# On the Error Analysis of a Continuous Implicit Hybrid One Step Method

T. A. Anake[1,*], S. O. Edeki[1]

[1] *Department of Mathematics, Covenant University, Canaanland, Ota, Nigeria*

**Abstract.** It is a known fact that in the application of a continuous linear multistep method, the global error at a particular point is influenced by the accumulation of local truncation errors at each step from the initial point and thereby reduces the accuracy of the approximated result. Hence, by controlling the growth of local errors it is expected that the accuracy of the approximations should improve. In this paper therefore, a method is derived for the bound on the local truncation error of continuous implicit hybrid one step method for the solution of initial value problems of second order ordinary differential equations by means of the generalized Lagrange form of the Taylor's remainder and the mean value theorem.

**2010 Mathematics Subject Classifications**: 34K34, 65L70, 34A45

**Key Words and Phrases**: truncation error, global error, mean value theorem, remainder theorem, one step method, hybrid multistep method

## 1. Introduction

An important property associated with the numerical solutions of the initial value problems of ordinary differential equations (ODEs), is the global error of computation. The size of this error on the interval of interest makes the effort put into developing the numerical method worthwhile or futile. If the global error is small, the numerical solution is considered to be accurate, and other properties of the solution become less important.

The requirement of a small global error will guarantee that the dynamics of the approximate solution is similar to that of the exact solution and facilitate step size control using the global error estimate, (Orel,2010).

The relationship between the local and the global errors and possible ways of controlling the global error directly has attracted a lot of interest. A number of researchers have studied this situation for different numerical methods; for instance, Highhman(1993), Dormund *et al.*(1994), Calvo *et al.*(1996) and Stuart(1997) all studied the connection between local truncation error and global error for Runge-Kutta methods while Lambert(1979),

---

*Corresponding author.

*Email addresses:* `timothy.anake@covenantuniversity.edu.ng` (T. A. Anake), `soedeki@yahoo.com` (S. O. Edeki)

Onumayi *et al.*(1999) as well as Kulikov and Shindin(2004) were concerned with estimates of these errors for multistep methods.

It is a well known that in the application of a linear multistep method at any given point, say $x_i$, the global error would usually be an accumulation of the local errors committed at each step from the initial point up to the reference point. This is usually a function of the problem to be solved, the numerical method adopted and step size selected, (Orel,2010). However, while it may be possible to control the local errors and step size during the implementation of the linear multistep method, it may not be possible to do the same for the global error directly.

The aim of this paper is to derive a method for a bound on the local truncation error, at each step of application, of a continuous implicit hybrid one step method(CIHOSM) as a means of managing the magnitude of the global error of computation. The result presented in this paper, extends an earlier result by Lambert(1979) for special second order initial value problems to the general case.

The paper is organized as follows: in Section 2 the CIHOSM is introduced. The main result is presented in Section 3. The error bound method derived in Section 3 is applied on the CIHOSM proposed in Anake and Adoghe(2013) for the solution of a simple electric circuit problem using different step sizes in Section 4. Finally in section 5, conclusions are presented.

## 2. The Continuous Implicit Hybrid One Step method

Consider the general form of the continuous implicit hybrid one step method proposed by Anake and Adoghe(2013):

$$y(x) = \sum_{j'=r-1,r} \alpha_{\nu_{j'}}(t)y_{n+\nu_{j'}} + h^2 \left[ \sum_{j=0}^{1} \beta_j(t)f_{n+j} + \sum_{j'=1}^{r} \beta_{\nu_{j'}}(t)f_{n+\nu_{j'}} \right], \qquad (1)$$

where $\alpha_{\nu'_j}$ $\beta_{\nu'_j}$ and $\beta_j$ are continuous in $t \in$; $y_{n+j}$ is the numerical approximation of the exact solution at the grid point $x_{n+j}$, and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$. The subscript, $\nu_{j'} \in (0,1)$, $\forall j'$ is a rational number representing the $j'^{th}$ off-grid point.

We note that by using off-grid points, the method mimics a continuous implicit multistep method and solves directly ordinary differential equations of the form:

$$\begin{cases} y'' = f(x,y,y') \\ y(a) = \eta_0, \\ y'(a) = \eta_1 \end{cases} \qquad (2)$$

where $x \in [a,b]$, $a,b \in$ and $f$ is continuously differentiable with respect to its arguments. Ordinary differential equations such as (2) are important in many application areas (see, Chapra and Canale(2010), Anake(2011)).

The following definition and theorem are useful results needed for the derivation in the next section.

**Definition 1** ([*Anake*, 2011, *p*.61] Local Truncation Error). *The local truncation error at* $x_{n+k}$ *of the numerical method* (2) *is defined to be the operator* $\mathcal{L}[y(x_n); h]$ *given by:*

$$\mathcal{L}[y(x_n); h] = \sum_{j=0}^{k} \left[ \alpha_j y(x_n + jh) - h^2 \beta_j y''(x_n + jh) \right] \tag{3}$$

*when* $y(x_n)$ *is the theoretical solution of the initial value problem* (2) *at* $x_n$.

In what follows, we shall assume that in the application of the method (1) to yield $y_{n+k}$, no previous truncation errors have been made. Furthermore, we shall assume the theoretical solution $y(x)$ possesses $p+2$ continuous derivatives, so that the local truncation error has the value;

$$\mathcal{L}[y(x_n); h] = C_{p+2} h^{p+2} y^{p+2}(x_n) + O(h^{p+3}) \tag{4}$$

where $p$ and $C_{p+2}$ are taken to be the order and error constant of method (1) respectively.

**Theorem 1** ([*Huang*, 2001, *p*.2] Generalized Mean Value Theorem). *If* $\varphi$ *is continuous and* $\vartheta$ *is integrable on an interval containing* $a$ *and* $x$, *and* $\vartheta$ *does not change sign in the interval* $[a, x]$, *then there exists a point* $\zeta$ *between* $a$ *and* $x$ *such that*

$$\int_a^x \varphi(t)\vartheta(t)\, dt = \varphi(\zeta) \int_a^x \vartheta(t)\, dt. \tag{5}$$

We note that the asymptotic method (4) can only predict the behaviour of the error in the limit as $h \to 0$. Furthermore, our knowledge of the order $p$ and error constant $C_{p+2}$ of the method is not sufficient to deduce a bound for the magnitude of the local truncation error when $h$ is fixed, since the magnitude of the term $O(h^{p+3})$ is unknown.

## 3. Main Result

We present the main result as a proposition as follows.

**Proposition 1.** *Given the operator* $\mathcal{L}$, *of order* $p$ *and error constant* $C_{p+2}$, *for* $h$ *fixed, the local truncation error takes the form:*

$$\mathcal{L}[y(x_n); h] = C_{p+2} h^{p+2} y^{p+2}(x_n + \xi h), \qquad 0 < \xi < k \tag{6}$$

*and*

$$|\mathcal{L}[y(x_n); h]| \leq h^{p+2} QY \tag{7}$$

*where*

$$Q = |C_{p+2}| = \frac{1}{p!} \left| \int_0^k G(t)\, dt \right| \tag{8a}$$

*and*

$$Y = \max_{x \in [a,b]} |y^{(p+2)}(x)|. \tag{8b}$$

*Proof.* Define the generalized form of the Lagrange remainder $R_{p+1}$ after $p + 2$ terms in a Taylor expansion of a sufficiently differentiable function $G(x)$ about $x = a$ as;

$$R_{p+1} = \frac{1}{(p+1)!} \int_0^h (h-z)^{p+1} G^{(p+2)}(a+z) \, dz. \tag{9}$$

Then we can now write the Taylor expansions of $y(x_n + jh)$ about $x_n$ truncated after $p + 2$ terms as

$$y(x_n + jh) = y(x_n) + jhy^{(1)} + \frac{(jh)^2}{2!} y^{(2)}(x_n) + \cdots + \frac{(jh)^{p+1}}{(p+1)!} y^{(p+1)}$$
$$+ \frac{1}{(p+1)!} \int_0^{jh} (jh-z)^{p+1} y^{(p+2)}(x_n + z) dz \tag{10}$$

and in the same manner, the Taylor expansion of $y''(x_n + jh)$ about $x_n$ truncated after $p$ terms may now be written as

$$y^{(2)}(x_n + jh) = y^{(2)}(x_n) + jhy^{(3)} + \frac{(jh)^2}{2!} y^{(4)}(x_n) + \cdots + \frac{(jh)^{p-1}}{(p-1)!} y^{(p+1)}$$
$$+ \frac{1}{(p-1)!} \int_0^{jh} (jh-z)^{p-1} y^{(p+2)}(x_n + z) dz. \tag{11}$$

Making $z = th$, (10) and (11) respectively become

$$y(x_n + jh) = y(x_n) + jhy^{(1)} + \frac{(jh)^2}{2!} y^{(2)}(x_n) + \cdots + \frac{(jh)^{p+1}}{(p+1)!} y^{(p+1)}$$
$$+ \frac{h^{p+1}}{(p+1)!} \int_0^j (j-t)^{p+1} y^{(p+2)}(x_n + th) dt \tag{12}$$

and

$$y^{(2)}(x_n + jh) = y^{(2)}(x_n) + jhy^{(3)} + \frac{(jh)^2}{2!} y^{(4)}(x_n) + \cdots + \frac{(jh)^{p-1}}{(p-1)!} y^{(p+1)}$$
$$+ \frac{h^{p-1}}{(p-1)!} \int_0^j (j-t)^{p-1} y^{(p+2)}(x_n + th) dt. \tag{13}$$

Then, using (12) and (13) in (3) yields after a little algebraic computation

$$\mathcal{L}[y(x_n); h] = \frac{h^{p+2}}{p!} \int_0^k \sum_{j=0}^k \left[ (p+1)^{-1} \alpha_j (j-t)_+^{p+1} - p\beta_j (j-t)_+^{p-1} \right] y^{(p+2)}(x_n + th) \, dt \tag{14}$$

where

$$s_+ = \begin{cases} s, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0. \end{cases} \tag{15}$$

Suppose the method is consistent, that is $p \geq 1$, also define the function $F(t)$:

$$F(t) = \sum_{j=0}^{k} \left[ (p+1)^{-1} \alpha_j (j-t)_+^{p+1} - p\beta_j (j-t)_+^{p-1} \right] \qquad (16)$$

such that $F(t)$ is a polynomial in each partition of the interval $[0, k]$ and independent of the theoretical solution $y(x)$. Then, (14) can be written in terms of $F(t)$ as follows:

$$\mathcal{L}[y(x_n); h] = \frac{h^{p+2}}{p!} \int_0^k F(t) y^{(p+2)}(x_n + th) \, dt. \qquad (17)$$

Furthermore, assume that $F(t)$ does not change sign in the interval $[0, k]$, then by Theorem 3.2, (17) becomes

$$\mathcal{L}[y(x_n); h] = h^{p+2} \frac{1}{p!} \left( \int_0^k F(t) \, dt \right) y^{(p+2)}(x_n + \xi h), \qquad 0 < \xi < k. \qquad (18)$$

Assume $y(x) = x^{p+2}$, such that all derivatives of $x^{p+2}$ of order greater than $p+2$ vanish identically. Then, by (18) and the expansion of the operator $\mathcal{L}$ we have;

$$C_{p+2} = \frac{1}{p!} \int_0^k F(t) \, dt. \qquad (19)$$

Thus, (19) becomes

$$\mathcal{L}[y(x_n); h] = h^{p+2} C_{p+2} y^{(p+2)}(x_n + \xi h), \qquad 0 < \xi < k \qquad (20)$$

which is the desired form. Hence,

$$|\mathcal{L}[y(x_n); h]| = |h^{p+2} C_{p+2} y^{(p+2)}(x_n + \xi h)|$$
$$\leq |h^{p+2}||C_{p+2}| \max_{x \in [a,b]} |y^{(p+2)}(x)|$$
$$\therefore \qquad |\mathcal{L}[y(x_n); h]| \leq h^{p+2} QY. \qquad (21)$$

## 4. Application of the Bound

Suppose the continuous implicit hybrid one step method proposed in Anake and Adoghe (2013), by evaluating (1) at the point $x_{n+1}$, given as follows:

$$y_{n+1} = 2y_{n+\frac{4}{5}} - y_{n+\frac{3}{5}} + \frac{h^2}{6000} \left[ 18 f_{n+1} + 209 f_{n+\frac{4}{5}} + 4 f_{n+\frac{3}{5}} + 14 f_{n+\frac{2}{5}} - 6 f_{n+\frac{1}{5}} + f_n \right], \quad (22)$$

is implemented for the solution of the initial value problem:

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = 0, q'(0) = 0$$

with exact solution:

$$q(t) = \frac{3}{4} - \frac{3}{4}e^{-10t}(\cos 10t + \sin 10t),$$

where $L = 1, C = 0.005, E(t) = 150$ and $R = 20$. According to Anake and Adoghe(2013), application of the convergent method, of order $p = 6$, at the first grid point $x_{n+1}$ yields the local error constant $C_{p+2} = -\frac{221}{23625000000}$.

To apply (21), we will need to find $Q$, $Y$ and choose a suitable step size. Thus, we checked that (19) is satisfied and the result of our computation agrees with the error constant reported by Anake and Adoghe (2013) as follows:

$$Q = |C_{p+2}| = \left|\frac{1}{p!}\int_0^1 F(t)\,dt\right| = \frac{221}{23625000000},$$

where

$$F(t) = \frac{1}{7}\left(\left(\frac{3}{5} - t\right)_+^7 - 2\left(\frac{4}{5} - t\right)_+^7 + (1 - t)_+^7\right) - \frac{1}{1000}\left(18(1 - t)_+^5 + 209\left(\frac{4}{5} - t\right)_+^5\right.$$
$$\left. + 4\left(\frac{3}{5} - t\right)_+^5 + 14\left(\frac{2}{5} - t\right)_+^5 - 6\left(\frac{1}{5} - t\right)_+^5 + (0 - t)_+^5\right).$$

By (17) given that $p = 6$, $Y$ is obtained as follows:

$$y^{(p+2)} = q^{(8)}(t) = 1200000000e^{-10t}(\cos 10t + \sin 10t). \tag{23}$$

Hence,

$$Y = \max_{x \in [0,1]}|y^{(p+2)}(x)| = 75351. \tag{24}$$

The bound on the application of the method (22) at the point $x_{n+1}$ obtained for different values of the step size $h$ is shown in the table below:

Table 1: Different values of $h$ and the corresponding bound

| $h$ | $h^{p+2}QY$ |
|---|---|
| 0.003125 | $6.4108 \times 10^{-24}$ |
| 0.01 | $7.0487 \times 10^{-20}$ |
| 0.03 | $4.6247 \times 10^{-16}$ |
| 0.05 | $2.7534 \times 10^{-14}$ |
| 0.10 | $7.0487 \times 10^{-12}$ |

## 5. Conclusion

A method for the bound on the local truncation error has been derived in this paper. The method has been applied to the continuous implicit hybrid one step method proposed by Anake and Adoghe (2013) and bounds were obtained, in the application of the method at the point $x_{n+1}$, on the local truncation error for different values of $h$. It is observed

that the magnitude of the bound computed reduces as $h \rightarrow 0$. This indicates that the size of the global error reduces with decrease in the step size and consequently, the accuracy of the approximation theoretically improves.

## Acknowledgements

## References

[1] Anake, T.A., *Continuous Implicit Hybrid One Step Methods for the Solutions of Initial Value Problems of General Second Order Ordinary Differential Equations,* Ph.D Thesis, Covenant University, Canaanland, Ota, Ogun State, Nigeria. 2011.

[2] Anake, T.A. and L. Adoghe. A four point integration method for the solutions of IVP in ODE. Australian Journal of Basic and Applied Sciences, 7(10), 467-473. 2013.

[3] Calvo, M., D.J. Higham, J.I. Monijano and L. Randez. Global error estimation with adaptive explicit Runge-Kutta methods. *IMA J. Numer Anal.*, 16, 47-63. 1996.

[4] Chapra, S.C. and R.P. Canale, *Numerical methods for Engineers(6th ed).* McGraw-Hill: New York. 2010.

[5] Dormund, J.R., J.P. Gilmore and P.J. Prince. Globally embedded Runge-Kutta Schemes. *Ann. Numer, Math.*, 1, 97-106. 1994.

[6] Higham, D.J. The tolerance proportionality of adaptive ODE solvers. *J. Comput. Appl. Math.*, 45, 227-236. 1993.

[7] Huang, GB. Topics on Mean Value Theorem. *Available at www.nsysu.edu.tw.* 2001.

[8] Kulikov, G. Y. and S.K. Shindin. On the efficient computation of asymptotically sharp estimates for local and global errors in multistep methods with constant coefficients. *Zh. Vychisl. Mat. Fiz.*, 44, 840-861. 2004.

[9] Lambert, J.D., *Numerical Methods for ordinary differential systems: The initial value problem.* John Wiley and Sons: New York. 1991.

[10] Onumanyi, P., U.W. Sirisena and S. N. Jator. Continuous finite difference approximations for solving differential equations.*Int. J. Comput. Math.*, 72, 15-27. 1999.

[11] Orel, B. Accumulation of global error in Lie group methods for linear ordinary differential equations. *Electron. Trans. Numer Anal.*, 37, 252-262. 2010.

[12] Stuart, A.M. Probabilistic and deterministic convergence proofs for software for initial value problems. *Numer. Algorithms*, 14, 227-260. 1997.