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On Minimal γ-open Sets

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Abstract. In this paper, we introduce and discuss minimal γ-open sets in topological spaces. We establish some basic properties of minimal γ-open sets and provide an example to illustrate that minimal γ-open sets are independent of minimal open sets introduced and discussed in [3]. We obtain some properties of pre γ-open sets using properties of minimal γ-open sets. As an application of a theory of minimal γ-open sets, we obtain a sufficient condition for a γ-locally finite space to be a pre γ-T_2 space.

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1. Introduction

Several known characterizations of compact spaces, nearly compact spaces and H-closed spaces are unified by generalizing the notion of compactness with the help of a certain operation γ of a topology τ into the power set P(X) of a space X introduced and discussed by S. Kasahara [2]. By using operation γ, H. Ogata [4], introduced the concept of γ-open sets and investigated the related topological properties of the associated topology τ_γ and τ. He introduced the notions of γ-T_i (i = 0, 1/2, 1, 2) spaces which generalize T_i - spaces (i = 0, 1/2, 1, 2) respectively. Moreover, he investigated general operator approaches of the closed graph mappings.

In 2003, B. Ahmad and S. Hussain [1] continued studying the properties of γ-operations on topological spaces and investigated many interesting results.

In this paper, we introduce and discuss minimal γ-open sets in topological spaces. We establish some basic properties of minimal γ-open sets and provide an example to illustrate that minimal γ-open sets are independent of minimal open sets introduced and investigated in [3]. We obtain some properties of pre γ-open sets using properties of minimal γ-open sets. As an application of a theory of minimal γ-open sets, we obtain a sufficient condition for a γ-locally finite space to be a pre γ-T_2 space.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. Preliminaries

Definition 2.1. *[2] Let (X,τ) be a space. An operation $\gamma : \tau \rightarrow P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ.*

Definition 2.2. *[4] Let $A \subseteq X$. A point $x \in A$ is said to be γ-interior point of A if there exists an open nbd N of x such that $N^{\gamma} \subseteq A$ and we denote the set of all such points by $int_{\gamma}(A)$. Thus*

$$int_{\gamma}(A) = \{x \in A : x \in N \in \tau \text{ and } N^{\gamma} \subseteq A\} \subseteq A.$$

Note that A is γ-open [1] iff $A = int_{\gamma}(A)$. A set A is called γ- closed [1] iff X-A is γ-open.

Definition 2.3. *[4] A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^{\gamma} \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_{\gamma}(A)$. A subset A of X is called γ-closed, if $cl_{\gamma}(A) \subseteq A$. Note that $cl_{\gamma}(A)$ is contained in every γ-closed superset of A.*

Definition 2.4. *[4] An operation γ on τ is said be regular, if for any open nbds U,V of $x \in X$, there exists an open nbd W of x such that $U^{\gamma} \cap V^{\gamma} \supseteq W^{\gamma}$.*

Definition 2.5. *[4] An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^{\gamma} \supseteq B$.*

3. Minimal γ-open Sets

In view of the definition of minimal open sets [3], we define minimal γ-open sets as:

Definition 3.1. *Let X be a space and $A \subseteq X$ a γ-open set. Then A is called a minimal γ-open set if ϕ and A are the only γ-open subsets of A.*

The following Example shows that minimal γ-open sets and minimal open sets are independent of each other.

Example 3.1. *Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by*

$$\gamma(A) = A^{\gamma} = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A. \end{cases}$$

The γ-open sets are ϕ, X, $\{b\}$,$\{a,b\}$, $\{a,c\}$ [4]. Here $\{a\}$ is a minimal open set which is not minimal γ-open. Also $\{a,c\}$ is minimal γ-open set which is not minimal open.

The following is immediate:

Proposition 3.1. *Let X is a space. Then*
(1) Let A be a minimal γ-open set and B a γ-open set. Then $A \cap B = \phi$ or $A \subseteq B$, where γ is regular.
(2) Let B and C be minimal γ-open sets. Then $B \cap C = \phi$ or $B = C$, where γ is regular.

Proposition 3.2. *Let X be a space and A a minimal γ-open set. If $a \in A$, then for any γ-open nbd B of a, $A \subseteq B$, where γ is regular.*

Proof. Suppose on the contrary that B is a γ-open nbd B of $a \in A$ such that $A \nsubseteq B$. Since γ is a regular operation, therefore $A \cap B$ is a γ-open set [4] with $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This is a contradiction to our supposition that A is a minimal γ-open set. Hence the proof.

The following example shows that the condition that γ is regular is necessary for the above Proposition.

Example 3.2. *Let X= $\{a,b,c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by*

$$\gamma(A) = A^{\gamma} = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A\,. \end{cases}$$

Then calculations show that the operation γ is not regular [4]. The γ-open sets are ϕ , X, $\{b\}$, $\{a,b\}$, $\{a,c\}$[4]. Clearly $A = \{a,c\}$ is a minimal γ-open set. Thus for $a \in A$, there does not exist γ-open nbd B of a such that $A \subseteq B$.

The following proposition easily follows from proposition 3.1.

Proposition 3.3. *Let X be a space and A a minimal γ-open set. Then for any $y \in A$, $A = \cap\{B : B$ is γ-open nbd of $y\}$, where γ is regular.*

Similarly we have:

Proposition 3.4. *Let A be a minimal γ-open set in X and $x \in X$ such that $x \notin A$. Then for any γ-open nbd C of x, $C \cap A = \phi$ or $A \subseteq C$.*

Corollary 3.1. *Let A be a minimal γ-open set in X and $x \in X$ such that $x \notin A$. If $A_x = \{B : B$ is a γ-open nbd of $x\}$. Then $A_x \cap A = \phi$ or $A \subseteq A_x$.*

If $\Gamma(X)$ denotes the class of monotone operators, then we have:

Corollary 3.2. *Let X be a space and $\gamma \in \Gamma(X)$. If A is a nonempty minimal γ-open set of X, then for a nonempty subset C of A, $A \subseteq cl_\gamma(C)$, where γ is regular.*

Proof. Let C be any nonempty subset of A. Let $y \in A$ and B be any γ-open nbd B of y. By Proposition 3.3, we have $A \subseteq B$. Also since γ is monotone, $C = A^\gamma \cap C \subseteq B^\gamma \cap C$. Thus we have $B^\gamma \cap C \neq \phi$ and hence $y \in cl_\gamma(C)[4]$. This implies that $A \subseteq cl_\gamma(C)$. This completes the proof.

Proposition 3.5. *Let A be a nonempty γ-open subset of a space X. If $A \subseteq cl_\gamma(C)$, then $cl_\gamma(A) = cl_\gamma(C)$, for any nonempty subset C of A, where γ is open.*

Proof. Since for any nonempty C such that $C \subseteq A$ implies $cl_\gamma(C) \subseteq cl_\gamma(A)$. On the other hand, by supposition we have $A \subseteq cl_\gamma(C)$. Since γ is open, $cl_\gamma(A) \subseteq cl_\gamma(cl_\gamma(C)) = cl_\gamma(C)[4]$ implies $cl_\gamma(A) \subseteq cl_\gamma(C)$. Hence the proof.

The following example shows that the condition that γ is open is necessary for the above Proposition.

Example 3.3. *Let X= $\{a,b,c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by*

$$\gamma(A) = A^{\gamma} = \begin{cases} cl(A), & \text{if } b \in A \\ int(cl(A)), & \text{if } b \notin A. \end{cases}$$

Then the operation γ is not open. The γ-open sets are ϕ, X, $\{b\}$, $\{a,c\}$. Let $A = \{b\}$ and $C = \{a,b\}$, then $cl_{\gamma}(A) = \{b\} \neq X = cl_{\gamma}(C)$.

Proposition 3.6. *Let A be a nonempty γ-open subset of a space X. If $cl_{\gamma}(A) = cl_{\gamma}(C)$, for any nonempty subset C of A, then A is a minimal γ-open set.*

Proof. We suppose on the contrary that A is not a minimal γ-open set. Then there exists a nonempty γ-open set D such that $D \subseteq A$ and hence there exists an element $x \in A$ such that $x \notin D$. Then we have $cl_{\gamma}(\{x\}) \subseteq D^{\gamma}$ implies that $cl_{\gamma}(\{x\}) \neq cl_{\gamma}(A)$. This contradiction proves the proposition.

Combining Propositions 3.4, 3.5 and 3.6, we have:

Theorem 3.1. *Let A be a nonempty γ-open subset of space X and $\gamma \in \Gamma(X)$. Then the following are equivalent:*

(1) A is minimal γ-open set, where γ is regular.

(2) For any nonempty subset C of A, $A \subseteq cl_{\gamma}(A)$, where γ is open.

(3) For any nonempty subset C of A , $cl_{\gamma}(A) = cl_{\gamma}(C)$.

Definition 3.2. *Let X be a space and $A \subseteq X$. Then A is called a pre-γ-open set, if $A \subseteq int_{\gamma}(cl_{\gamma}(A))$. The family of all pre-γ-open sets of X will be denoted by $PO_{\gamma}(X)$.*

In view of the definition of a pre-Hausdorff space [3], we define a γ-T_2 space as:

Definition 3.3. *A space X is called a pre γ-T_2 space, if for any $x, y \in X, x \neq y$, there exist subsets U and V of $PO_{\gamma}(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.*

Proposition 3.7. *Let X be a space and $\gamma \in \Gamma(X)$. If $A \subseteq X$ is a minimal γ-open set, then $\phi \neq C \subseteq A$ is a pre-γ-open set, where γ is regular.*

Proof. Let A be a minimal γ-open set and $\phi \neq C \subseteq A$. By Proposition 3.6, we have $A \subseteq cl_{\gamma}(C)$ implies $int_{\gamma}(A) \subseteq int_{\gamma}(cl_{\gamma}(C))$. Since A is a γ-open set, therefore $C \subseteq A = int_{\gamma}(A) \subseteq int_{\gamma}(cl_{\gamma}(C))$ or $C \subseteq int_{\gamma}(cl_{\gamma}(C))$, that is, C is pre-$\gamma$-open. Hence the proof.

We use Theorem 3.1(3) and prove the following:

Theorem 3.2. *Let B be a nonempty subset of a space X. Let A be a minimal γ-open set in X and $\gamma \in \Gamma(X)$. If there exists a γ-open set C containing B such that $C \subseteq cl_{\gamma}(B \cup A)$, then for any nonempty subset D of A, $B \cup D$ is a pre-γ-open set, where γ is regular and open.*

Proof. Suppose A is a minimal γ-open set in X. Since γ is regular, therefore for any nonempty subset D of A, we have
$cl_{\gamma}(B \cup D) = cl_{\gamma}(B) \cup cl_{\gamma}(D) = cl_{\gamma}(B) \cup cl_{\gamma}(A) = cl_{\gamma}(B \cup A)$.
By supposition, we have $C \subseteq cl_{\gamma}(B \cup A) = cl_{\gamma}(B \cup D)$ implies $int_{\gamma}(C) \subseteq int_{\gamma}(cl_{\gamma}(B \cup D))$, C being γ-open set such that $B \subseteq C$. It follows that
$B \subseteq C = int_{\gamma}(C) \subseteq int_{\gamma}(cl_{\gamma}(B \cup D))$ or $B \subseteq int_{\gamma}(cl_{\gamma}(B \cup D))$ .....(1)
and
$int_{\gamma}(A) = A \subseteq cl_{\gamma}(A) \subseteq cl_{\gamma}(B) \cup cl_{\gamma}(A) = cl_{\gamma}(B \cup A)$ implies $int_{\gamma}(A) \subseteq int_{\gamma}(cl_{\gamma}(B \cup A))$ .....(2)
Since A is a γ-open set, therefore
$D \subseteq A = int_{\gamma}(A) \subseteq int_{\gamma}(cl_{\gamma}(B \cup A)) =\subseteq int_{\gamma}(cl_{\gamma}(B \cup D))$ .....(3)
From (1) and (3),
$B \cup D \subseteq int_{\gamma}(cl_{\gamma}(B \cup D))$ implies $B \cup D$ is a pre-γ-open set. This completes the proof.

Corollary 3.3. *Let X be a space, $\phi \neq B \subseteq X$, A a minimal γ-open set of a space X and $\gamma \in \Gamma(X)$. If there exists a γ-open set C containing B such that $C \subseteq cl_{\gamma}(A)$, then for any nonempty subset D of A, $B \cup D$ is a pre γ-open set, where γ is regular and open.*

Proof. Let A be a minimal γ-open set and $B \subseteq X$. Suppose there exists a γ-open set C containing B such that $C \subseteq cl_\gamma(A)$. Then we have $C \subseteq cl_\gamma(B) \cup cl_\gamma(A) = cl_\gamma(A \cup B)$[4]. By Theorem 3.2, it follows that for any nonempty subset D of A, $B \cup D$ is a pre γ-open set. This completes the proof.

4. Finite γ-open Sets

Proposition 4.1. *Let X be a space and $\phi \neq B$ a finite γ-open set in X. Then there exists at least one (finite) minimal γ-open set A such that $A \subseteq B$.*

Proof. Suppose that B is a finite γ-open set in X. Then we have the following two possibilities:
(1) B is a minimal γ-open set.
(2) B is not a minimal γ-open set.

In case (1), if we choose $B = A$, then the theorem is proved. If the case (2) is true, then there exists a nonempty (finite) γ-open set B_1 which is properly contained in B. If B_1 is minimal γ-open, we take $A = B_1$. If B_1 is not a minimal γ-open set, then there exists a nonempty (finite) γ-open set B_2 such that $B_2 \subset B_1 \subset B$. We continue this process and have a sequence of γ-open sets $... \subset B_m \subset ... \subset B_2 \subset B_1 \subset B$. Since B is a finite, this process will end in a finite number of steps. That is, for some natural number k, we have a minimal γ-open set B_k such that $B_k = A$. This completes the proof.

In view of the Definition of locally finite space [3], we define γ-locally finite space as:

Definition 4.1. *A space X is said to be a γ-locally finite space, if for each $x \in X$ there exists a finite γ-open set A in X such that $x \in A$.*

Example 4.1. *Let X=* $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ *[4]. For* $b \in X$*, define an operation* $\gamma : \tau \to P(X)$ *by*

$$\gamma(A) = A^{\gamma} = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A. \end{cases}$$

Then calculations show that the γ*-open sets are* ϕ *, X,* $\{b\}$*,* $\{a, b\}$*,* $\{a, c\}$*[4]. Clearly X is* γ*-locally finite space.*

Proposition 4.2. *Let* $\phi \neq B$ *be a* γ*-open set in a* γ*-locally finite space X. Then there exists at least one (finite) minimal* γ*-open set A which is contained in B, where* γ *is regular.*

Proof. Let $y \in B$. Since X is a γ-locally finite space, then there exists a finite γ-open set B_y such that $y \in B_y$. Since $B \cap B_y$ is a finite γ-open set [4], therefore by proposition 4.1 there exists a minimal γ-open set A such that $A \subseteq B \cap B_y \subseteq B$. This completes the proof.

Proposition 4.3. *Let X be a* γ*-locally finite space and for any* $\alpha \in I$*,* B_α *a* γ*-open set and* $\phi \neq A$ *a finite* γ*-open set. Then* $A \cap (\bigcap_{\alpha \in I} B_\alpha)$ *is a finite* γ*-open set, where* γ *is regular.*

Proof. Since X is a γ-locally finite space, then there exists an integer k such that $A \cap (\bigcap_{\alpha \in I} B_\alpha) = A \cap (\bigcap_{i=1}^{k} B_i)$. Since γ is regular [4], $A \cap (\bigcap_{\alpha \in I} B_\alpha)$ is a finite γ-open set. This completes the proof.

Using Proposition 4.3, we can prove the following:

Theorem 4.1. *Let X be a space and for any* $\alpha \in I$ *,* B_α *a* γ*-open set and for any* $\beta \in J$ *,* A_β *a nonempty finite* γ*-open set. Then* $(\bigcup_{\beta \in J} A_\beta) \cap (\bigcap_{\alpha \in I} B_\alpha)$ *is a* γ*-open set, where* γ *is regular.*

5. Applications

Let A be a nonempty finite γ-open set. It is clear, by Proposition 3.1 and Proposition 4.3, that if γ is regular, then there exists a natural number m such that $\{A_1,A_2,...,A_m\}$ is the class of all minimal γ-open sets in A satisfying the following two conditions:

(1) For any l,n with $1 \leq l,n \leq m$ and $l \neq n$, $A_l \cap A_n = \phi$.

(2) If C is a minimal γ-open set in A, then there exists l with $1 \leq l \leq m$ such that $C = A_l$.

Theorem 5.1. *Let X be a space and $\phi \neq A$ a finite γ-open set such that A is not a minimal γ-open set. Let $\{A_1,A_2,...,A_m\}$ be a class of all minimal γ-open sets in A and $y \in A-(A_1 \cup A_2 \cup ... \cup A_m)$. If $A_y = \cap\{B$: B is a γ-open nbd of y $\}$. Then there exists a natural number $k \in \{1,2,...,m\}$ such that A_k is contained in A_y, where γ is regular.*

Proof. Suppose on the contrary that for any natural number $k \in \{1,2,...,m\}$, A_k is not contained in A_y. By Corollary 3.1, for any minimal γ-open set A_k in A, $A_k \cap A_y = \phi$. By Proposition 4.3, $\phi \neq A_y$ is a finite γ-open set. Therefore by Proposition 4.1, there exists a minimal γ-open set C such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, then C is a minimal γ-open set in A. By supposition, for any minimal γ-open set A_k, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore for any natural number $k \in \{1,2,...,m\}$, $C \neq A_k$. This is a contradiction to our supposition. Hence the proof.

Proposition 5.1. *Let X be a space and $\phi \neq A$ be a finite γ-open set which is not a minimal γ-open set. Let $\{A_1,A_2,...,A_m\}$ be a class of all minimal γ-open sets in A and $y \in A-(A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1,2,...,m\}$ such that for any γ-open nbd B_y of y, A_k is contained in B_y, where γ is regular.*

Proof. This follows from Theorem 5.1 as $\cap\{B$: B is a γ-open nbd of y $\} \subseteq B_y$. Hence the proof.

Theorem 5.2. *Let X be a space and $\phi \neq A$ be a finite γ-open set which is not a minimal γ-open set. Let $\{A_1, A_2, ..., A_m\}$ be the class of all minimal γ-open sets in A and $y \in A - (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, ..., m\}$ such that $y \in cl_\gamma(A_k)$, where γ is regular.*

Proof. It follows from Proposition 5.1 that there exists a natural number $k \in \{1, 2, ..., m\}$ such that $A_k \subseteq B$ for γ-open nbd B of x. Therefore $\phi \neq A_k \cap A_k \subseteq A_k \cap B \subseteq A_k \cap B^\gamma$ implies $y \in cl_\gamma(A_k)$. This completes the proof.

Proposition 5.2. *Let $\phi \neq A$ be a finite γ-open set in a space X, $\gamma \in \Gamma(X)$ and for each $k \in \{1, 2, ..., m\}$, A_k a minimal γ-open set in A. If the class $\{A_1, A_2, ..., A_m\}$ contains all minimal γ-open sets in A, then for any $\phi \neq B_k \subseteq A_k$, $A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, where γ is regular and open.*

Proof. Let $\phi \neq A$ be a finite γ-open set. We consider the following two cases:

Case 1. If A is a minimal γ-open set, then this follows directly from Proposition 3.6.

Case 2. If A is not a minimal γ-open set. $y \in A - (A_1 \cup A_2 \cup ... \cup A_m)$. Then by Theorem 5.2, it follows that $y \in cl_\gamma(A_1) \cup cl_\gamma(A_2) \cup ... \cup cl_\gamma(A_m)$. Therefore by Proposition 3.6, we have

$$A \subseteq cl_\gamma(A_1) \cup cl_\gamma(A_2) \cup ... \cup cl_\gamma(A_m) = cl_\gamma(B_1) \cup cl_\gamma(B_2) \cup ... \cup cl_\gamma(B_m) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m).$$

This completes the proof.

Proposition 5.3. *Let X be a space and $\phi \neq A$ a finite γ-open set and A_k a minimal γ-open set in A, for each $k \in \{1, 2, ..., m\}$. If for any $\phi \neq B_k \subseteq A_k$, $A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$ then $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, where γ is open.*

Proof. For any $\phi \neq B_k \subseteq A_k$, $k \in \{1, 2, ..., m\}$, we have $cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m) \subseteq cl_\gamma(A)$. Also, we have

$$cl_\gamma(A) \subseteq cl_\gamma(cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m) .$$

This implies that for any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$. Hence the proof.

Proposition 5.4. *Let X be a space and $\phi \neq A$ be a finite γ-open set and for each $k \in \{1,2,...,m\}$, A_k a minimal γ-open set in A. If for any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, then the class $\{A_1, A_2, ..., A_m\}$ contains all minimal γ-open sets in A.*

Proof. Suppose on the contrary that C is a minimal γ-open set in A and for $k \in \{1,2,...,m\}$, $C \neq A_i$. Therefore, for each $k \in \{1,2,...,m\}$, $C \cap cl_\gamma(A_k) = \phi$. This implies that any element of C is not contained in $cl_\gamma(A_1 \cup A_2 \cup ... \cup A_m)$. This is a contradiction to the fact that $C \subseteq A \subseteq cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$. This completes the proof.

Combining Propositions 5.2, 5.3 and 5.4, we have the following theorem:

Theorem 5.3. *Let X be a space and $\phi \neq A$ be a finite γ-open set and for each $k \in \{1,2,...,m\}$, A_k a minimal γ-open set in A. Then the following three conditions are equivalent:*

(1) The class $\{A_1, A_2, ..., A_m\}$ contains all minimal γ-open sets in A.

(2) For any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$.

(3) For any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, where γ is regular and open.

Remark 5.1. *Suppose that $\phi \neq A$ is a finite γ-open set and $\{A_1, A_2, ..., A_m\}$ is a class of all minimal γ-open sets in A such that for each $k \in \{1,2,...,m\}$, $y_k \in A_k$. Then by Theorem 5.3, it is clear that $\{y_1, y_2, ..., y_m\}$ is a pre-γ-open set.*

Theorem 5.4. *Let X be a space. Suppose that $\phi \neq A$ is a finite γ-open set and $\{A_1, A_2, ..., A_m\}$ is a class of all minimal γ-open sets in A. If for any $B \subseteq A - \{A_1, A_2, ..., A_m\}$and $\phi \neq B_k \subseteq A_k$, for each $k \in \{1,2,...,m\}$, then $B \cup B_1 \cup B_2 \cup ... \cup B_m$ is a pre-γ-open set, where γ is regular and open.*

Proof. Suppose that $\phi \neq A$ is a finite γ-open set and $\{A_1, A_2, ..., A_m\}$ is a class of all minimal γ-open sets in A. Then by Proposition 5.2
$A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m) \subseteq cl_\gamma(B \cup B_1 \cup B_2 \cup ... \cup B_m)$.

Also, A is γ-open implies

$B \cup B_1 \cup B_2 \cup ... \cup B_m \subseteq A = int_\gamma(A) \subseteq int_\gamma(cl_\gamma(B \cup B_1 \cup B_2 \cup ... \cup B_m))$.

This follows that $B \cup B_1 \cup B_2 \cup ... \cup B_m$ is a pre-γ-open set. This completes the proof.

Theorem 5.5. *Let X be a γ-locally finite space and $\gamma \in \Gamma(X)$. If a minimal γ-open set $A \subseteq X$ has more than one element, then X is a pre γ-T_2 space, where γ is regular and open.*

Proof. Let $a, b \in X$ such that $a \neq b$. Since X is γ-locally finite, therefore there exist finite γ-open sets V and W containing a and b respectively. Proposition 4.1 implies that there exist a class $\{V_1, V_2, ..., V_m\}$ of all minimal γ-open sets in V and a class $\{W_1, W_2, ..., W_l\}$ of all minimal γ-open sets in W. We consider three possibilities:

1. Suppose there exist $k \in \{1, 2, ..., m\}$ and $i \in \{1, 2, ..., l\}$ such that $a \in V_k$ and $b \in W_i$. Then Proposition 3.7 implies that $\{a\}$ and $\{b\}$ are pre-γ-open sets such that $a \in \{a\}$, $b \in \{b\}$ and $\{a\} \cap \{b\} = \phi$.

2. Suppose there exist $k \in \{1, 2, ..., m\}$ and $i \in \{1, 2, ..., l\}$ such that $a \in V_k$ and $b \notin W_i$. Then by supposition, proposition 3.7 and Theorem 5.4, we can find for each i, $b_i \in W_i$ such that $\{a\}$ and $\{b, b_1, b_2, ..., b_l\}$ are pre-γ-open sets and $\{a\} \cap \{b, b_1, b_2, ..., b_l\} = \phi$.

3. Suppose that there exist $k \in \{1, 2, ..., m\}$ and $i \in \{1, 2, ..., l\}$ such that $a \notin V_k$ and $b \notin W_i$. Then by supposition and Theorem 5.4, for each k and i, we can find elements $a_k \in V_k$ and $b_i \in W_i$ such that $\{a, a_1, a_2, ..., a_m\}$ and $\{b, b_1, b_2, ..., b_l\}$ are pre-γ-open sets and $\{a, a_1, a_2, ..., a_m\} \cap \{b, b_1, b_2, ..., b_l\} = \phi$. Hence X is a pre γ-T_2 space. This completes the proof.

References

[1] B. Ahmad and S. Hussain: *Properties of γ-Operations on topological spaces, Aligarh Bull. Math.* 22(1) (2003), 45-51.

[2] S. Kasahara: *Operation-compact spaces, Math. Japon.,* 24(1979), 97-105.

[3] F. Nakaoka and N. Oda: *Some applications of minimal open sets, Internat. Jr. Math. Math. Sci.,* 27(2001), no. 8 , 471-476.

[4] H. Ogata: *Operations on topological spaces and associated topogy, Math. Japon.,* 36(1)(1991), 175-184.