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Solving the Ivancevic Pricing Model Using the He's Frequency Amplitude Formulation

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Abstract. In financial mathematics, option pricing theory remains a core area of interest that requires effective models. Thus, the Ivancevic option pricing model (IOPM) is a nonlinear adaptive-wave alternative for the classical Black-Scholes option pricing model; it represents a controlled Brownian motion (BM) in an adaptive setting with relation to nonlinear Schrödinger equation. The importance of the IOPM cannot be overemphasized; though, it seems difficult and complex to obtain the associated exact solutions if they exist. Therefore, this paper provides exact solutions of the IOPM by means of a proposed analytical method referred to as He's frequency amplitude formulation. Cases of nonzero adaptive market potential are considered. The method is shown to effective, efficient, simple and direct in application, even without loss of generality.

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1. Introduction

The classical Black-Scholes model (BSM) serves as a remarkable financial model for option pricing and valuation. The BSM describes the time-evolution of the market value of financial equity such as European or stock option [4, 17, 19]. The main assumptions associated with this classical arbitrage pricing theory (BSM) include the following: the asset price S (or the underlying asset) following a geometric Brownian motion (GBM), the drift parameter, μ and the volatility rate, σ are assumed constants, lack of arbitrage opportunities (no risk-free profit), frictionless and competitive markets [12, 8, 2]. Thus,

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the stock price $S = S(t)$, at time t, $(0 \leq t \leq T)$ follows the stochastic differential equation (SDE):

$$dS = S(\mu dt + \sigma dW_t), \ S \in [0, \infty) \tag{1}$$

where $\mu, \sigma > 0$, and W_t are mean rate of return of S, the volatility, and a standard Brownian motion respectively.
So, for an option value $u = u(S,t)$, the Black-Scholes partial differential equation (PDE) associated to (1) can be expressed as:

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 u}{\partial S^2} - ru = 0 \tag{2}$$

with $u(0,t) = 0$, $u(S,t) \to 0$ as $S \to \infty$, $u(S,T) = max(S - E, 0)$, E is a constant and

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \ \ S_0 = S(0). \tag{3}$$

In literature, detailed and extensive work on the importance of (2) with respect to exact, analytical, approximate or numerical methods of solutions have been referred [21, 9, 5, 10]. Vukovic [24] in a recent study, established the interconnectedness of the Schrödinger and the Black-Scholes equations via the tools of quantum physics in the sense of Hamiltonian operator. It was noted that while the Black-Scholes Hamiltonian was anti-Hermitian causing the eigenvalues to be complex, the Schrödinger Hamiltonian was Hermitian. It was further showed that the Black-Scholes equation can be derived from the Schrödinger equation via the application of quantum mechanics tools [1, 7]. In [3], [25] and [27] the solution of linear and nonlinear Schrödinger equations was obtained by Homotopy perturbation method, variational iteration method and He's frequency formulation respectively. Recently, in [22] an analytical option pricing model based on the nonlinear Schrödinger partial differential equation with vanishing external potential has been considered. The facts incorporated include the points that: the Schrödinger equation requires a complex state function while the Black-Scholes equation is a real PDE that yields a real valued expression for the option price at all time.
The Black-Scholes model (2) can be applied to a reasonable number of one dimensional option models ascribed to u and S, say for puts/calls and stocks/dividends respectively [17]. As noted in [11, 23], one could consider the associated probability density function (PDF) resulting from the backward Fokker-Planck equation using the classical Kolmogorov probability method instead of the market value of an option obtained via the Black-Scholes equation.

2. The Ivancevic Option Pricing Model (IOPM) [13]

As an alternative method for obtaining the same PDF for the market value of a stock option, Ivancevic [18] applied the quantum-probability formation as a solution to a time-dependent Schrödinger equation (linear or nonlinear) for the evolution of the complex-valued wave function, and proposed an adaptive, wave-form nonlinear model [6, 20].

Henceforth, such nonlinear adaptive model is referred to as Ivancevic option pricing model as follows:

$$i\frac{\partial w}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial S^2}+\beta|w|^2w=0,\ \ i^2=-1 \tag{4}$$

where $w=w(S,t)$ denotes the option pricing wave-function at time t , $|w|^2=|w(S,t)|^2$ represents the PDF for the option price with regard to stock price and time, σ represents a constant or stochastic process as the dispersion frequency volatility coefficient while β is referred to as the Landau coefficient representing adaptive market potential. The model (4) becomes linear if $\beta=0$. In this work, a case of non-zero adaptive market potential ($\beta\neq 0$) will be considered in terms of analytical solutions using a proposed semi-analytical method referred to as He's frequency amplitude formulation.

3. He's amplitude frequency formulation

He's frequency amplitude formulation is the development of an ancient Chinese algorithm [14, 15, 16]; it is very effective to nonlinear oscillators as shown by the authors in [26].

We consider a generalized nonlinear oscillator in the form

$$u''+f(u)=0,\quad u(0)=A,\quad u'(0)=0. \tag{5}$$

We use two trial functions $u_1(t)=Acost$ and $u_2(t)=Acos\omega t$, which are, respectively, the solutions of the following linear oscillator equations:

$$u''+\omega_1^2u=0,\quad \omega_1^2=1 \tag{6}$$

and

$$u''+\omega_2^2u=0,\quad \omega_2^2=\omega^2. \tag{7}$$

where ω is assumed to be the frequency of the nonlinear oscillator, equation (5).

For the case of equation (5), the residuals are

$$R_1(t)=-cost+f(Acost) \tag{8}$$

and

$$R_2(\omega t)=-\omega^2cos\omega t+f(Acos\omega t). \tag{9}$$

We will use the following frequency-amplitude formulation:

$$\omega^2=\frac{\omega_1^2R_2(\omega t=0)-\omega_2^2R_1(t=0)}{R_2-R_1}. \tag{10}$$

In order to use He's amplitude frequency formulation, we choose two trial functions $u_1(S,t)$ and $u_2(S,t)$ with which we calculate the residuals $R_1(S,t)$ and $R_2(S,t)$ respectively. The trial functions $u_1(S,t)$ and $u_2(S,t)$ for the application of the present study are solutions of the following linear Schrödinger equation

$$i\frac{\partial w}{\partial t}+\omega\frac{\partial^2 w}{\partial S^2}=0. \tag{11}$$

He's amplitude frequency formulation reads [16]

$$\omega^2 = \frac{R_2(S,0) - \omega^2 R_1(S,0)}{R_2(S,0) - R_1(S,0)}. \tag{12}$$

From Eq (12) we can obtain approximate solutions for a certain type of differential equations whose solutions are a priori periodic.

3.1. Numerical Illustrative Examples

In this subsection, we consider the following cases for numerical examples.

Example 1. *Suppose $\beta = 2$ and $\sigma = \sqrt{2}$. Then the corresponding Ivancevic option pricing model given by Eq (4) is:*

$$\begin{cases} \frac{\partial w}{\partial t} = i\Big(\frac{\partial^2 w}{\partial S^2} + 2|w|^2 w\Big), \\ w(S,0) = e^{2iS}. \end{cases} \tag{13}$$

We use the trial functions

$$w_1(S,t) = e^{i(2S+t)}, \quad w_2(S,t) = e^{i(2S+\omega t)}. \tag{14}$$

By calculation, we obtain

$$w_{1,t} = ie^{i(2S+t)},\ w_{1,S} = 2ie^{i(2S+t)},\ w_{1,SS} = -4e^{i(2S+t)} \tag{15}$$

$$w_{2,t} = i\omega e^{i(2S+\omega t)},\ w_{2,S} = 2ie^{i(2S+\omega t)},\ w_{2,SS} = -4e^{i(2S+\omega t)}. \tag{16}$$

Substituting the two trial functions and their derivatives into Eq. (13) gives the residuals:

$$R_1(S,t) = -3e^{i(2S+t)}, \quad R_2(S,t) = -(2+\omega)e^{i(2S+\omega t)} \tag{17}$$

According to He's frequency formulation, we have

$$\omega^2 = \frac{R_2(S,0) - \omega^2 R_1(S,0)}{R_2(S,0) - R_1(S,0)} = \frac{3\omega^2 e^{2iS} - \omega e^{2iS} - 2e^{2iS}}{e^{2iS} - \omega e^{2iS}}. \tag{18}$$

Simplifying, we obtain $\omega = -2$. The solution is obtained as follows

$$w(S,t) = e^{2i(S-t)}. \tag{19}$$

Showing that (19) satisfies (13) is obvious and straightforward.

Example 2. *Suppose $\beta = 3$ and $\sigma = 1$. Then the corresponding Ivancevic option pricing model given by Eq (4) is:*

$$\begin{cases} \frac{\partial w}{\partial t} = i\Big(\frac{\partial^2 w}{\partial S^2} + 6|w|^2 w\Big), \\ w(S,0) = e^{2iS}. \end{cases} \tag{20}$$

We use the trial functions

$$w_1(S,t) = e^{i(2S+t)}, \quad w_2(S,t) = e^{i(2S+\omega t)}. \tag{21}$$

Now, by calculating, we obtain

$$w_{1,t} = ie^{i(2S+t)},\ w_{1,S} = 2ie^{i(2S+t)},\ w_{1,SS} = -4e^{i(2S+t)} \tag{22}$$

$$w_{2,t} = i\omega e^{i(2S+\omega t)},\ w_{2,S} = 2ie^{i(2S+\omega t)},\ w_{2,SS} = -4e^{i(2S+\omega t)}. \tag{23}$$

Substituting the two trial functions and their derivatives into Eq. (20) gives the residuals:

$$R_1(S,t) = -3e^{i(2S+t)}, \quad R_2(S,t) = -(2+\omega)e^{i(2S+\omega t)} \tag{24}$$

According to He's frequency formulation, we have

$$\omega^2 = \frac{R_2(S,0) - \omega^2 R_1(S,0)}{R_2(S,0) - R_1(S,0)} = \frac{(2-\omega)e^{2iS} - \omega^2 e^{2iS}}{(2-\omega)e^{2iS} - e^{2iS}}. \tag{25}$$

Simplifying, we obtain $\omega = 2$. *The solution is obtained as follows*

$$w(S,t) = e^{2i(S+t)}. \tag{26}$$

It is straightforward to verify that $w(S,t) = e^{2i(S+t)}$ *satisfies the problem given by Eq. (20).*

4. Concluding Remarks

In this work, the Ivancevic option pricing model (IOPM) is considered. This nonlinear adaptive-wave model serves as alternative for the classical Black-Scholes option pricing model based on a controlled Brownian motion in an adaptive setting relating to nonlinear Schrödinger equation. By considering cases of nonzero adaptive market potential, exact solutions of the IOPM by means of a proposed He's frequency amplitude formulation method (HFAFM) were obtained. The results revealed that the proposed HFAFM is simple, direct, and effective as the obtained solutions coincide exactly with the exact solutions without any form of linearization, perturbation, or discretization.

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