $\{v_n\}$, are defined for $n > 0$ and nonzero integer numbers p, q by

$$u_{n+1} = pu_n + qu_{n-1} \text{ and } v_{n+1} = pv_n + qv_{n-1}$$

in which $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = p$, respectively. When $p = q = 1$, $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number).

If α and β are the roots of equation $x^2 - px - q = 0$, the Binet formulae of the sequences $\{u_n\}$ and $\{v_n\}$ have the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

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respectively. From [7], E. Kılıç and P. Stanica derived the following recurrence relations for the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ for $k \geq 0$, $n > 0$. It is clearly that

$$u_{k(n+1)} = v_k u_{kn} + (-1)^{k+1} q^k u_{k(n-1)} \text{ and } v_{k(n+1)} = v_k v_{kn} + (-1)^{k+1} q^k v_{k(n-1)},$$

where the initial conditions of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are 0, u_k, and 2, v_k, respectively. The Binet formulae of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$u_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \text{ and } v_{kn} = \alpha^{kn} + \beta^{kn},$$

respectively. From the Binet formulas, one can see that $u_{-kn} = (-1)^{kn+1} u_{kn}$ and $u_{2kn} = u_{kn} v_{kn}$.

In [1] and [2], the authors obtained some new identities for the sequence $\{u_n\}$. For example, for $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\alpha}{q}\right)^k u_k = \frac{\alpha}{q} \left(\frac{p\alpha}{q} + 2\right)^{n-1},$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\beta}{q}\right)^k u_k = \frac{\beta}{q} \left(\frac{p\beta}{q} + 2\right)^{n-1}.$$

Let $\{a_k\}$ and $\{b_k\}$ be sequences with the property that a_k is the finite difference of b_k, that is, $a_k = \Delta b_k := b_{k+1} - b_k$, for $k \geq 0$. We take

$$g_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \text{and} \quad h_n = \sum_{k=0}^{n} \binom{n}{k} b_k. \tag{1}$$

In [11], Komatsu obtained several sequences of binomial sums of generalized Fibonacci numbers. For example,

$$\sum_{k=0}^{n} \binom{n}{k} c^k u_k = r_n \, (n \geq 0)$$

satisfies the recurrence relation

$$r_n = (ac + 2) \, r_{n-1} + \left(bc^2 - ac - 1\right) r_{n-2} \quad (n \geq 2)$$

with $r_0 = 0$, $r_1 = c$ and

$$\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k u_k = \lambda_n \, (n \geq 0)$$

satisfies the recurrence relation

$$\lambda_n = (ad + 2c) \, \lambda_{n-1} + \left(bd^2 - acd - c^2\right) \lambda_{n-2} \quad (n \geq 2)$$

with initial conditions $\lambda_0 = 0$ and $\lambda_1 = d$, where c, d are nonzero real numbers.

In [4], the authors obtain some binomial summation identities of sequences $\{r_n\}$ and $\{\lambda_n\}$:

$$\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k\left(bc^2-ac-1\right)^{2n-k}r_{2k+1} = (ac+2)^{2n}r_{2n+1},$$
$$\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k\left(bc^2-ac-1\right)^{2n-k}r_{2k} = (ac+2)^{2n}r_{2n}.$$

2. Some Results

In this section, firstly, we define sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ and then derive some new combinatorial identities for these sequences.

Lemma 1. *For $n \geq 0$, the sum $\sum\limits_{i=0}^{n}\binom{n}{i}c^{ki}u_{ki} = g_{kn}$ satisfies the recurrence relation*

$$g_{k(n+2)} = \left(c^k v_k + 2\right)g_{k(n+1)} - \left(c^{2k}(-q)^k + c^k v_k + 1\right)g_{kn},$$

where initial conditions $g_0 = 0$, $g_k = c^k u_k$.

Proof. Considering $a_{kn} = c^{kn}u_{kn}$ and $b_{kn} = c^{k(n+1)}u_{k(n+1)}$ in (1), the proof is completed as similar to proof technique in [11].

Lemma 2. *The generating function $U(z)$ of $\sum\limits_{i=0}^{n}\binom{n}{i}c^{ki}u_{ki} = g_{kn}$ is given by*

$$U(z) = \frac{z^k g_k}{1-\left(c^k v_k+2\right)z^k + \left(c^{2k}(-q)^k + c^k v_k + 1\right)z^{2k}}.$$

Proof. Observed that

$$\begin{aligned}
U(z) &= g_0 z^0 + g_k z^k + g_{2k}z^{2k} + ... + g_{kn}z^{kn} + ...\\
z^k U(z) &= g_0 z^k + g_k z^{2k} + g_{2k}z^{3k} + ... + g_{k(n-1)}z^{kn} + ...\\
z^{2k} U(z) &= g_0 z^{2k} + g_k z^{3k} + g_{2k}z^{4k} + ... + g_{k(n-2)}z^{kn} + ...\\
&\vdots
\end{aligned}$$

From here, we have

$$\begin{aligned}
&U(z)\left(1-\left(c^k v_k+2\right)z^k - \left(c^{2k}(-q)^k + c^k v_k + 1\right)z^{2k}\right)\\
=\ & z^k g_k + \sum_{i=2}^{\infty}\left(g_{ki} - \left(c^k v_k + 2\right)g_{k(i-1)} + \left(c^{2k}(-q)^k + c^k v_k + 1\right)g_{k(i-2)}\right)z^{ki}.
\end{aligned}$$

From the recurrence relation in Lemma 1, we complete the proof for $U(z)$.

Similarly, the proofs of the following lemmas are given as the proofs of Lemmas 1 and 2.

Lemma 3. *For $n \geq 0$, the sum $\sum_{i=0}^{n} \binom{n}{i} c^{ki} v_{ki} = h_{kn}$ satisfies the recurrence relation*

$$h_{k(n+2)} = \left(c^k v_k + 2\right) h_{k(n+1)} - \left(c^{2k}(-q)^k + c^k v_k + 1\right) h_{kn},$$

where initial conditions $h_0 = 2, h_k = 2 + c^k v_k$.

Lemma 4. *The generating function $V(z)$ of $\sum_{i=0}^{n} \binom{n}{i} c^{ki} v_{ki} = h_{kn}$ is given by*

$$V(z) = \frac{z^k \left(h_k - 2\left(c^k v_k + 2\right)\right) + 2}{1 - \left(c^k v_k + 2\right) z^k + \left(c^{2k}(-q)^k + c^k v_k + 1\right) z^{2k}}.$$

If $c^k \alpha^k + 1$ and $c^k \beta^k + 1$ are the roots of equation

$$x^2 - \left(c^k v_k + 2\right) x + \left(c^{2k}(-q)^k + c^k v_k + 1\right) = 0,$$

Binet formulae of sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ are

$$g_{kn} = \frac{\left(c^k \alpha^k + 1\right)^n - \left(c^k \beta^k + 1\right)^n}{\alpha - \beta} \text{ and } h_{kn} = \left(c^k \alpha^k + 1\right)^n + \left(c^k \beta^k + 1\right)^n,$$

respectively. It is clear that

$$g_{-kn} = -\left(c^{2k}(-q)^k + c^k v_k + 1\right)^{-n} g_{kn}, \quad g_{2kn} = g_{kn} h_{kn} \tag{2}$$

and $h_{-kn} = \left((-q)^k c^{2k} + c^k v_k + 1\right)^{-n} h_{kn}$.

Now, we define a 2×2 matrix A and then we give some new results for the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ by matrix methods.

Consider the 2×2 matrix A as follows:

$$A = \begin{bmatrix} c^k v_k + 2 & -\left(c^{2k}(-q)^k + c^k v_k + 1\right) \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix A are

$$\lambda_1 = c^k \alpha^k + 1, \quad \lambda_2 = c^k \beta^k + 1.$$

Also λ_1, λ_2 are distinct. Let V be the 2×2 matrix defined as follows:

$$V = \begin{bmatrix} c^k \alpha^k + 1 & c^k \beta^k + 1 \\ 1 & 1 \end{bmatrix}.$$

One can easily verify that

$$AV = VD_1,$$

where $D_1 = diag\left(\lambda_1,\ \lambda_2\right)$. Since $\det V \neq 0$, the matrix V invertible. So, we write

$$D_1 = V^{-1}AV.$$

Thus, the matrix A is similar to the diagonal matrix D_1. We obtain

$$\begin{aligned} A^n &= VD_1^nV^{-1} \\ &= \frac{1}{c^k u_k}\begin{bmatrix} g_{k(n+1)} & -\left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{kn} \\ g_{kn} & -\left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{k(n-1)} \end{bmatrix}. \end{aligned}$$

Clearly, the matrix A^n satisfies the recurrence relation: for $n > 0$,

$$A^{n+1} = \left(c^k v_k + 2\right) A^n - \left(c^{2k}(-q)^k + c^k v_k + 1\right) A^{n-1},$$

where initial conditions $A^0 = 0,\ A^1 = A$.

Also by matrix methods, it is clearly that

$$A^n \begin{bmatrix} g_k \\ g_0 \end{bmatrix} = \begin{bmatrix} g_{k(n+1)} \\ g_{kn} \end{bmatrix}. \tag{3}$$

For $n \geq 0$, if we consider the fact that $\det(A^n) = (\det A)^n$, then we obtain the Cassini identity

$$g_{kn}^2 - g_{k(n+1)}g_{k(n-1)} = \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right)^{n-1} g_k^2.$$

For example, for $k = c = p = q = 1$, we write $F_{2n}^2 - F_{2n+1}F_{2n-1} = -1$(see page 74, [12]).

Similarly

$$\begin{aligned} A^{-n} = & \frac{1}{c^k u_k \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right)^n} \times \\ & \begin{bmatrix} -\left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{k(n-1)} & \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{kn} \\ -g_{kn} & g_{k(n+1)} \end{bmatrix} \end{aligned}$$

and

$$A^{-n} \begin{bmatrix} g_k \\ g_0 \end{bmatrix} = \begin{bmatrix} g_{k(-n+1)} \\ g_{-kn} \end{bmatrix}. \tag{4}$$

By considering sequence $\{h_{kn}\}$, we write the simple relation between the vector of sequence $\{h_{kn}\}$ and generating matrix of sequence $\{g_{kn}\}$:

$$\begin{bmatrix} h_{k(n+1)} \\ h_{kn} \end{bmatrix} = \frac{1}{c^k u_k}\begin{bmatrix} g_{k(n+1)} & -\left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{kn} \\ g_{kn} & -\left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{k(n-1)} \end{bmatrix}\begin{bmatrix} c^k v_k + 2 \\ 2 \end{bmatrix}.$$

Theorem 1. *For all $n, m \in \mathbb{Z}$, we have*

$$c^k u_k g_{k(n+m)} = g_{kn} g_{k(m+1)} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{k(n-1)} g_{km}. \tag{5}$$

Proof. After some simplications,$(2,1)-$entries of $A^n A^m = A^{n+m}$ give the conclusion.

For example, when $n = m$ in (5), we have

$$c_k u_k g_{2kn} = g_{kn} g_{k(n+1)} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{kn} g_{k(n-1)},$$

or

$$c_k u_k h_{kn} = g_{k(n+1)} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g_{k(n-1)}.$$

Theorem 2. *For all $n \in \mathbb{Z}$, we have*

$$\begin{aligned} & c^k u_k \left(g_{k(2n+1)} + g_{k(2n-1)}\right) \\ = \; & g^2_{k(n+1)} - \left(c^{2k}\left(-q\right)^k + c^k v_k\right) g^2_{kn} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g^2_{k(n-1)}, \\ & c^k u_k \left(g_{k(2n+1)} - g_{k(2n-1)}\right) \\ = \; & g^2_{k(n+1)} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 2\right) g^2_{kn} + \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g^2_{k(n-1)}. \end{aligned}$$

Proof. Considering the $(1,1)$ and $(2,2)-$entries of the matrix equation $A^{2n} = (A^n)^2$, we have

$$g^2_{k(n+1)} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g^2_{kn} = c^k u_k g_{k(2n+1)}, \tag{6}$$

$$g^2_{kn} - \left(c^{2k}\left(-q\right)^k + c^k v_k + 1\right) g^2_{k(n-1)} = c^k u_k g_{k(2n-1)}. \tag{7}$$

By adding and substracting of (6) and (7) side by side, we have the conclusion.

Theorem 3. *For $n > 0$, we have*

$$g_{kn} = \sum_{t=0}^{n} \binom{n}{t} \left(c^k v_k + 2\right)^t (-1)^{n-t} \left(c^{2k}(-q)^k + c^k v_k + 1\right)^{-n+t} g_{k(n-t)}.$$

Proof. From the matrix relation, we write

$$\begin{aligned} A^n &= \left(\left(c^k v_k + 2\right) I - \left(c^{2k}(-q)^k + c^k v_k + 1\right) A^{-1}\right)^n \\ &= \sum_{t=0}^{n} \binom{n}{t} \left(c^k v_k + 2\right)^t \left(c^{2k}(-q)^{k+1} - c^k v_k - 1\right)^{n-t} A^{-(n-t)}. \end{aligned} \tag{8}$$

Then equating $(2,1)-$entries of the equality (8), we get

$$g_{kn} = \sum_{t=0}^{n} \binom{n}{t} \left(c^k v_k + 2\right)^t (-1)^{n-t} \left(c^{2k}(-q)^k + c^k v_k + 1\right)^{-n+t} g_{k(n-t)}.$$

Theorem 4. *For $n>0$, we have*

$$\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{k(2i+1)} = \left(c^kv_k+2\right)^{2n}g_{k(2n+1)},$$
$$\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{2ki} = \left(c^kv_k+2\right)^{2n}g_{2kn}.$$

Proof. The matrix A^2 is

$$\begin{bmatrix} \left(c^kv_k+2\right)^2+q^k(-1)^{k+1}c^{2k}-c^kv_k-1 & \left(c^kv_k+2\right)\left(q^k(-1)^{k+1}c^{2k}-c^kv_k-1\right) \\ c^kv_k+2 & q^k(-1)^{k+1}c^{2k}-c^kv_k-1 \end{bmatrix}$$

and the characteristic equation for A^2 is

$$\lambda\left(c^kv_k+2\right)^2=\left(\lambda+\left(c^{2k}(-q)^k+c^kv_k+1\right)\right)^2. \tag{9}$$

From the Caley-Hamilton Theorem for A^2, we have

$$A^2\left(c^kv_k+2\right)^2=\left(A^2+\left(c^{2k}(-q)^k+c^kv_k+1\right)I\right)^2.$$

Thus

$$A^{2n}\left(c^kv_k+2\right)^{2n}=\left(A^2+\left(c^{2k}(-q)^k+c^kv_k+1\right)I\right)^{2n}.$$

By Binomial Theorem and (3), we have

$$\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}A^{2i} = A^{2n}\left(c^kv_k+2\right)^{2n}$$
$$\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}\begin{bmatrix} g_{k(2i+1)} \\ g_{2ki}\end{bmatrix} = \left(c^kv_k+2\right)^{2n}\begin{bmatrix} g_{k(2n+1)} \\ g_{2kn}\end{bmatrix},$$

as claimed.

Theorem 5. *For $n>0$, we have*

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{k(2i+1)} = c^{2kn}\left(v_k^2+4q^k\left(-1\right)^{k+1}\right)^n g_{k(2n+1)},$$
$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{2ki} = c^{2kn}\left(v_k^2+4q^k\left(-1\right)^{k+1}\right)^n g_{2kn}.$$

Proof. Writing $4\lambda\left(q^k(-1)^{k+1}c^{2k}-c^kv_k-1\right)$ to each side of (3), we write

$$\lambda c^{2k}\left(v_k^2-4\left(-q\right)^k\right)^2=\left(\lambda+\left(q^k(-1)^{k+1}c^{2k}-c^kv_k-1\right)\right)^2. \tag{10}$$

Similarly, using (10), as the proof of Theorem 1, the proof is completed.

Theorem 6. *For $n > 0$, we have*

$$\begin{aligned}\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^i g_{k(-2i+1)} &= \left(c^kv_k+2\right)^{2n} g_{k(-2n+1)},\\ \sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^i g_{-2ki} &= \left(c^kv_k+2\right)^{2n} g_{-2kn}.\end{aligned}$$

Proof. The characteristic equation for A^{-2} is

$$\left(\lambda\left(c^{2k}(-q)^k+c^kv_k+1\right)+1\right)^2=\lambda\left(c^kv_k+2\right)^2.$$

From the Caley-Hamilton Theorem for A^{-2}, we have

$$\left(A^{-2}\left(c^{2k}(-q)^k+c^kv_k+1\right)+1\right)^2=\left(c^kv_k+2\right)^2 A^{-2}.$$

Thus

$$\left(A^{-2}\left(c^{2k}(-q)^k+c^kv_k+1\right)+1\right)^{2n}=\left(c^kv_k+2\right)^{2n} A^{-2n}.$$

By Binomial Theorem and (4), we have

$$\begin{aligned}\sum_{i=0}^{2n}\binom{2n}{i}\left((-q)^kc^{2k}+c^kv_k+1\right)^i A^{-2i} &= \left(c^kv_k+2\right)^{2n} A^{-2n}\\ \sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^i \begin{bmatrix} g_{k(-2i+1)}\\ g_{-2ki}\end{bmatrix} &= \left(c^kv_k+2\right)^{2n}\begin{bmatrix} g_{k(-2n+1)}\\ g_{-2kn}\end{bmatrix}.\end{aligned}$$

Thus, the proof is completed.

Let C be an arbitrary 2×2 matrix, T and D denote the trace and determinant of C, respectively. For the distinct eigenvalues a and b of matrix C, the following result can be found in [5, 10]:

Lemma 5.

$$z_n := \frac{a^n-b^n}{a-b}=\frac{1}{2^{n-1}}\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}T^{n-2i-1}(T^2-4D)^i,$$

then $C^n = z_nC - z_{n-1}DI_2$, where I_2 is the identity matrix of order 2.

As a consequence of Lemma 5, we obtain that

$$2^{n-1}g_{kn}=u_kc^k\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}\left(c^kv_k+2\right)^{n-2i-1}(c^{2k}v_k^2-4c^{2k}(-q)^k)^i.$$

Let w be a complex number such that $w^2+Tw+D\neq 0, w\neq 0$. For a positive integer n,

$$C^n=\left(\frac{wD}{w^2+Tw+D}\right)^n\sum_{t=0}^{2n}\sum_{i=0}^{t}\binom{n}{i}\binom{n}{t-i}\left(\frac{D}{w^2}\right)^i\left(\frac{w}{D}\right)^t C^t. \tag{11}$$

Therefore we get the following result of equality (11).

Theorem 7. *For $n>0$ and any complex number w different from* 0, $c^k\alpha^k+1$ *and* $c^k\beta^k+1$,

$$\begin{aligned} g_{kn} &= \left(\frac{w\left(c^{2k}(-q)^k+c^kv_k+1\right)}{w^2+(c^kv_k+2)\,w+(c^{2k}(-q)^k+c^kv_k+1)}\right)^n\times \\ &\quad \sum_{t=0}^{2n}\sum_{i=0}^{t}\binom{n}{i}\binom{n}{t-i}\left(q^k(-1)^kc^{2k}+c^kv_k+1\right)^{i-t}w^{t-2i}g_{kt}. \end{aligned}$$

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