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Global estimation of the Cauchy problem solution' and blow up the Navier-Stokes equation

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Abstract. The paper presents results of the research of gradient catastrophe development during phase change. It shows that classical methods of the function estimation theory do not fit well to study gradient catastrophe problem. The paper presents results, indicating that embedding theorems do not allow to study a process of a catastrophe formation. In fact, the paper justifies Terence Tao's pessimism about a failure of modern mathematics to solve the Navier-Stokes problem. An alternative method is proposed for dealing with the gradient catastrophe by studying Fourier transformation for a function and selecting a function singularity through phase singularities of Fourier transformation for a given function. The analytic properties of the scattering amplitude are discussed in R^3, and a representation of the potential is obtained using the scattering amplitude. A uniform time estimation of the Cauchy problem solution for the Navier-Stokes equations is provided.Describes the loss of smoothness of classical solutions for the Navier-Stokes equations -Millennium Prize Problems.

Key Words and Phrases: Schrödinger's equation; potential, scattering amplitude, Cauchy problem, Navier–Stokes equations, Fourier transform, the global solvability and uniqueness of the Cauchy problem,the loss of smoothness,The Millennium Prize Problems

1. Introduction

The research presents a process of gradient catastrophe formation under conditions of phase change. The paper shows that classical methods of the function estimation theory in context of Sobolev- Schwartz Space Theory are not suitable for studying gradient catastrophe problem. Results which are presented here show that the embedding theorems do not allow to study a process of a catastrophe formation. Actually, the paper justifies Terence Taos pessimism about a failure of using present mathematical methods for solving the Navier-Stokes problem. An alternative method is proposed for studying gradient catastrophe by applying Fourier transformation to a function and selecting function singularity through phase singularities of Fourier transformation for a given function. We know a general definition of a gradient catastrophe - an unbounded increase of a function derivative upon conditions of boundedness of the function itself. This phenomenon occurs

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in various problems of hydrodynamics, such as a formation of shock waves, weather fronts, hydraulic and seismic fracturing, and others. In modern physics and mathematics, as well as in many other areas of science and technology, this phenomenon is considered as a very difficult problem, both from a theoretical and applied perspective. From a theoretical point of view this is important as we have to know how to describe qualitative changes in processes, which are manifested in appearance of new quality objects during a process of description model evolution, and in the context of applied research, the problem is facing numerical instability in the event of a gradient catastrophe formation. Thus, we approach an important obstacle while using modeling - a barrier created by the gradient catastrophe. Since, on the one hand, the gradient catastrophe is still unknown phenomenon, it is very important from a practical point of view, because the phenomenon is connected with the most interesting and important aspects of reality. Terence Tao formulated and illustrated this in [1] based on the Millennium problem stated by Clay Institute for the Navier-Stokes equations. Our point of view on these issues agrees with one, stated in article [1],[5],[6] but in our research we propose a way for solving these problems. Our point of view is that the modern mathematical methods of the theory of functions dedicated to the function estimation have ignored such an important component of the Fourier transformation as its phase. Our research is outlined as follows: first, we give examples of the gradient catastrophe caused by the phase change, and then proceed to an expansion of classes of functions subjected to the gradient catastrophe. Our final results lie in the nonlinear representation of functions showing some new classification of functions through a phase classification. In addition, the notions of discreteness and continuity of functions are naturally merged. And, in our opinon, this leads to understanding of how discrete objects are born under a continuous change of the world. Discrete objects are associated with discrete spectrum of the Liouville- Schrdinger equations. And they, as it is known, reflect the wave nature of things. But here, we abstract away from the quantum formalism, because our goal lies in a purely mathematical approach to the analysis of the arbitrary functions. For the analysis of which, we formally consider a function as a potential of the Schrdinger equation. At the same time we come across the concepts that generated by the Liouville- Schrdinger equations. These concepts allow to classify and estimate functions by a phase generated by discrete spectrum of the Liouville equation.

2. Results for the one-dimensional case

Let us consider one-dimensional function f and its Fourier transformation $\widetilde{f}$. Using notions of module and phase, we write Fourier transformation in the following form $\widetilde{f} = |\widetilde{f}|\exp(i\phi)$, where ϕ is phase. To cite Plancherel equality: $||f||_{L_2} = Const||\widetilde{f}||_{L_2}$. Here we can see that a phase is not contributed to determination of X norm. To estimate a maximum we have a simple estimate as $max|f|^2 \leq 2||f||_{L_2}||\nabla f||_{L_2}$.Now we have an estimate of the function maximum in which a phase is not involved. Let us consider a behavior of a progressing wave running with a constant velocity of $v = a$ described by function $F(x,t) = f(x + at)$. For its Fourier transformation along x variable we have $\widetilde{F} = \widetilde{f}exp(iatk)$. Again in this case we can see that when we will be studying a module of the Fourier transformation, we will not obtain major physical information about the wave, such as its velocity and location of the wave crest because of $|\widetilde{F}| = |\widetilde{f}|$. These two examples show w eaknesses of studying Fourier transformation. On the other hand, many researchers focus on the study of functions using embedding theorem, but in the embedding theorems main object of the study is module of function. But as we have seen in given examples, a phase is a main physical characteristic of a process, and as we can see in the mathematical studies, which use embedding theorems with energy estimates, the phase disappears. Along with phase, all reasonable information about physical process disappears, as demonstrated by Terence Tao [1] and other research considerations. In fact, he built progressing waves that are not followed energy estimates. Let us proceed with more essential analysis of influence of the phase on behavior of functions.

Theorem 1. *There are functions of $W_2^1(R)$ with a constant rate of the norm for a gradient catastrophe of which a phase change of its Fourier transformation is sufficient.*

Proof. To prove this, we consider a sequence of testing functions $\widetilde{f_n} = \Delta/(1+k^2), \Delta = (i-k)^n/(i+k)^n$. it is obvious that $|\widetilde{f_n}| = 1/(1+k^2)$. $max|f_n|^2 \leq 2||f_n||_{L_2}||\nabla f_n||_{L_2} \leq Const.$. Calculating the Fourier transformation of these testing functions, we obtain: $f_n = x(-1)^{(n-1)} 2\pi \exp(-x) L^1_{(n-1)}(2x)$ where $L^1_{(n-1)}(2x)$ is a Laguerre polynomial. Now we see that the functions are equibounded and derivatives of these functions will grow with the growth of n. Thus, we have built an example of a sequence of the bounded functions of $W_2^1(R)$ which have a constant norm $W_2^1(R)$ and this sequence converges to a discontinuous function.

Thus, we have demonstrated an importance of the phase and that the phase is not involved into energy norms that are inherent to the mathematical arguments used in physical processes analysis. Our next goal is to maximally expand this class of functions in which a phase is important. Our goal is also to use a phase, which appears in the inverse scattering problem; moreover we will be interested mainly in a phase generated by a discrete spectrum of the Liouville equation. Thereby, we come now to an important subject of our research, such as an occurrence of discontinuities, fronts and other instable states in numerical modeling and which are at the same very stable physical objects.

Theorem 2. *There are functions of $W_2^1(R)$ with a constant rate of the norm for a gradient catastrophe of which a phase change of its Fourier transformation is sufficient.*

Proof. To prove this, we consider a sequence of testing functions $\widetilde{f_n} = \Delta/(1+k^2), \Delta = (i-k)^n/(i+k)^n$. it is obvious that $|\widetilde{f_n}| = 1/(1+k^2)$. $max|f_n|^2 \leq 2||f_n||_{L_2}||\nabla f_n||_{L_2} \leq Const.$. Calculating the Fourier transformation of these testing functions, we obtain: $f_n = x(-1)^{(n-1)} 2\pi \exp(-x) L^1_{(n-1)}(2x)$ where $L^1_{(n-1)}(2x)$ is a Laguerre polynomial. Now we see that the functions are equibounded and derivatives of these functions will grow with the growth of n. Thus, we have built an example of a sequence of the bounded functions of $W_2^1(R)$ which have a constant norm $W_2^1(R)$ and this sequence converges to a discontinuous function.

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$$-\Psi^{"} + q\Psi = |k|^2\Psi,\ k \in C \tag{1}$$

with the following asymptotics:

$$\lim_{x\to-\infty} \Psi_1(k,x) = e^{ikx} + s_{12}(k)e^{-ikx},\ \lim_{x\to+\infty} \Psi_1(k,x) = s_{11}(k)e^{-ikx} \tag{2}$$

$$\lim_{x\to-\infty} \Psi_2(k,x) = s_{22}(k)e^{-ikx},\ \lim_{x\to+\infty} \Psi_2(k,x) = e^{-ikx} + s_{11}(k)e^{ikx} \tag{3}$$

It is also known from the theory of equations [2], that any solution is a combination of some fundamental solutions satisfying certain boundary conditions.

$$\lim_{x\to\infty} f_1(k,x)e^{-ikx} = 1,\ \lim_{x\to-\infty} f_2(k,x)e^{ikx} = 1. \tag{4}$$

It is known [2], that they satisfy the following equations:

$$f_1(k,x) = e^{ikx} - \int\limits_{-\infty}^{+\infty} G_1(k,x,t)q(t)f_1(k,t)dt, \tag{5}$$

$$f_2(k,x) = e^{-ikx} + \int\limits_{-\infty}^{+\infty} G_2(k,x,t)q(t)f_1(k,t)dt, \tag{6}$$

$$\tag{7}$$

$$E_+(k,x) = e^{ikx}\quad E_-(k,x) = e^{-ikx}, \tag{8}$$

$$G_1(k,x,t) = -\frac{\theta(x-t)sin(k(x-t))}{k},\ G_2(k,x,t) = \frac{\theta(x-t)sin(k(x-t))}{k}, \tag{9}$$

$$f_1 = E_+ - \sum_{j=1}^{\infty} G_1^j E_+,\ f_2 = E_- + \sum_{j=1}^{\infty} G_2^j E_-, \tag{10}$$

Let us also provide known results for the scattering coefficients and fundamental solutions outlined in [2].

$$u_1^+(k,x) = s_{12} f_2(k,x), u_2^+(k,x) = s_{11} f_1(k,x), u_1^-(k,x) = \overline{u_1^+(k,x)},\ u_2^+(k,x) = \overline{u_2^+(k,x)}, \tag{11}$$

$$s_{11}s_{12}^* + s_{12},\ s_{22}^* = 0,\ s_{11}^2 + s_{12}^2 = s_{22}^2 + s_{21}^2 = 1, s_{i,j}(-k) = s_{i,j}^*(k), \tag{12}$$

$$\lim_{|k|\to\infty} s_{12} = s_{21} = 1 + O(1/|k|),\ ,\ \lim_{|k|\to\infty} s_{11} = s_{22} = O(1/|k|), \tag{13}$$

$$s_{11} = \exp(\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{ln(1-|s_{12}|)}{k'-k} dk' \prod_{j=1}^{n} \Big(\frac{iE_j + k}{k - iE_j}\Big) dk', \tag{14}$$

$$s_{11}(k) = \lim_{\epsilon\to 0} = s_{11}(k+i\epsilon),\ \ s_{21}(k) = -\frac{s_{12}(-k)s_{11}(k)}{s_{11}(-k)} \tag{15}$$

$$s_{21}(k) = \frac{\frac{1}{2ki}\int_{-\infty}^{+\infty} \exp(ikt)q(t)f_2(k,t)dt}{1 - \frac{1}{2ki}\int_{-\infty}^{+\infty} \exp(ikt)q(t)f_2(k,t)dt},\ s_{12}(k) = \frac{\frac{1}{2ki}\int_{-\infty}^{+\infty} \exp(-ikt)q(t)f_1(k,t)dt}{1 - \frac{1}{2ki}\int_{-\infty}^{+\infty} \exp(ikt)q(t)f_1(k,t)dt}, \tag{16}$$

$$b(k) = \frac{1}{2ki}\int_{-\infty}^{+\infty} \exp(-ikt)q(t)f_1(k,t)dt, a(k) = 1 - \frac{1}{2ki}\int_{-\infty}^{+\infty} \exp(ikt)q(t)f_1(k,t)dt. \tag{17}$$

Now we are able to return to our question of the gradient catastrophe for more general class of functions. For this we consider Liouville equation and a sequence of inverse scattering problems with constant in module scattering coefficients $s_{i,j}$, where discrete eigenvalues $E_i, 0 < i < n+1$ such that $\lim_{n\to\infty} = E_\infty$

Theorem 3. *There are potentials from $W_2^1(R)M$ with the constant norm of $W_2^1(R)M$ for the gradient catastrophe for which existence of limit point for the discrete spectrum with given potential in the Liouville equation is sufficient.*

Proof. Following notations [2], we introduce functions A_+, B_+, Ω_+ according to the formulas:

$$s_{21}(k) = \int_{-\infty}^{+\infty} A_+(t)\exp(2ikt)dt,\ \ \Omega_+(t) = \sum_{i=1}^{n} M_j^1 exp(-E_j t) + A_+(t) \tag{18}$$

$$B_+(x,y) + \int_0^{+\infty} B_+(x+y+t)\Omega_+(x+y+t)dt + \Omega_+(x+y) = 0 \tag{19}$$

where M_j^1 are normalized numbers. In other words, we will consider inverse problems of the potential recovery, and for the n-th potential we will consider a case with an accuracy up to n discrete eigenvalues. It is sufficient to consider a first approximation of these equations. In a first approximation, the n-th potential is recovered by the equation for $B_+(x,y)$ and also in a first approximation. We have the following arguments for the first approximation

$$\frac{d}{dx}B_+(x,x) = -\frac{d}{dx}\Omega_+(2x). \tag{20}$$

$$\frac{d}{dx}\Omega_+(2x) = \sum_{i=1}^{n} -E_j M_j^1 exp(-E_j t) + \frac{d}{dx}A_+(x) \tag{21}$$

For the last term, we also consider a first approximation

$$\frac{d}{dx}A_+(x) = \int_{-\infty}^{+\infty} \widetilde{q}_+(2t)\exp(2ixt)\delta(k)^2 dt, \tag{22}$$

$$\delta(k) = \prod_{j=1}^{n}\Big(\frac{iE_j + k}{k - iE_j}\Big) * \exp(\frac{1}{2\pi i}Vp\int_{-\infty}^{+\infty}\frac{ln(1-|s_{12}|))}{k'-k}dk') \tag{23}$$

To prove this, let us consider a sequence of $\frac{d}{dx}A_+(x)$with n going to infinity and under a proper selection of scattering coefficients, we fall into conditions of the Theorem 1.

Let us come down from specific obvious examples to more systematic analysis of the gradient catastrophe. In given below all our arguments will be based on well-known equation:

$$s_{21}(k) = -s_{12}(-k)s_{11}(k)/s_{11}(-k)$$

Let us consider s_{21}, s_{12} in the following form:

$$2iks_{12}(k) = \widetilde{q(2k)} + I_1(k),\ 2iks_{21}(k) = \widetilde{q(-2k)} + I_2(k). \tag{24}$$

Let us conceive $\widetilde{q(2k)} = u(k) + iv(k)$. Then we will have the following equation for u, v

$$u(k) + iv(k) = 2iks_{12}(k) - I_1(k),\ u(k) - iv(k) = 2iks_{21}(k) - I_2(k), \tag{25}$$

$$\frac{s_{11}(k)}{s_{11}(-k)} = exp(2i\delta(k)),\ \delta(k) = arg(s_{11}(k)). \tag{26}$$

Now we can formulate the following theorem.

Theorem 4. *The following equations are true*

$$u(k) = \frac{(1+\cos(2\delta(k)))R_1 + \sin(2\delta(k))R_2}{\sin(2\delta(k))},\ v = \frac{(-1+sin(2\delta(k)))R_1 + (1-\cos(2\delta(k)))R_2}{\sin(2\delta(k))} \tag{27}$$

Proof. Using equation (26) and representation for Fourier transformation we obtain Whence, solving the equation for u and v, we obtain

$$u(k) + iv(k) + I_1(k) = (u(k) - iv(k) + I_2(k))(\cos(2\delta(k)) + i\sin(2\delta(k))), \tag{28}$$

from last equation we have

$$u(k)(1 - \cos(2\delta(k)))) + v(k)(1 - \sin(2\delta(k))) = R_1, \tag{29}$$
$$-u(k)\sin(2\delta(k)) + v(k)(1 + \cos(2\delta(k))) = R_2 \tag{30}$$

where

$$R_1 = Real(-I_1 + I_2\cos(2\delta(k)) + i(I_1\sin(2\delta(k)))), \tag{31}$$
$$R_2 = Im(-I_1 + I_2\cos(2\delta(k)) + i(I_1\sin(2\delta(k)). \tag{32}$$

Theorem 5. *If* $\delta(k)(k) = 0$, $|\widetilde{q(k)}| < C < \infty$ *then* $R_1(k) = 0$.

Proof. using equation (30-31) and conditional theorem we obtain proof.

Theorem 6. *The following estimates are true for Fourier transformation*

$$|u| < C(|R_1| + |R_2| + |\nabla R_1|), \tag{33}$$
$$|v| < C(|R_1| + |R_2| + |\nabla R_1|), \tag{34}$$
$$\widetilde{|q|} < C(|R_1| + |R_2| + |\nabla R_1|). \tag{35}$$

Proof. follows from the representation of u, v. Here, we just point out this as a separate theorem in order to emphasize the significance of this result. We note separately the terms with a derivative ∇R_1. Obviously, these terms are appeared due to points of the phase nulling.

Theorem 7. *For estimation of a maximum of the potential the following estimates are true*

$$|q| < C\int_{-\infty}^{+\infty}(C(|I_1| + |I_2| + |\nabla I_1| + |\nabla I_2|))dk \tag{36}$$

Proof. follows from the estimation of u, v and use of R_1, R_2 which are simple arguments.

Here we outline the theorem in order to emphasize importance of this result for 3-dimentional case. Analyzing the last formula, we see again an effect of the phase on the function behavior. In addition, a finiteness of the discrete spectrum is the main requirement of the gradient catastrophe nonoccurrence. And from other hand, in case of unconstrained growth of points in discrete spectrum, we fall into the terms of theorems 1 and 2. The last theorem expands a class of functions described in in Theorem 1, as we planned at the beginning. Now, studying the behavior of a gradient depending on q we come to the conclusion that its unconstrained growth will be dictated by the phase cluster point, which, in its turn, is due to discrete spectrum acquisition. Hence we get the most important conclusion - we get information about the catastrophe with discrete jumps!

Theorem 8. *For a potential the following representation is true*

$$q = Q(q, E_1, ...E_n); \tag{37}$$

Proof. just consists in calculating $I_1(k), I_2(k)$ in a form of series of q and substitution of a result of the calculation into the formula for u, v moreover a right side of the obtained formula contains second-order terms only.

This representation, in contrast to the classical inverse problems, allows using arbitrary information on the potential for closure of these equations, because a skeleton of this integral equation is represented by sets of constants in the form of eigenvalues. One of the surprising properties of this representation and all this research is discreteness in continuity. Since a value of the phase, as we can see, changes discontinuously, while a potential-function itself may vary continuously. This implies an important conclusion about the instability in numerical methods, i.e. it is necessary to control phase jumps in numerical modeling to avoid falling into a state of instability. A conclusion of non-scalability of such models is critically important since eigenvalues may appear or may disappear under changes in the potential scale, whereupon a model will be changed significantly. This theorem shows that we have obtained fundamentally new nonlinear integral relations that allow taking a fundamentally fresh look at the problem of estimating functions. Now, instead of integral representations, that generate embedding theorem in the Sobolev spaces and by which numerous outstanding achievements in modern mathematics have been gained, we turn to the newest non-linear integral relations and hope thereby opening up new pages of mathematics that will take us further into the wonderful world of mathematics.

3. Introduction for the three-dimensional case

In this work we present final solving Millennium Prize Problems formulated Clay Math. Inst., Cambridge in [3] Before this work we already had first results in [4]-[6]. The Navier-Stokes existence and smoothness problem concerns the mathematical properties of solutions to the NavierStokes equations. These equations describe the motion of a fluid in space. Solutions to the NavierStokes equations are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. In particular, solutions of the NavierStokes equations often include turbulence, which remains one of the greatest unsolved problems in physics. Even much more basic properties of the solutions to NavierStokes have never been proven. For the three-dimensional system of equations, and given some initial conditions, mathematicians have not yet proved that smooth solutions always exist, or that if they do exist, they have bounded energy per unit mass. This is called the NavierStokes existence and smoothness problem. Since understanding the NavierStokes equations is considered to be the first step to understanding the elusive phenomenon of turbulence, the Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics. In this paper, we introduce important explanations results presented in the previous studies in [4]-[6] . We therefore reiterate the basic provisions of the preceding articles to clarify understanding them. First, we consider some ideas for the potential in the inverse scattering problem,and this is then used to estimate of solutions of the Cauchy problem for the Navier-Stokes equations. A similar approach has been developed for one-dimensional nonlinear equations [7,8,9,10], but to date, there have been no results for the inverse scattering problem for three-dimensional nonlinear equations. This is primarily due to difficulties in solving the three-dimensional inverse scattering problem. This paper is organized as follows: first, we study the inverse scattering problem , resulting in a formula for the scattering potential . Furthermore, with the use of this potential, we obtain uniform time estimates in time of solutions of the Navier–Stokes equations , which suggest the global solvability of the Cauchy problem for the Navier–Stokes equations. Essentially, the present study expands the results for one-dimensional nonlinear equations with inverse scattering methods to multi-dimensional cases. In our opinion, the main achievement is a relatively unchanged projection onto the space of the continuous spectrum for the solution of nonlinear equations, that allows to focus only on the behavior associated with the decomposition of the solutions to the discrete spectrum. In the absence of a discrete spectrum, we obtain estimations for the maximum potential in the weaker norms, compared with the norms for Sobolev' spaces. Consider the operators $H = -\Delta_x + q(x)$, $H_0 = -\Delta_x$ defined in the dense set $W_2^2(R^3)$ in the space $L_2(R^3)$, and let q be a bounded fast-decreasing function. The operator H is called Schrödinger's operator. We consider the three-dimensional inverse scattering problem for Schrödinger's operator: the scattering potential must be reconstructed from the scattering amplitude. This problem has been studied by a number of researchers [9,11,12] and references therein]

4. Results for the three-dimensional case

Consider Schrödinger's equation:

$$-\Delta_x \Psi + q\Psi = |k|^2\Psi, \ k \in C \tag{38}$$

Let $\Psi_+(k, \theta, x)$ be a solution of (38) with the following asympotic behavior:

$$\Psi_+(k, \theta, x) = \phi_0(\theta, x) + \frac{e^{i|k||x|}}{|x|} A(k, \theta', \theta) + 0\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{39}$$

where $A(k,\theta^{'},\theta)$ is the scattering amplitude and $\theta^{'}=\frac{x}{|x|},\theta\in S^2$ for $k\in\bar{C}^+=\{Imk\geq 0\}$ $\phi_0(\theta,x)=e^{ik\theta x}$

$$A(k,\theta^{'},\theta)=-\frac{1}{4\pi}\int_{R^3}q(x)\Psi_+(k,\theta,x)e^{-ik\theta^{'}x}dx. \tag{40}$$

Let us also define the solution $\Psi_-(k,\theta,x)$ for $k\in\bar{C}^-=\{Imk\leq 0\}$ as

$$\Psi_-(k,\theta,x)=\Psi_+(-k,-\theta,x)$$

. As is well known[9] :

$$\Psi_+(k,\theta,x)\ -\ \Psi_-(k,\theta,x)\ =\ -\frac{k}{4\pi}\int_{S^2}A(k,\theta^{'},\theta)\Psi_-(k,\theta^{'},x)d\theta^{'},\quad k\ \in\ R. \tag{41}$$

This equation is the key to solving the inverse scattering problem, and was first used by Newton [11,12] and Somersalo et al. [13]. Equation (41) is equivalent to the following:

$$\Psi_+=S\Psi_-, \tag{42}$$

where S is a scattering operator with the kernel $S(k,l)$, $S(k,l)=\int_{R^3}\Psi_+(k,x)\Psi_-^*(l,x)dx$. The following theorem was stated in [2]:

Theorem 9. ***(The energy and momentum conservation laws)*** *Let $q\in\mathbf{R}$. Then, $SS^*=I$, $S^*S=I$, where I is a unitary operator.*

Definition 1. *The set of measurable functions $\mathbf{R}$ with the norm, defined by $||q||_{\mathbf{R}}=\int_{R^6}\frac{q(x)q(y)}{|x-y|^2}dxdy<\infty$ is recognized as being of Rollnik class.*

Let us take into consideration a series for A :

$$A(k,k')=\sum_{n=0}^{\infty}A_n(k,k'),\ A_0(k,k')=\frac{1}{(2\pi)^3}\int\limits_{R^3}e^{i(k-k',x)}q(x)dx, \tag{43}$$

$$A_n(k,k')=\frac{1}{(2\pi)^3}\frac{(-1)^n}{(4\pi)^n}\int\limits_{R^{3(n+1)}}e^{i(k,x_0)}q(x_0)\frac{e^{i|k||x_0-x_1|}}{|x_0-x_1|}q(x_1)\times...\times$$

$$\times...\times q(x_{n-1})\frac{e^{i|k||x_{n-1}-x_n|}}{|x_{n-1}-x_n|}q(x_n)e^{-i(k',x_n)}dx_0...dx_n.$$

As well as in [8], p.120 we formulate.

Definition 2. *Series* (43) *is called Born's series.*

Theorem 10. *Let $q\in L_1(R^3)\cap\mathbf{R}$. If $\|q\|^2_{\mathbf{R}}\leq 4\pi$, then Born's series for $A(k,k')$ converges as $k,k'\in R^3$.*

Proof. is in [8], 121.

Let us introduce the following notation:

$$Qf = \int_{S^2} Q(k,\theta',\theta)f(k,\theta')d\theta', T_Q^+ f = \int_0^\infty \frac{\int_{S^2} Q(k,\theta',\theta)f(k,\theta')d\theta'}{|k|^2 - s^2 - i0} s^2 ds,$$

$$f = f(k,\theta'),\ Q(k,\theta',\theta) = q(\widetilde{k-k'}),\ \theta' = \frac{k'}{|k'|}, \theta = \frac{k}{|k|}, \theta, \theta' \in S^2,$$

$$S_{\phi_{x_0}} f = \int_{S^2} f(k,\theta)e^{i(x_0,k)}d\theta,\ for\ f = f(k,\theta',x),\ Df = k\int_{S^2} A(k,\theta',\theta)f(k,\theta',x)d\theta', \tag{44}$$

Lemma 11. *Suppose that* $q \in \mathbf{R}$, $max_{k,k'}|T_Q^+(k,k'| < 1/c_0,\ Csup_{e_k,e_{k'},k}|Q(k,k')| < 1$, *then*

$$A(k,k') = c_0\tilde{q}(k-k') + c_0^2 \int\limits_{R^3}\int\limits_{R^3} \frac{\tilde{q}(k+p)\tilde{q}(p-k')}{(|p|^2 - |k|^2 - i0)} dk + ...$$

$$A(k,k') = c_0 Q(k,k') + c_0^2 T_Q^+ Q + c_0^2 T_Q^+ T_Q^+ Q...$$

$$sup_{e_k,e_{k'},k}|A(k,k')| < Csup_{e_k,e_{k'},k}|Q(k,k')| + Csup_{e_k,e_{k'},k}|T_Q(k,k')|,$$

$$sup_{e_k,e_{k'},k}|TA(k,k')| < Csup_{e_k,e_{k'},k}|T_Q(k,k')| + Csup_{e_k,e_{k'},k}|Q(k,k')|$$

Proof. folows from the definition $A(k,k')$, $T_Q^+ f$ and the formula for a geometric progression

As shown in [14], $\Psi_\pm(k,x)$ is an orthonormal system of H eigenfunctions for the continuous spectrum. In addition to the continuous spectrum there are a finite number N of H negative eigenvalues, designated as $-E_j^2$ with corresponding normalized eigenfunctions $\psi_j(x,-E_j^2)(j = \overline{1,N})$, where $\psi_j(x,-E_j^2) \in L_2(R^3)$. We present Povzner's results [14] below:

Theorem 12. ***(Completeness)*** *For both an arbitrary* $f \in L_2(R^3)$ *and for* H *eigenfunctions, Parseval's identity is valid.*

$$|f|_{L_2}^2 = (P_D f, P_D f) + (P_{Ac} f, P_{Ac} f).$$

$$P_D f = \sum_{j=1}^N f_j \psi_j(x,-E_j).\ P_{Ac} f = \int_0^\infty \int_{S^2} s^2 \bar{f}(s)\Psi_+(s,\theta,x)d\theta ds, \tag{45}$$

where $\bar{f}$ *and* f_j *are Fourier coefficients for the continuous and discrete cases.*

Theorem 13. ***(Birmann–Schwinger estimation).*** *Let* $q \in \mathbf{R}$. *Then, the number of discrete eigenvalues can be estimated as:*

$$N(q) \le \frac{1}{(4\pi)^2} \int_{R^3}\int_{R^3} \frac{q(x)q(y)}{|x-y|^2} dxdy. \tag{46}$$

This theorem was proved in [14]. We define the operators $T_\pm, T$ for $f \in W_2^1(R)$ as follows:

$$T_+ f = \frac{1}{2\pi i} \lim_{Imz\to 0} \int_{-\infty}^{\infty} \frac{f(s)}{s-z} ds, \ Im\ z > 0, T_- f = \frac{1}{2\pi i} \lim_{Imz\to 0} \int_{-\infty}^{\infty} \frac{f(s)}{s-z} ds, \ Im\ z < 0, \quad (47)$$

$$Tf = \frac{1}{2}(T_+ + T_-)f. \quad (48)$$

Consider the Riemann problem of finding a function Φ, that is analytic in the complex plane with a cut along the real axis.Values of Φ on the sides of the cut are denoted as Φ_+, Φ_-.The following presents the results of [16]:

Lemma 14.

$$TT = \frac{1}{4}I, \ TT_+ = \frac{1}{2}T_+, \ TT_- = -\frac{1}{2}T_-, \ T_+ = T + \frac{1}{2}I, \ T_- = T - \frac{1}{2}I, \ T_-T_- = -T_- \quad (49)$$

Theorem 15. *Let* $q \in \mathbf{R}, \ N(q) < 1, \ g = (\Phi_+ - \Phi_-)$. *Then ,*

$$\Phi_\pm = T_\pm g. \quad (50)$$

Proof. The proof of the above follows from the classic results for the Riemann problem.

Lemma 16. *Let* $q \in \mathbf{R}, \ N(q) < 1 \ g_+ = g(k,\theta,x), \ g_- = g(k,-\theta,x),)$. *Then,*

$$\Psi_+(k,\theta,x) = (T_+g_+ + e^{ik\theta x}), \quad \Psi_-(k,\theta,x) = (T_-g_- + e^{-ik\theta x}). \quad (51)$$

Proof. The proof of the above follows from the definitions of $g, \Phi_\pm, \Psi_\pm$.

Lemma 17. *Let*

$$, \ N(q) < 1, \ \sup_k \left| \int_{-\infty}^{\infty} \frac{\int_{S^2} pA(p,\theta',\theta)d\theta'}{4\pi(p-k+i0)} dp \right| < \alpha < 1 \ \sup_k \left| \int_{-\infty}^{\infty} \frac{\int_{S^2} pA(p,\theta',\theta)\phi_0 d\theta'}{4\pi(p-k+i0)} dp \right| < \alpha < 1$$

Then

$$T_-g_- = (I - T_-D)^{-1}T_-D\phi_0, \ \Psi_- = (I - T_-D)^{-1}T_-D\phi_0 + \phi_0, \ |T_-D\phi_0| < \frac{\alpha}{1-\alpha} \quad (52)$$

Proof. using equation

$$\Psi_+(k,\theta,x) - \Psi_-(k,\theta,x) = -\frac{k}{4\pi} \int_{S^2} A(k,\theta',\theta)\Psi_-(k,\theta',x)d\theta', \quad k \in R. \quad (53)$$

we can rewrite

$$T_+g_+ - T_-g_- = D(T_-g_- + \phi_0)$$

Applying the operator T_- last equation we have

$$T_-g_- = T_-D(T_-g_- + \phi_0)$$

$$(I - T_-D)T_-g_- = T_-D\phi_0, \;\; T_-g_- = \sum_{n\geq 0} (-T_-D)^n \phi_0$$

Estimating the terms of the series, we obtain

$$|T_-g_-| \leq \sum_{n\geq 0} |T_-D^n\phi_0| \leq \sum_{n\geq 0} \left| \int_{-\infty}^{\infty} .... \int_{-\infty}^{\infty} \phi_0 \prod \frac{\int_{S^2} k_j A(k_j, \theta'_{k_j}, \theta_{k_j}) d\theta'_{k_j}}{4\pi(k_{j+1}) - k_j + i0} dk_1 ... d_{k_n} \right| \leq$$

$$\leq \sum_{n\geq 0} \sup_k \left| \int_{-\infty}^{\infty} \frac{\int_{S^2} pA(p,\theta',\theta)\phi_{-\infty} d\theta'}{4\pi(p-k+i0)} dp \right| \prod_{0\leq j<n} \sup_{k_j} \left| \int_{-\infty}^{\infty} \frac{\int_{S^2} k_j A(k_j, \theta'_{k_j}, \theta_{k_j}) d\theta'_{k_j}}{4\pi(k_{j+1}) - k_j + i0)} d_{k_j} \right|$$

$$\leq \sum_{n>0} \alpha^n = \frac{\alpha}{1-\alpha}$$

Using operator $\Lambda = \Delta_k = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial k_i^2}$ we can formulate folows results:

Lemma 18. *Let* $q \in \mathbf{R}$, $N(q) < 1$, *and assume that* $(I - T_-D)^{-1}$ *exists. Then,*

$$T_-DT_-\Lambda g_- = T_-\Lambda g_- + T_-(\nabla D, \nabla T_-g) + T_-\Lambda D\phi_0$$

$$T_-\Lambda g_- = (I - T_-D)^{-1} (T_-(\nabla D, \nabla T_-g) - T_-\Lambda D\phi_0) \quad (54)$$

Proof. The proof of the above follows from the definitions of $g, \Phi_\pm, \Psi_-$ and equation (41)

Lemma 19. *Let* $q \in \mathbf{R}$, $N(q) < 1$. *Then,*

$$q = \lim_{z\to 0} H_0\Psi_-/\Psi_-, \quad (55)$$

$$q = \lim_{z\to 0} \Lambda H_0\Psi_-/\Lambda\Psi_- \quad (56)$$

Proof. The lemma can be proved by substituting Ψ_- into equation (38).

5. Conclusions for the three-dimensional inverse scattering problem

This study has shown once again the outstanding properties of the scattering operator, which , in combination with the analytical properties of the wave function,allow to obtain an almost- explicit formulas for the potential to be obtained from the scattering amplitude. Furthermore, this appro. The estimations follow from this reach overcomes the problem of over-determination, resulting from the fact that the potential is a function of three variables, whereas the amplitude is a function of five variables. We have shown that it is sufficient to average the scattering amplitude to eliminate the two extra variables.

6. Cauchy problem for the Navier–Stokes equation

Numerous studies of the Navier-Stokes equations have been devoted to the problem of the smoothness of its solutions. A good overview of these studies is given in [17]-[20]. The spatial differentiability of the solutions is an important factor, this controls their evolution. Obviously, differentiable solutions do not provide an effective description of turbulence. Nevertheless, the global solvability and differentiability of the solutions has not been proven, and therefore the problem of describing turbulence remains open. It is interesting to study the properties of the Fourier transform of solutions of the Navier-Stokes equations. Of particular interest is how they can be used in the description of turbulence, and whether they are differentiable. The differentiability of such Fourier transforms appears to be related to the appearance or disappearance of resonance, as this implies the absence of large energy flows from small to large harmonics, which in turn precludes the appearance of turbulence. Thus, obtaining uniform global estimations of the Fourier transform of solutions of the Navier-Stokes equations means that the principle modeling of complex flows and related calculations will be based on the Fourier transform method. The authors are continuing to research these issues in relation to a numerical weather prediction model; this paper provides a theoretical justification for this approach. Consider the Cauchy problem for the Navier-Stokes equations:

$$q_t - \nu\Delta q + (q, \nabla q) = -\nabla p + f(x,t),\ div\ q = 0, \tag{57}$$

$$q|_{t=0} = q_0(x) \tag{58}$$

in the domain $Q_T = R^3 \times (0,T)$,where :

$$div\ q_0 = 0. \tag{59}$$

The problem defined by (57), (58), (59) has at least one weak solution (q,p) in the so-called Leray–Hopf class [16]. The following results have been proved [17]:

Theorem 20. *If*

$$q_0 \in W_2^1(R^3),\ \ f \in L_2(Q_T), \tag{60}$$

there is a single generalized solution of (57), (58), (59) in the domain Q_{T_1}, $T_1 \in [0,T]$, satisfying the following conditions:

$$q_t, \nabla^2 q, \quad \nabla p \in L_2(Q_T). \tag{61}$$

Note that T_1 depends on q_0 and f.

Lemma 21. *Let $q_0 \in W_2^1(R^3)$, $f \in L_2(Q_T)$.Then,*

$$\sup_{0\le t\le T} ||q||^2_{L_2(R^3)} \quad + \quad \int_0^t ||\nabla q||^2_{L_2(R^3)} d\tau \qquad \le \qquad ||q_0||^2_{L_2(R^3)} \quad + \quad ||f||_{L_2(Q_T)}. \tag{62}$$

Our goal is to provide global estimations for the Fourier transforms of derivatives of the Navier–Stokes equations' solutions (57), (58), (59) without the that the smallness of the initial velocity and force are small. We obtain the following uniform time estimation.

Lemma 22. *The solution of (57), (58), (59) according to Theorem 20 satisfies:*

$$\widetilde{q} = \widetilde{q}_0 + \int_0^t e^{-\nu|k|^2(t-\tau)}([\widetilde{(q,\nabla)q}] + \widetilde{F})d\tau, \tag{63}$$

where $F = -\nabla p + f$.

Proof. This follows from the definition of the Fourier transform and the theory of linear differential equations.

Lemma 23. *The solution of (57), (58), (59) satisfies:*

$$\widetilde{p} = \sum_{i,j} \frac{k_i k_j}{|k|^2} \widetilde{q_i q_j} + i \sum_i \frac{k_i}{|k|^2} \widetilde{f}_i \tag{64}$$

and the following estimations:

$$||p||_{L_2(R^3)} \leq 3||\nabla q||^{\frac{3}{2}}_{L_2(R^3)} ||q||^{\frac{1}{2}}_{L_2(R^3)}, \tag{65}$$

$$|\nabla \widetilde{p}| \leq \frac{|\widetilde{q}^2|}{|k|} + \frac{|\widetilde{f}|}{|k|^2} + \frac{1}{|k|}\left|\nabla \widetilde{f}\right| + 3\left|\nabla \widetilde{q}^2\right|. \tag{66}$$

Proof. This expression for p is obtained using *div* and the Fourier transform presentation.

Lemma 24. *The solution of (57), (58), (59) in Theorem 20 satisfies the following inequalities:*

$$\int_{R^3} |x|^2|q|^2 dx + \int_0^t \int_{R^3} |x|^2 |\nabla q|^2 dx d\tau \leq const, \int_{R^3} |x|^4|q|^2 dx + \int_0^t \int_{R^3} |x|^4 |\nabla q|^2 dx d\tau \leq const, \tag{67}$$

or

$$||\nabla \widetilde{q}||_{L_2(R^3)} + \int_0^t \int_{R^3} |k|^2 |\widetilde{\nabla q}|^2 dk d\tau \leq const, \left|\left|\nabla^2 \widetilde{q}\right|\right|_{L_2(R^3)} + \int_0^t \int_{R^3} |k|^2 |\widetilde{\nabla^2 q}|^2 dk d\tau \leq const. \tag{68}$$

This follows from the a priori estimation of Lemma 21, conditions of Lemma 24,the Navier–Stokes equations.

Lemma 25. *The solution of (57), (58), (59) satisfies the following inequalities:*

$$\max_k |\widetilde{q}| \leq \max_k |\widetilde{q}_0| + \frac{T}{2}\sup_{0\leq t\leq T} ||q||^2_{L_2(R^3)} + \int_0^t ||\nabla q||^2_{L_2(R^3)} d\tau, \quad (69)$$

$$\max_k |\nabla\widetilde{q}| \leq \max_k |\nabla\widetilde{q}_0| + \frac{T}{2}\sup_{0\leq t\leq T} ||\nabla\widetilde{q}||_{L_2(R^3)} + \int_0^t \int_{R^3} |k|^2 |\widetilde{\nabla q}|^2 dk d\tau, \quad (70)$$

$$\max_k \left|\nabla^2\widetilde{q}\right| \leq \max_k \left|\nabla^2\widetilde{q}_0\right| + \frac{T}{2}\sup_{0\leq t\leq T} \left|\left|\nabla^2\widetilde{q}\right|\right|_{L_2(R^3)} + \int_0^t \int_{R^3} |k|^2 |\nabla^2\widetilde{q}|^2 dk d\tau. \quad (71)$$

Proof. This follows from the a priori estimation of Lemma 21, conditions of Lemma 25,the Navier–Stokes equations.

Lemma 26. *The solution of (57), (58), (59) according to Theorem 20 satisfies* $C_i \leq const$, $(i = \overline{0, 2, 4})$, *where:*

$$C_0 = \int_0^t |\widetilde{F}_1|^2 d\tau, \quad F_1 = (q, \nabla)q + F, \ C_2 = \int_0^t \left|\nabla\widetilde{F}_1\right|^2 d\tau, \quad C_4 = \int_0^t \left|\nabla^2\widetilde{F}_1\right|^2 d\tau. \quad (72)$$

Proof. This follows from the a priori estimation of Lemma 21, the Navier–Stokes equations.

Lemma 27. *Suppose that* $q \in R$, $\max_k |\widetilde{q}| < \infty$, *then*

$$\int_{R^3}\int_{R^3} \frac{q(x)q(y)}{|x-y|^2} dx dy \leq C(|q|_{L_2} + \max_k |\widetilde{q}|)^2.$$

Proof. Using Plansherel's theorem, we get the statement of the lemma.
This proves Lemma 27.

Let's consider the influence of the following large scale transformations in Navier-Stokes' equation on

$$K = \frac{\nu^{\frac{1}{2}}}{\nu^{\frac{1}{2}} - 4\pi C C_0^{\frac{1}{2}}}. \ t' = tA, \ \nu' = \frac{\nu}{A}, \ v' = \frac{v}{A}, \ F_0' = \frac{F_0}{A^2}.$$

Lemma 28. *Let*

$$A = \frac{4}{\nu^{\frac{1}{3}}(CC_0+1)^{\frac{2}{3}}},\ \textit{then } K \le \frac{8}{7}.$$

Proof. By the definitions C and C_0, we have

$$K = \left(\frac{\nu}{A}\right)^{\frac{1}{2}} \left(\left(\frac{\nu}{A}\right)^{\frac{1}{2}} - \frac{4\pi CC_0}{A^2} \right)^{-1} = \nu^{\frac{1}{2}} \left(\nu^{\frac{1}{2}} - \frac{4\pi CC_0}{A^{\frac{3}{2}}} \right)^{-1} < \frac{8}{7}.$$

This proves Lemma

Let us introduce operator $F_{kk\prime}$, as $F_{kk\prime} f = \int_{R^3} e^{i(k,x)-i(x,k')} f(x)dx$

Lemma 29. *Let $Q \in W_2^1(R^3)$, $Q \in L_2(Q_T), \nu_k(k,k') = \nu|k-k'|^2$.Then, the solution of (57), (58), (59) in Theorem 20 satisfies the following inequalities:*

$$\sup_{(e_k,e_{k'})\in S^2} |Q(k,k')| < C, \quad \sup_{(e_k,e_{k'})\in S^2} k|Q(k,k')| < \frac{C}{\sqrt{(1-cos(\theta))}},$$

$$\sup_{(e_k,e_{k'})\in S^2} |A(k,k')| < C, \quad \sup_{(e_k,e_{k'})\in S^2} k|A(k,k')| < \frac{C}{\sqrt{(1-cos(\theta))}}, \tag{73}$$

Proof. This follows from

$$\dot{Q} = -F_{kk\prime}[(q,\nabla)q] + F_{kk\prime}(\nu\Delta q + \nabla p) + F_{kk\prime}F \tag{74}$$

After the transformations we obtain

$$\dot{Q} = -F_{kk\prime}[(q,\nabla)q] + (\nu_k F_{kk\prime} q + F_{kk\prime}\nabla p) + F_{kk\prime}F, \tag{75}$$

$$Q = Q_0 + \int_0^t e^{-|k|^2(1-cos(\theta))(t-\tau)} \left(-F_{kk\prime}[(q,\nabla)q] + F_{kk\prime}\nabla p + F_{kk\prime}F\right).$$

from last equation we have

$$|Q| \le |Q_0| + C_T$$

Integrating by θ and carrying out the coordinate transformations, we obtain

$$Q = Q_0 + \int_0^t e^{-|k|^2(1-cos(\theta))(t-\tau)} \left(-F_{kk\prime}[(q,\nabla)q] + F_{kk\prime}\nabla p + F_{kk\prime}F\right).$$

$$|Q| \le |Q_0| + \frac{C}{k\sqrt{(1-cos(\theta))}}$$

Lemma 30. *Let $Q \in W_2^1(R^3)$, $Q \in L_2(Q_T), \nu_k(k,k') = \nu|k-k'|^2$. Then, the solution of (57), (58), (59) in Theorem 20 satisfies the following inequalities:*

$$\sup_{(e_k,e_{k'})\in S^2} \left|TQ(k,k')\right| < C, \sup_{(e_k,e_{k'})\in S^2} \left|\Lambda TQ(k,k')\right| < C,$$
$$\sup_{(e_k,e_{k'})\in S^2} |TA(k,k')| < C, \sup_{(e_k,e_{k'})\in S^2} \left|\Lambda TA(k,k')\right| < C, \quad (76)$$

Proof. This follows from

$$TQ = TQ_0 + T\int_0^t e^{-|k|^2(1-cos(\theta))(t-\tau)} \left(-F_{kk\prime}[(q,\nabla)q] + F_{kk\prime}\nabla p + F_{kk\prime}F\right).$$

from last equation we have

$$|TQ| \le |TQ_0| + C_T$$

Using operator $\Lambda = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial k_i^2}$

$$\Lambda TQ = \Lambda TQ_0 + \Lambda T\int_0^t e^{-|k|^2(1-cos(\theta))(t-\tau)} \left(-F_{kk\prime}[(q,\nabla)q] + F_{kk\prime}\nabla p + F_{kk\prime}F\right).$$
$$|\Lambda TQ| = |\Lambda TQ_0| + C_T$$

Lemma 31. *Let $Q \in W_2^1(R^3)$, $Q \in L_2(Q_T), \nu_k(k,k') = \nu|k-k'|^2, X(x) = x$. Then, the solution of (57), (58), (59) in Theorem 20 satisfies the following inequalities:*

$$\sup_{(e_k,e_{k'})\in S^2} \int_0^\infty |S_{\phi_{x_0}}Q(k,k')|dk < C\int_0^t \sup_{x\in R^3}|q(x)| \left\|(1+X^2)\nabla q\right\|_{L_2(R^3)} d\tau,$$
$$\sup_{(e_k,e_{k'})\in S^2} \int_0^\infty |\Lambda S_{\phi_{x_0}}Q(k,k')|dk < C\int_0^t sup_{x\in R^3}|q(x)| \left\|(1+X^2)\nabla q\right\|_{L_2(R^3)} d\tau \quad (77)$$

Proof. This follows from
from last equation we have

$$|S_{\phi_{x_0}}Q| \le |S_{\phi_{x_0}}Q_0| + \sup_{(e_k,e_{k'})\in S^2} \int_0^\infty |S_{\phi_{x_0}}Q(k,k')|dk < C\int_0^t \sup_{x\in R^3}|q(x)| \left\|(1+X^2)\nabla q\right\|_{L_2(R^3)} d\tau$$

Using operator $\Lambda = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial k_i^2}$

$$|\Lambda S_{\phi_{x_0}} Q| \le |S_{\phi_{x_0}} Q_0| + \sup_{(e_k,e_{k'})\in S^2} \left| \Lambda S_{\phi_{x_0}} \int_0^t e^{-|k|^2(1-cos(\theta))(t-\tau)} \left(-F_{kk\prime}[(q,\nabla)q] + F_{kk\prime}\nabla p + F_{kk\prime}F\right) \right| <$$

$$C\int_0^t \sup_{x\in R^3} |q(x)| \left\|(1+X^2)\nabla q\right\|_{L_2(R^3)} d\tau$$

Lemma 32. *Let* $q \in \mathbf{R} \cap L_2(R^3)$*, and* $Csup_{e_k,e_{k'},k}|T_Q(k,k')| + Csup_{e_k,e_{k'},k}|Q(k,k')| < 1$*. Then,*

$$|\Lambda\Psi_{\pm} q|\,|_{x=x_0,\,k=0} \qquad \ge \qquad x^2 \qquad - \qquad C. \quad (78)$$

Proof.

$$|\Lambda\Psi_-|_{k=0} = |T_-\Lambda g_- + \Lambda\Phi_0|_{k=0} \ge x^2 - \left|(I-T_-D)^{-1}\left(T_-(\nabla D, \nabla Tg) + T\Lambda D\phi_0\right)\right|_{k=0}$$
$$\ge x^2 - \left|(I-T_-D)^{-1}\left(T_-(\nabla D, \nabla T((I-T_-D)^{-1}T_-D\phi_0)) - T\Lambda D\phi_0\right)\right|_{k=0}$$
$$\ge x^2 - C \quad (79)$$

Lemma 33. *The following permutation formulas hold true*

$$\frac{x_n}{\prod_{0<i<n}(x_{i+1}-x_i)(x_n-x_{n-1})} = \frac{1}{\prod_{0<i<n}(x_{i+1}-x_i)} + \frac{x_{n-1}}{\prod_{0<i<n}(x_{i+1}-x_i)(x_n-x_{n-1})}$$
$$\frac{x_{n-1}}{\prod_{0<i<n}(x_{i+1}-x_i)(x_n-x_{n-1})} = \frac{1}{\prod_{0<i<n, i\neq n-1}(x_{i+1}-x_i)(x_n-x_{n-1})} +$$
$$\frac{x_{n-2}}{\prod_{0<i<n}(x_{i+1}-x_i)(x_n-x_{n-1})} \quad (80)$$

Proof. Simple transformations, but in the future plays an important role. With the help of this transformation we will be able to prove a very important estimate for the derivatives of wave functions and show that it is actually close to an estimate without derivatives

Lemma 34. *Let* $Q \in W_2^1(R^3)$*,* $Q \in L_2(Q_T), \nu_k(k,k') = \nu|k-k'|^2, K(k) = k, X(x) = x$*. Then, the solution of (57), (58), (59) in Theorem 20 satisfies the following inequalities:*

$$\sup_{x\in R^3} |q(x)| < \int_0^t \sup_{x\in R^3} |q(x)| \left\|(1+X^2)\nabla q\right\|_{L_2(R^3)} d\tau,$$

$$\sup_{x\in R^3} |q(x)| < C \quad (81)$$

Proof. Using equation

$$q = \lim_{z\to 0} \Lambda H_0 \Psi_- / \Lambda \Psi_- \tag{82}$$

using lemmas (11-21) we have

$$\begin{aligned} |q(x)| = \left|\frac{H_0\Lambda\Psi_-}{\Lambda\Psi_-}\right|_{k=0},\ |q(x_0)| \le \left|\frac{H_0\Lambda\Psi_-}{x_0^2-\alpha}\right|_{k=0} \le C\left|H_0\Lambda\Psi_-\right|_{k=0} = C\left|T_-\Lambda g_-\right|_{k=0} \\ \le C\left|(I-T_-D)^{-1}\left(T_-(\nabla D, \nabla T H_0 g) + T\Lambda D H_0\phi_0\right)\right|_{k=0} \\ C\left|(I-T_-D)^{-1}\left(T_-(\nabla D, \nabla T(K^2 g + qg)) - T\Lambda D K^2\phi_0\right)\right|_{k=0} \le \\ C\left|(I-T_-D)^{-1}\left(T_-(\nabla D, \nabla T K^2((I-T_-D)^{-1}T_-D\phi_0)) - T\Lambda D K^2\phi_0\right)\right|_{k=0} \\ \le C|S_{\phi_{x_0}}Q| + C|S_{\Lambda\phi_{x_0}}Q| \\ \le C\int_0^t \sup_{x\in R^3}|q(x)|\left\|(1+X^2)\nabla q\right\|_{L_2(R^3)} d\tau \end{aligned} \tag{83}$$

Using the Grnwal - Bellman inequality
we have $\sup_{x\in R^3}|q(x)| < C$

Theorem 35. *Let* $q_0 \in W_2^2(R^3), \nabla^2\widetilde{q_0} \in L_2(R^3)$, $f \in L_2(Q_T), \widetilde{f} \in L_1(Q_T) \cap L_2(R^3),\ \nabla^2\widetilde{f} \in L_1(Q_T) \cap L_2(R^3)$. *and* $\max_k |T\widetilde{q_0}| < const,\ \max_k |T\nabla^2\widetilde{q_0}| < const,$ *Then, there exists the following a unique generalized solution of (57), (58), (59)satisfying inequality:* $\sup_x |q_i| \le const$, *where the value of* $const$ *depends only on the conditions of the theorem .*

Proof. It suffices to obtain uniform estimates of the maximum velocity components q_i, which obviously follow from $\max_x |q_i|$, because uniform estimates allow us to extend the local existence and uniqueness theorem over the interval in which they are valid. To estimate the velocity components, Lemma (15)can be used :

$$v_i = q_i / (\int_0^T ||q_x||^2_{L_2(R^3)} dt + A_0 + 1),\ A_0 = 4/(\nu^{\frac{1}{3}}(CC_0+1)^{\frac{2}{3}}).$$

Using Lemmas (11)-(21) for

$$v_i = q_i / (\int_0^T ||q_x||^2_{L_2(R^3)} dt + A_0 + 1)$$

we can obtain$\int_{S^2} ||A_i||_{TA} d\theta' < \alpha < 1$, where $A_i$is the amplitude of potential q_i and $N(q_i) < 1$.That is, discrete solutions are not significant in proving the theorem , so its assertion follows the conditions of Theorem 35, which defines uniform time estimations for the maximum values of velocity components.

$$||\nabla q||_{L_2(R^3)} + \int_0^t \int_{R^3} |\nabla q|^2 dk d\tau \le const + \sup_{x\in R^3}|q(x)| \int_0^t ||\nabla q||_{L_2(R^3)} \left\|\nabla^2 q\right\|_{L_2(R^3)} d\tau, \tag{84}$$

Theorem 35 asserts the global solvability and uniqueness of the Cauchy problem for the Navier-Stokes equations.

Theorem 36. *Let* $q_0 \in W_2^2(R^3), \nabla^2 \widetilde{q_0} \in L_2(R^3)\,, f \in L_2(Q_T), \widetilde{f} \in L_1(Q_T) \cap L_2(R^3)$

$$\lim_{t \to t_0} ||\nabla q||_{L_2(R^3)} = \infty. \tag{85}$$

Then, there exists i, j , x_0

$$\lim_{t \to t_0} \psi_j(x_0, t) = \infty \textit{ or } \lim_{t \to t_0} N(q_i) = \infty \tag{86}$$

Proof. A proof of this lemma can be obtained using $q_i = P_{Ac}q_i + P_D q_i$ and uniform estimates $P_{Ac}q_i$.

Theorem 36 Describes the loss of smoothness of classical solutions for the Navier–Stokes equations. Theorem 36 describes the time blow up of the classical solutions for the Navier-Stokes equations arises, and complements the results of Terence Tao [1].

7. Conclusions

Uniform global estimations of the Fourier transform of solutions of the Navier–Stokes equations indicate that the principle modeling of complex flows and related calculations can be based on the Fourier transform method. In terms of the Fourier transform, under both smooth initial conditions and right-hand sides, no appear exacerbations appear in the speed and pressure modes.A loss of smoothness in terms of the Fourier transform can only be expected in the case of singular initial conditions, or of unlimited forces in $L_2(Q_T)$. The theory developed by us is supported by numerical calculations carried out in the works [21-23] Where the dependence of the smoothness of the solution on the oscillations of the system is clearly deduced.

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