EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 1, 2018, 90-109
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Some common fixed points of six mappings on G_b-metric spaces using $(E.A)$ property

Z. Mustafa[1,*], M.M.M. Jaradat[2], H. Aydi[3,4], A. Alrhayyel[5]

[1] *Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar*
Department of Mathematics, The Hashemite University, Zarqa- Jordan
[2] *Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar*
[3] *Imam Abdulrahman Bin Faisal University, Department of Mathematics,*
College of Education of Jubail, P.O: 12020, Industrial Jubail 31961, Saudi Arabia
[4] *Department of Medical Research, China Medical University Hospital,*
China Medical University, Taichung, Taiwan
[5] *Department of Mathematics. Faculty of Science, Yarmouk University, Irbid, Jordan*

Abstract. The aim of this manuscript is to present a unique common fixed point theorem for six mappings satisfying (ϕ, ψ)-contractions using $(E.A)$ property in the framework of G_b- metric spaces. An illustrative example is also given to justify the established result.

2010 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Complete G_b-metric space, Cauchy sequence, (ϕ, ψ)-contraction, $(E.A)$ property, common fixed point, weakly compatible.

1. Introduction

In recent years, many authors studied common fixed points of mappings having different contractive conditions. This area has variety of important applications in applied mathematics and sciences.

In 1976, Jungck [17] proved a common fixed point theorem for commuting maps under the assumption that one of maps must be continuous.

In 1982, the concept of weak commutativity for a pair of self maps was introduced by Sessa [47]. He also proved that weakly commuting pairs of maps in a metric space are commuting, but the converse need not be true. Later, Jungck [18] introduced the notion of compatible mappings in order to generalize the concepts of weak commutativity and showed that weak commuting maps are compatible, but the reverse implication may not hold.

*Corresponding author.

Email addresses: `zead@qu.edu.qa, zmagablh@hu.edu.jo` (Z. Mustafa),
`mmjst4@qu.edu.qa` (M.M.M. Jaradat), `hmaydi@iau.edu.sa, hassen.aydi@isima.rnu.tn` (H. Aydi),
`Al-Rhayyel@yu.edu.jo` (A. Alrhayyel)

In 1996, Jungck [20] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points.

Therefore, we have one way implication namely, Commuting maps $\Rightarrow$ Weakly Commuting maps $\Rightarrow$ Compatible maps $\Rightarrow$ Weakly Compatible maps. Recently, various authors have introduced a coincidence points results for various classes of mappings on metric spaces. For more details on coincidence point theory and related results, see [19, 21, 43].

However, the study of common fixed points of non-compatible mappings has recently been initiated by Pant [44].

In 2002, Amari and El Moutawakil [1] defined a new property called $(E.A)$ property which generalizes the concept of non-compatible mappings and they proved some common fixed point theorem.

Yan et al. [48] gave the idea of (ϕ, ψ)-contractions and proved a fixed point theorem of a contraction mapping in a complete metric space endowed with a partial order by using altering distance functions [22]. Different authors used (ϕ, ψ)-contractions to obtain common fixed point results in different spaces. Some of the works on (ϕ, ψ)-contractions are given in [4, 5, 8, 10, 26, 27, 42, 23, 41].

Mustafa and Sims [28] introduced a new generalizations of a metric space by assigning to every $(x, y, z) \in X \times X \times X$ a real number and is named as a G-metric space. In 2008, Mustafa et al. [29] obtained some fixed point results in G-metric spaces for mappings satisfying different contractive conditions. After that several fixed point results were obtained. Among these works, we mention ([6],[7],[11],[14],[15], [16],[24]-[40]). In 2014, Aghajani et al. [2] introduced a new generalization of a metric space. They combined the definition of a G-metric and a b-metric and generated a new definition called a G_b-metric space. They also pointed out that the class of G_b-metric spaces is effectively larger than that of G-metric spaces. Note that a G-metric space becomes a particular case of a G_b-metric space when $s = 1$. Further, they showed that every G_b-metric space is equivalent to a b-metric space topologically.

In the current work, we will obtain a unique common fixed point result in G_b- metric spaces involving (ϕ, ψ)-contractions and using the $(E.A)$ property. Also, an example to illustrate the main result is given.

2. Preliminaries

First, we present some definitions from the literature.

Definition 1. *([13]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric provided that, for all $a, b, c \in X$, the following conditions are satisfied:*

$(B1)$ $d(a, b) = 0$ *if and only if* $a = b$;

$(B2)$ $d(a, b) = d(b, a)$;

$(B3)$ $d(a, c) \leq s[d(a, b) + d(b, c)]$.

The pair (X, d) is called a b-metric space with parameter s.

The following definition was given by Mustafa and Sims [28]

Definition 2. *([28]) Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ satisfies:*
(G1) $G(a, b, c) = 0$ *if* $a = b = c$;
(G2) $G(a, a, b) > 0$ *for all* $a, b \in X$ *with* $a \neq b$;
(G3) $G(a, b, b) \leq G(a, b, c)$ *for all* $a, b, c \in X$ *with* $a \neq c$;
(G4) $G(a, b, c) = G(b, c, a) = G(c, a, b) = \cdots$ *(symmetry in* a, b, c*);*
(G5) $G(a, b, c) \leq G(a, d, d) + G(d, b, c)$ *for all* $a, b, c, d \in X$.

Then function G is called a G-metric on X, and the pair (X, G) is called a G-metric space. As a combination of the two above definitions, Aghajani et al. [2] (see also [3]) introduced the following.

Definition 3. *([2]) Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G_b : X \times X \times X \to [0, \infty)$ satisfies:*
(G_b1) $G_b(x, y, z) = 0$ *if* $x = y = z$;
(G_b2) $G_b(x, x, y) > 0$ *for all* $x, y \in X$ *with* $x \neq y$;
(G_b3) $G_b(x, y, y) \leq G_b(x, y, z)$ *for all* $x, y, z \in X$ *with* $x \neq z$;
(G_b4) $G_b(x, y, z) = G_b(p\{x, y, z\})$ *where* p *is a permutation of* x, y, z *(symmetry);*
(G_b5) $G_b(x, y, z) \leq s(G_b(x, a, a) + G_b(a, y, z))$ *for all* $x, y, z, a \in X$.

Then G_b is called a generalized b-metric (named as a G_b-metric) on X, and the pair (X, G_b) is called a G_b-metric space.

Note that every G-metric space is a G_b-metric space, but the converse need not to be true as its clear from the following example.

Example 1. *([46]) Let X={1, 2, 3, 4}. Define $G_b : X \times X \times X \to [0, \infty)$ by*
$G_b(1, 1, 1) = G_b(2, 2, 2) = G_b(3, 3, 3) = G_b(4, 4, 4) = 0,$
$G_b(1, 1, 2) = G_b(1, 2, 2) = G_b(1, 1, 3) = G_b(1, 3, 3) = G_b(1, 1, 4) = G_b(1, 4, 4) = 1,$
$G_b(2, 2, 3) = G_b(2, 3, 3) = G_b(2, 4, 4) = G_b(2, 2, 4) = 2,$
$G_b(3, 4, 4) = G_b(3, 3, 4) = 3,$
$G_b(1, 2, 3) = 4,\ G_b(1, 3, 4) = 5,\ G_b(1, 2, 4) = 6,\ G_b(2, 3, 4) = 7.$
Evidently, the above is a G_b-metric on X with $s = \frac{7}{5}$, but not a G-metric. In fact, the rectangle inequality is violated, for instant $7 = G_b(2, 3, 4) \nleq G_b(2, 1, 1) + G_b(1, 3, 4) = 1 + 5$.

The following example can be founded in [45].

Example 2. *Let (X, G) be a G-metric space. Take $G_b(x, y, z) = G^p(x, y, z)$, where $p > 1$ is a real number. Note that G_b is a G_b-metric with $s = 2^{p-1}$. In general (X, G_b) is not necessary a G-metric space. For instant, let $X = \mathbb{R}$ and the G-metric be defined by $G(x, y, z) = \frac{1}{3}(|x-y|+|y-z|+|x-z|)$ for all $x, y, z \in \mathbb{R}$. Then $G_b(x, y, z) = G^2(x, y, z) = \frac{1}{9}(|x - y| + |y - z| + |x - z|)^2$ is a G_b-metric on $\mathbb{R}$ with $s = 2^{2-1} = 2$, but it is not a G-metric on $\mathbb{R}$.*

Example 3. *([2]) Let $X = \mathbb{R}$. Take the G_b-metric defined by*

$$G_b(x, y, z) = \max\left\{|x - y|^2, |y - z|^2, |z - x|^2\right\}, \quad \forall\ x, y, z \in X.$$

Then (X, G_b) is a complete G_b-metric space with $s = 2$, but not a G-metric.

Proposition 1. *([2]) Let X be a G_b-metric space. Then for each $x, y, z, a \in X$, it follows that*

(1) If $G_b(x, y, z) = 0$, then $x = y = z$,
(2) $G_b(x, y, z) \leq s(G_b(x, x, y) + G_b(x, x, z))$,
(3) $G_b(x, y, y) \leq 2sG_b(y, x, x)$,
(4) $G_b(x, y, z) \leq s(G_b(x, a, z) + G_b(a, y, z))$.

Definition 4. *([2]) Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:*
(1) G_b-Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0, G_b(x_n, x_m, x_l) < \epsilon$;
(2) G_b-convergent to a point $x \in X$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0, G_b(x_n, x_m, x) < \epsilon$.

Proposition 2. *([2, 9]) Let X be a G_b-metric space. The following are equivalent:*
(1) $\{x_n\}$ is G_b-convergent to x;
(2) $G_b(x_n, x_n, x) \to 0$ as $n \to \infty$;
(3) $G_b(x_n, x, x) \to 0$ as $n \to \infty$.

Definition 5. *([2]) A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.*

The following definition was given by Jungck [19].

Definition 6. *([19]) Two maps f and g are said to be weakly compatible if they commute at their coincidence points, that is if $f(x) = g(x)$ for some $x \in X$, then $f(g(x)) = g(f(x))$.*

The following definition was introduced by Amari and El Moutawakil [1] in 2002.

Definition 7. *([1]) Two self mappings S and T of a metric space (X, d) are said to satisfy an (E.A) property if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = r \quad for \quad some \quad r \in X.$$

This concept was extended to G-metric spaces in [24]. The following lemma is useful in the proof of our main result.

Lemma 1. *([45]) Let (X, G_b) be a G_b-metric space with $s > 1$. Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent sequences to x, y and z, respectively. Then we have*

(i)

$$\frac{1}{s^3}G_b(x, y, z) \leq \liminf_{n\to\infty} G_b(x_n, y_n, z_n) \leq \limsup_{n\to\infty} G_b(x_n, y_n, z_n) \leq s^3 G_b(x, y, z).$$

(ii) If $\{z_n\} = c$ is constant, then

$$\frac{1}{s^2}G_b(x, y, c) \leq \liminf_{n\to\infty} G_b(x_n, y_n, c) \leq \limsup_{n\to\infty} G_b(x_n, y_n c) \leq s^2 G_b(x, y, c).$$

(iii) If $\{z_n\} = c$ and $\{y_n\} = b$ are constant, then

$$\frac{1}{s}G_b(x,b,c) \le \liminf_{n\to\infty} G_b(x_n,b,c) \le \limsup_{n\to\infty} G_b(x_n,b,c) \le sG_b(x,b,c).$$

In particular, if $x = y = z$, then we have $\lim_{n\to\infty} G_b(x_n, y_n, z_n) = 0$.

3. Main results

We start this section with the following definition and lemma which will play a major role in our main result.

Lemma 2. *Let (X, G_b) be a G_b-metric space with $s > 1$. Suppose that $\{x_n\}$ is a G_b-convergent sequence to x. Then for $y \in X$ we have*

$$\frac{1}{s}G_b(y,x,x) \le \liminf_{n\to\infty} G_b(y,x_n,x_n) \le \limsup_{n\to\infty} G_b(y,x_n,x_n) \le sG_b(y,x,x).$$

Proof. Using the rectangle inequality for the G_b-metric, we obtain that

$$G_b(y,x,x) \le s[G_b(y,x_n,x_n) + G_b(x_n,x,x)] \tag{1}$$

and

$$G_b(y,x_n,x_n) \le s[G_b(y,x,x) + G_b(x,x_n,x_n)]. \tag{2}$$

Taking the limit inferior as $n \to \infty$ in (1) and the limit superior as $n \to \infty$ in (2), the proof is completed.

Definition 8. *A mapping $\psi : [0,\infty) \to [0,\infty)$ is called a super-altering distance function if the following properties are satisfied:*

1. ψ is continuous and increasing.

2. $\psi(t) = 0$ if and only if $t = 0$.

We denoted by Ψ to be the set of all super-altering distance functions. Note that the class of altering distance functions was defined in [22], where ψ is considered non-decreasing (not necessarily increasing). Any super-altering distance function is of course a function in the sense of [22].

In the following example, the given mapping is just an altering distance function, but not in Ψ.

Example 4. *Let $\psi : [0,\infty) \to [0,\infty)$ be such that*

$$\begin{cases} \psi(t) = t & \text{if } t \in [0,1] \\ \psi(t) = 1 & \text{if } t \ge 1. \end{cases}$$

Theorem 1. *Let (X, G_b) be a complete G_b-metric space and let $f, g, h, R, S, T : X \to X$ be self mappings such that* *(i)* (f,S) *and* (g,R) *satisfy the* $(E.A)$ *property;*

(ii) $f(X)\subseteq T(X)$, $g(X)\subseteq S(X)$ *and* $h(X)\subseteq R(X)$*;*

(iii) $R(X)$ *is a closed subspace of* X*;*

(iv) (f,S), (g,R) *and* (h,T) *are weakly compatible pairs of mappings;*

(v)
$$\psi\big(s^2G_b(fx,gy,hz)\big)\leq\psi\big(M(x,y,z)\big)-\phi\big(M(x,y,z)\big),\forall x,y,z\in X \tag{3}$$
where $\psi,\phi\in\Psi$ *and*

$$\begin{aligned}M(x,y,z)=\max\big\{&G_b(fx,Sx,Tz),G_b(gy,Ry,Ry),\\ &G_b(fx,fx,hz),\frac{G_b(Tz,Tz,hz)+G_b(fx,Sx,Sx)}{2s}\big\}.\end{aligned}$$

Then f,g,h,R,S *and* T *have a unique common fixed point in* X*.*

Proof. Since the pair (f,S) satisfies the $(E.A)$ property, there exists a sequence $\{x_n\}$ such that
$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}Sx_n=q_1,\ \text{for some } q_1\in X.$$
As $f(X)\subseteq T(X)$, there exists a sequence $\{z_n\}\in X$ such that
$$fx_n=Tz_n \text{ and } \lim_{n\to\infty}fx_n=\lim_{n\to\infty}Tz_n=\lim_{n\to\infty}Sx_n=q_1. \tag{4}$$
Again the pair (g,R) satisfies the $(E.A)$ property, so there exists a sequence $\{y_n\}$ such that
$$\lim_{n\to\infty}gy_n=\lim_{n\to\infty}Ry_n=q_2,\ \text{for some } q_2\in X. \tag{5}$$
But $g(X)\subseteq S(X)$, so there exists a sequence $\{\alpha_n\}\in X$ such that
$$gy_n=S\alpha_n,\ \text{and } \lim_{n\to\infty}gy_n=\lim_{n\to\infty}S\alpha_n=\lim_{n\to\infty}Ry_n=q_2. \tag{6}$$

Now, we shall show that $\lim_{n\to\infty}hz_n=q_1$. From (3), (G_b3) and the fact that ψ is an increasing mapping, we have
$$\begin{aligned}\psi\big(sG_b(fx_n,fx_n,hz_n)\big)&\leq\psi\big(s^2G_b(fx_n,gy_n,hz_n)\big)\\&\leq\psi\big(M(x_n,y_n,z_n)\big)-\phi\big(M(x_n,y_n,z_n)\big)\end{aligned} \tag{7}$$
where,
$$\begin{aligned}M(x_n,y_n,z_n)&=\max\big\{G_b(fx_n,Sx_n,Tz_n),G_b(gy_n,Ry_n,Ry_n),\\&\quad G_b(fx_n,fx_n,hz_n),\frac{G_b(Tz_n,Tz_n,hz_n)+G_b(fx_n,Sx_n,Sx_n)}{2s}\big\},\end{aligned}$$

$$= \max\big\{G_b(fx_n,Sx_n,fx_n),G_b(gy_n,Ry_n,Ry_n), \\ G_b(fx_n,fx_n,hz_n),\frac{G_b(fx_n,fx_n,hz_n)+G_b(fx_n,Sx_n,Sx_n)}{2s}\big\}.$$

Taking $\limsup_{n\to\infty}$ and using (4) together with (6), we obtain

$$\limsup_{n\to\infty} M(x_n,y_n,z_n)=\limsup_{n\to\infty} G_b(fx_n,fx_n,hz_n). \tag{8}$$

Taking again $\limsup_{n\to\infty}$ in (7) and substituting (8), we get

$$\begin{aligned}\psi\big(\limsup_{n\to\infty} sG_b(fx_n,fx_n,hz_n)\big) &\leq \psi\big(\limsup_{n\to\infty} s^2G_b(fx_n,gy_n,hz_n)\big)\\ &\leq \psi\big(\limsup_{n\to\infty} G_b(fx_n,fx_n,hz_n)\big)\\ &- \phi\big(\liminf_{n\to\infty} M(x_n,y_n,z_n)\big).\\ &\leq \psi\big(\limsup_{n\to\infty} G_b(fx_n,fx_n,hz_n)\big).\end{aligned} \tag{9}$$

Since $s>1$ and being ψ is an increasing mapping, we deduce from (9) that $\limsup\limits_{n\to\infty} G_b(fx_n,fx_n,hz_n)=0$, which implies that

$$\lim_{n\to\infty} G_b(fx_n,fx_n,hz_n)=0, \tag{10}$$

and so by (8), we conclude that

$$\lim_{n\to\infty} M(x_n,y_n,z_n)=0. \tag{11}$$

Now, by (G_b4), (10) and (4), we have

$$G_b(hz_n,q_1,q_1)\leq s\big[G_b(hz_n,fx_n,fx_n)+G_b(fx_n,q_1,q_1)\big]\to 0 \text{ as } n\to\infty. \tag{12}$$

Thus, $\lim\limits_{n\to\infty} G_b(hz_n,q_1,q_1)=0$ which gives that $\lim hz_n=q_1$ as $n\to\infty$. Now, we shall prove that $q_1=q_2$. By applying (3) and using (G_b3), we find that

$$\begin{aligned}\psi\big(sG_b(fx_n,gy_n,gy_n)\big) &\leq \psi\big(s^2G_b(fx_n,gy_n,hz_n)\big)\\ &\leq \psi\big(M(x_n,y_n,z_n)\big)-\phi\big(M(x_n,y_n,z_n)\big).\end{aligned} \tag{13}$$

Taking the limit as $n\to\infty$ in (13) and recalling (11), we obtain

$$\lim_{n\to\infty} G_b(fx_n,gy_n,gy_n)=0. \tag{14}$$

Thus, by using (G_b4), (4) and (14),

$$G_b(q_1,S\alpha_n,S\alpha_n) \quad = \quad G_b(q_1,gy_n,gy_n)$$

$$\leq \quad s\big[G_b(q_1, fx_n, fx_n) + G_b(fx_n, gy_n, gy_n)\big] \to 0 \text{ as } n \to \infty.$$

This implies that $\lim_{n\to\infty} S\alpha_n = q_1$. On the other hand, from (6) we have $\lim\limits_{n\to\infty} S\alpha_n = q_2$, hence by uniqueness of limits, we obtain that $q_1 = q_2$. Therefore

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hz_n = \lim_{n\to\infty} Tz_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} S\alpha_n = \lim_{n\to\infty} Ry_n = q \tag{15}$$

for some $q \in X$. Since $R(X)$ is a closed subspace of X, there exists $u \in X$ such that $Ru = q$. Now we shall prove that $gu = q$. Observe that

$$\begin{aligned} M(x_n, u, z_n) &= \max\big\{G_b(fx_n, Sx_n, Tz_n), G_b(gu, Ru, Ru), \\ &\quad G_b(fx_n, fx_n, hz_n), \frac{G_b(Tz_n, Tz_n, hz_n) + G_b(fx_n, Sx_n, Sx_n)}{2s}\big\}, \\ &= \max\big\{G_b(fx_n, Sx_n, Tz_n), G_b(gu, Ru, Ru), \\ &\quad G_b(fx_n, fx_n, hz_n), \frac{G_b(fx_n, fx_n, hz_n) + G_b(fx_n, Sx_n, Sx_n)}{2s}\big\}, \\ &= \max\big\{G_b(fx_n, Sx_n, Tz_n), G_b(gu, q, q), \\ &\quad G_b(fx_n, fx_n, hz_n), \frac{G_b(fx_n, fx_n, hz_n) + G_b(fx_n, Sx_n, Sx_n)}{2s}\big\}. \end{aligned} \tag{16}$$

By taking limit superior as $n \to \infty$ and taking into account (4), (6) and (15), then (16) becomes

$$\limsup_{n\to\infty} M(x_n, u, z_n) = G_b(gu, q, q). \tag{17}$$

By the help of Lemma 2, we obtain that

$$\begin{aligned} \frac{1}{s} G_b(q, gu, q) &\leq \liminf_{n\to\infty} G_b(gu, fx_n, fx_n) \\ &\leq \limsup_{n\to\infty} G_b(gu, fx_n, fx_n) \\ &\leq sG_b(gu, q, q). \end{aligned} \tag{18}$$

Also from (G_b3), we have

$$G_b(gu, fx_n, fx_n) \leq G_b(fx_n, gu, hz_n). \tag{19}$$

Thus, from (3), together with (17), (18), (19) and properties of ψ, we get that

$$\begin{aligned} \psi\big(sG_b(q, gu, q)\big) &\leq \psi\big(\limsup_{n\to\infty} s^2 G_b(gu, fx_n, fx_n)\big) \\ &\leq \psi\big(\limsup_{n\to\infty} s^2 G_b(fx_n, gu, hz_n)\big) \\ &= \limsup_{n\to\infty} \psi\big(s^2 G_b(fx_n, gu, hz_n)\big) \\ &\leq \limsup_{n\to\infty} \psi\big(M(x_n, u, z_n)\big) - \liminf_{n\to\infty} \phi\big(M(x_n, u, z_n)\big), \end{aligned}$$

$$\begin{aligned}
&= \psi\big(\limsup_{n\to\infty} M(x_n,u,z_n)\big) - \phi\big(\liminf_{n\to\infty} M(x_n,u,z_n)\big),\\
&\le \psi\big(G_b(q,gu,q)\big) - \phi\big(\liminf_{n\to\infty} M(x_n,u,z_n)\big),\\
&\le \psi(G_b(q,gu,q)). \qquad (20)
\end{aligned}$$

Since $s>1$ and ψ is increasing, the above inequality gives that $G_b(q,gu,q)=0$, which implies that $gu=q$. But $g(X)\subseteq S(X)$, so there exists a point $p\in X$ such that $gu=Sp=q$. We shall show that $fp=q$. Now

$$\begin{aligned}
M(p,u,z_n) &= \max\big\{G_b(fp,Sp,Tz_n), G_b(gu,Ru,Ru),\\
&\quad G_b(fp,fp,hz_n), \frac{G_b(Tz_n,Tz_n,hz_n)+G_b(fp,Sp,Sp)}{2s}\big\},\\
&= \max\big\{G_b(fp,q,Tz_n), G_b(q,q,q),\\
&\quad G_b(fp,fp,hz_n), \frac{G_b(Tz_n,Tz_n,hz_n)+G_b(fp,q,q)}{2s}\big\}\\
&= \max\big\{G_b(fp,q,Tz_n), G_b(fp,fp,hz_n), \frac{G_b(Tz_n,Tz_n,hz_n)+G_b(fp,q,q)}{2s}\big\} \qquad (21)\\
&\le \max\big\{G_b(fp,q,Tz_n), G_b(fp,Tz_n,hz_n), \frac{G_b(fp,Tz_n,hz_n)+G_b(fp,q,Tz_n)}{2s}\big\},\\
&\le \max\big\{G_b(fp,q,Tz_n), G_b(fp,Tz_n,hz_n)\big\}. \qquad (22)
\end{aligned}$$

Now, taking the limit superior in (22) as $n\to\infty$ and using Lemma 1, parts (2) and (3), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} M(p,u,z_n) &= \limsup_{n\to\infty} \max\big\{G_b(fp,q,Tz_n), G_b(fp,Tz_n,hz_n)\big\}\\
&= \max\big\{\limsup_{n\to\infty} G_b(fp,q,Tz_n), \limsup_{n\to\infty} G_b(fp,Tz_n,hz_n)\big\}\\
&\le \max\{sG_b(fp,q,q), s^2G_b(fp,q,q)\}\\
&= s^2G_b(fp,q,q). \qquad (23)
\end{aligned}$$

Now, taking the limit infimum in (21) as $n\to\infty$ and using Lemma 1, parts (2) and (3), we get

$$\begin{aligned}
\liminf_{n\to\infty} M(p,u,z_n) &= \liminf_{n\to\infty} \max\big\{G_b(fp,q,Tz_n), G_b(fp,fp,hz_n),\\
&\quad \frac{G_b(Tz_n,Tz_n,hz_n)+G_b(fp,q,q)}{2s}\big\}\\
&= \max\big\{\liminf_{n\to\infty} G_b(fp,q,Tz_n), \liminf_{n\to\infty} G_b(fp,fp,hz_n),\\
&\quad \frac{\liminf_{n\to\infty} G_b(Tz_n,Tz_n,hz_n)+\liminf_{n\to\infty} G_b(fp,q,q)}{2s}\big\}
\end{aligned}$$

$$\begin{aligned} &\geq \max\{\frac{1}{s}G_b(fp,q,q), \frac{1}{s}G_b(fp,fp,q), \frac{G_b(fp,q,q)}{2s}\} \\ &= \max\{\frac{1}{s}G_b(fp,q,q), \frac{1}{s}G_b(fp,fp,q)\}. \end{aligned} \tag{24}$$

Thus, from (3), (G_b3) and the fact that ψ and ϕ are increasing, we have

$$\begin{aligned} \psi\big(s^2G_b(fp,q,q)\big) &\leq \psi\big(s^2G_b(fp,q,hz_n)\big) \\ &= \psi\big(s^2G_b(fp,gu,hz_n)\big) \\ &\leq \psi\big(M(p,u,z_n)\big) - \phi\big(M(p,u,z_n)\big). \end{aligned} \tag{25}$$

Therefore, by taking the limit superior in (25) as $n \to \infty$ and using (23) and (24),

$$\begin{aligned} \psi\big(s^2G_b(fp,q,q)\big) &\leq \psi\big(\limsup_{n\to\infty} M(p,u,z_n)\big) - \phi\big(\liminf_{n\to\infty} M(p,u,z_n)\big), \\ &\leq \psi\big(s^2G_b(fp,q,q)\big) - \phi\big(\max\{\frac{1}{s}G_b(fp,q,q), \frac{1}{s}G_b(fp,fp,q)\}\big),. \end{aligned} \tag{26}$$

So,

$$\phi\big(\max\{\frac{1}{s}G_b(fp,q,q), \frac{1}{s}G_b(fp,fp,q)\}\big) = 0,$$

or equivalently,

$$\max\{\frac{1}{s}G_b(fp,q,q), \frac{1}{s}G_b(fp,fp,q)\} = 0,$$

which implies that $G_b(fp,q,q) = G_b(fp,fp,q) = 0$. Hence $fp = Sp = q$. We conclude that p is a coincidence point of f and S. Also

$$fp = Sp = gu = Ru = q. \tag{27}$$

Again, since $h(X) \subseteq R(X)$, there exists $w \in X$ such that $hw = Ru = q$. Now, we shall show that $Tw = hw$. From the definition of $M(x,y,z)$ and by the help of (27), we get

$$\begin{aligned} M(p,u,w) &= \max\Big\{G_b(fp,Sp,Tw), G_b(gu,Ru,Ru), G_b(fp,fp,hw), \\ &\quad \frac{G_b(Tw,Tw,hw) + G_b(fp,Sp,Sp)}{2s}\Big\}, \\ &= \max\Big\{G_b(q,q,Tw), G_b(q,q,q), G_b(q,q,q), \\ &\quad \frac{G_b(Tw,Tw,q) + G_b(q,q,q)}{2s}\Big\}, \\ &= \max\big\{G_b(q,q,Tw), \frac{G_b(Tw,Tw,q)}{2s}\big\}. \end{aligned}$$

But, by part 3 of Proposition 1, we have $\frac{G_b(Tw,Tw,q)}{2s} \leq G_b(q,q,Tw)$ and so the above inequality becomes

$$M(p,u,w) = G_b(q,q,Tw). \tag{28}$$

Thus, applying (3) for $x = q, y = q$ and $z = Tw$ and using (G_b3), (28) and properties of ψ, we obtain

$$\begin{aligned}\psi\big(G_b(q,q,Tw)\big) &\leq \psi\big(s^2G_b(q,q,Tw)\big)\\ &= \psi\big(s^2G_b(fp,gu,Tw)\big)\\ &\leq \psi\big(M(p,u,w)\big) - \phi\big(M(p,u,w)\big),\\ &= \psi\big(G_b(q,q,Tw)\big) - \phi\big(G_b(q,q,Tw)\big). \end{aligned} \tag{29}$$

So, $\phi\big(G_b(q,q,Tw)\big) = 0$, which implies that $G_b(q,q,Tw) = 0$. Hence $Tw = q = hw$ and so w is a coincidence point of h and T. Therefore

$$fp = Sp = gu = Ru = Tw = hw = q. \tag{30}$$

Now, we shall show that q is a common fixed point of f, g, h, R, S and T. Since the pairs (f,S), (g,R) and (h,T) are weakly compatible, the functions of each pair commute at their coincidence point, that is

$$\left.\begin{array}{l} f(q) = f(Sp) = S(fp) = S(q),\\ R(q) = R(gu) = g(Ru) = g(q),\\ T(q) = T(hw) = h(Tw) = h(q). \end{array}\right\} \tag{31}$$

Using (30) and (31), we obtain

$$\begin{aligned} M(q,u,w) &= \max\Big\{G_b(fq,Sq,Tw), G_b(gu,Ru,Ru), G_b(fq,fq,hw),\\ &\qquad \frac{G_b(Tw,Tw,hw) + G_b(fq,Sq,Sq)}{2s}\Big\},\\ &= \max\big\{G_b(fq,Sq,q), 0, G_b(fq,fq,q), 0\big\},\\ &= G_b(fq,fq,q). \end{aligned}$$

Also, from (3) and (G_b3), we get

$$\begin{aligned}\psi\big(s^2G_b(fq,fq,q)\big) &\leq \psi\big(s^2G_b(fq,gu,q)\big)\\ &= \psi\big(s^2G_b(fq,gu,hw)\big)\\ &\leq \psi\big(M(q,u,w)\big) - \phi\big(M(q,u,w)\big),\\ &= \psi\big(G_b(fq,fq,q))\big) - \phi\big(G_b(fq,fq,q)\big),\\ &\leq \psi\big(G_b(fq,fq,q)\big). \end{aligned} \tag{32}$$

Since $s^2 > s > 1$ and ψ is increasing, the inequality above yields that $G_b(fq,fq,q) = 0$ and so $fq = q = Sq$. We shall prove that $gq = Rq = q$. As in the above, using (30) and (31), we find that

$$\begin{aligned} M(p,q,w) &= \max\Big\{G_b(fp,Sp,Tw), G_b(gq,Rq,Rq), G_b(fp,fp,hw),\\ &\qquad \frac{G_b(Tw,Tw,hw) + G_b(fp,Sp,Sp)}{2s}\Big\}, \end{aligned}$$

$$
\begin{aligned}
&= \max\big\{G_b(q,q,q), G_b(Rq,Rq,Rq), G_b(q,q,q), \frac{G_b(q,q,q)+G_b(q,q,q)}{2s}\big\},\\
&= 0.
\end{aligned}
$$

Applying (3),

$$
\begin{aligned}
\psi\big(s^2 G_b(fp,gq,hw)\big) &\le \psi\big(M(p,q,w)\big) - \phi\big(M(p,q,w)\big),\\
&= \psi(0) - \phi(0) = 0. \qquad (33)
\end{aligned}
$$

Consequently, $G_b(fp,gq,hw) = G_b(q,gq,q) = 0$ and so $gq = q$. Hence $gq = Rq = q$. Now we shall prove that $hq = Tq = q$. Similarly, using (30) and (31), we obtain that

$$
\begin{aligned}
M(p,u,q) &= \max\big\{G_b(fp,Sp,Tq), G_b(gu,Ru,Ru), G_b(fp,fp,hq),\\
&\qquad \frac{G_b(Tq,Tq,hq)+G_b(fp,Sp,Sp)}{2s}\big\},\\
&= \max\big\{G_b(q,q,Tq), G_b(q,q,q), G_b(q,q,hq), \frac{G_b(Tq,Tq,Tq)+G_b(q,q,q)}{2s}\big\},\\
&= \max\{G_b(q,q,Tq), 0, G_b(q,q,Tq), 0\}\\
&= G_b(q,q,Tq).
\end{aligned}
$$

By specifying $x = z = q$ and $y = u$ in (3) and using (27),

$$
\begin{aligned}
\psi\big(s^2 G_b(q,q,Tq)\big) &= \psi\big(s^2 G_b(fq,gu,hq)\big)\\
&\le \psi\big(M(p,u,q)\big) - \phi\big(M(p,u,q)\big),\\
&= \psi\big(G_b(q,q,Tq))\big) - \phi\big(G_b(q,q,Tq))\big),\\
&\le \psi\big(G_b(q,q,Tq))\big). \qquad (34)
\end{aligned}
$$

Again, $s^2 > s > 1$ and ψ is increasing, so $G_b(q,q,Tq) = 0$, that is, $Tq = q = Rq$. Thus

$$fq = Sq = gq = Rq = hq = Tq = q.$$

Then q is a common fixed point of f, g, h, R, S and T.

Now, we shall prove that the obtained fixed point is unique. Suppose that v is another common fixed point of f, g, h, R, S and T, that is $fv = gv = hv = Rv = Sv = Tv = v$. Then

$$
\begin{aligned}
M(q,q,v) &= \max\big\{G_b(fq,Sq,Tv), G_b(gq,Rq,Rq), G_b(fq,fq,hv),\\
&\qquad \frac{G_b(Tv,Tv,hv)+G_b(fq,Sq,Sq)}{2s}\big\},\\
&= \max\big\{G_b(q,q,v), 0, G_b(q,q,v), 0\big\},\\
&= G_b(q,q,v).
\end{aligned}
$$

From (3) we have that

$$
\psi\big(s^2 G_b(q,q,v)\big) = \psi\big(s^2 G_b(fq,gq,hv)\big)
$$

$$
\begin{aligned}
&\leq \psi\big(M(q,q,v)\big) - \phi\big(M(q,q,v)\big), \\
&= \psi\big(G_b(q,q,v))\big) - \phi\big(G_b(q,q,v))\big), \\
&\leq \psi\big(G_b(q,q,v))\big). \qquad (35)
\end{aligned}
$$

Again, since $s^2 > s > 1$ and being ψ is increasing, the above inequality implies that $G_b(q,q,v) = 0$ and so $q = v$. That is, q is the unique common fixed point for f, g, h, S, R and T.

The following result is an immediate consequence of Theorem 1 by taking $\phi(t) = t$.

Corollary 1. *Let (X, G_b) be a complete G_b-metric space and let $f, g, h, R, S, T : X \to X$ be self mappings such that*

(1) *(f, S) and (g, R) satisfy the $(E.A)$ property;*
(2) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and $h(X) \subseteq R(X)$;*
(3) *$R(X)$ is a closed subspace of X;*
(4) *(f, S), (g, R) and (h, T) are weakly compatible pairs of mappings;*
(5) *$\psi\big(s^2 G_b(fx, gy, hz)\big) \leq \psi\big(M(x,y,z)\big) - M(x,y,z)$ for all $x, y, z \in X$ where $\psi \in \Psi$ and*

$$
\begin{aligned}
M(x,y,z) = \max\big\{&G_b(fx, Sy, Tz), G_b(gy, Ry, Ry), \\
&G_b(fx, fx, hz), \frac{G_b(Tz, Tz, hz) + G_b(fx, Sx, Sx)}{2s}\big\}.
\end{aligned}
$$

Then f, g, h, R, S and T have a unique common fixed point in X.

As in the above corollary, the following result follows from Theorem 1 by taking $\psi(t) = t$.

Corollary 2. *Let (X, G_b) be a complete G_b-metric space and let $f, g, h, R, S, T : X \to X$ be self mappings such that*

(1) *(f, S) and (g, R) satisfy the $(E.A)$ property;*
(2) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and $h(X) \subseteq R(X)$;*
(3) *$R(X)$ is a closed subspace of X;*
(4) *$(f, S), (g, R)$ and (h, T) are weakly compatible pairs of mappings;*
(5) *$s^2 G_b(fx, gy, hz) \leq M(x,y,z) - \phi\big(M(x,y,z)\big)$ for each $x, y, z \in X$ where $\phi \in \Psi$ and*

$$
\begin{aligned}
M(x,y,z) = \max\big\{&G_b(fx, Sy, Tz), G_b(gy, Ry, Ry), \\
&G_b(fx, fx, hz), \frac{G_b(Tz, Tz, hz) + G_b(fx, Sx, Sx)}{2s}\big\}.
\end{aligned}
$$

Then f, g, h, R, S and T have a unique common fixed point in X.

By specifying $\psi(t) = t$ and $\phi(t) = \frac{t}{k}$ with $k > 1$ in Theorem 1, we get the following corollary.

Corollary 3. *Let (X, G_b) be a complete G_b-metric space and let $f, g, h, R, S, T : X \to X$ are self mappings such that*

(1) *(f, S) and (g, R) satisfy the (E.A) property;*

(2) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and $h(X) \subseteq R(X)$;*

(3) *$R(X)$ is a closed subspace of X;*

(4) *$(f, S), (g, R)$ and (h, T) are weakly compatible pairs of mappings;*

(5) *$\psi\big(s^2 G_b(fx, gy, hz)\big) \leq \frac{k-1}{k} M(x, y, z)$ for each $x, y, z \in X$ where k is a positive integer and*

$$M(x,y,z) = \max\big\{G_b(fx, Sy, Tz), G_b(gy, Ry, Ry), G_b(fx, fx, hz), \frac{G_b(Tz, Tz, hz) + G_b(fx, Sx, Sx)}{2s}\big\}.$$

Then f, g, h, R, S and T have a unique common fixed point in X.

By taking $f = g$ and $R = S$ in Theorem 1, we get the following result.

Corollary 4. *Let (X, G_b) be a complete G_b-metric space and let $f, g, h, R, S, T : X \to X$ be self mappings such that*

(1) *(g, S) satisfies the (E.A) property;*

(2) *$g(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and $h(X) \subseteq S(X)$;*

(3) *$S(X)$ is a closed subspace of X;*

(4) *(g, S) and (h, T) are weakly compatible pairs of mappings;*

(5) *$\psi\big(s^2 G_b(gx, gy, hz)\big) \leq \psi\big(M(x, y, z)\big) - \phi\big(M(x, y, z)\big)$ for each $x, y, z \in X$ where $\phi \in \Psi$ and*

$$M(x,y,z) = \max\big\{G_b(gx, Sy, Tz), G_b(gy, Sy, Sy), G_b(gx, gx, hz), \frac{G_b(Tz, Tz, hz) + G_b(gx, Sx, Sx)}{2s}\big\}.$$

Then g, h, S and T have a unique common fixed point in X.

The following example is to illustrate Theorem 1.

Example 5. *Let $X = [0, \infty)$ and $G : X \times X \times X \to [0, \infty)$ be the complete G-metric which is defined by*

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Define the G_b metric by

$$G_b(x,y,z) = (G(x,y,z))^2.$$

Then it is clear that (X, G_b) is a complete G_b-metric with $s = 2$. Also, define the mappings f, g, h, R, S and T by

$$fx = \frac{x}{32}, g(x) = \frac{x}{36}, h(x) = \frac{x}{48},$$

and

$$R(x) = \frac{4x}{9}, S(x) = \frac{x}{2}, \ \textit{and}\ T(x) = \frac{x}{3}$$

for all $x \in X$. *Further, define* $\psi(t) = 4\sqrt{t}$ *and* $\phi(t) = \frac{\sqrt{t}}{3}$ *for all* $t \in [0, \infty)$. *Then* f, g, h, R, S *and* T *have a unique common fixed point.*

Proof.

(1) (f, S) and (g, R) satisfy the $(E.A)$ property with $x_n = \frac{1}{n}$.

(2) $f(X) \subseteq T(X), g(X) \subseteq S(X)$ and $h(X) \subseteq R(X)$. In fact, $f(X) = g(x) = S(x) = R(x) = T(X) = [0, \infty)$.

(3) $R(X) = [0, \infty)$ is a closed subspace of X.

(4) $(f, S), (g, R)$ and (h, T) are weakly compatible pairs of mappings. In fact, the only coincident point for f and R is 0 and $f(R(0)) = R(f(0)) = 0$. Similarly for the other two pairs.

(5) We shall show that the above mappings satisfy the contractive condition (3). On one hand, we observe that

$$\begin{aligned}
\psi(s^2 G_b(fx, gy, hz)) &= \psi(4(\max\{\frac{x}{32}, \frac{y}{36}, \frac{z}{48}\})^2) \\
&= \psi(4(\max\{(\frac{x}{32})^2, (\frac{y}{36})^2, (\frac{z}{48})^2\}) \\
&= \psi(\max\{(\frac{2x}{32})^2, (\frac{2y}{36})^2, (\frac{2z}{48})^2\}) \\
&= \psi(\max\{(\frac{x}{16})^2, (\frac{y}{18})^2, (\frac{z}{24})^2\}) \\
&= 4\max\{(\frac{x}{16}), \frac{y}{18}, \frac{z}{24}\} \\
&= \max\{\frac{x}{4}, \frac{2y}{9}, \frac{z}{6}\}. \qquad (36)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
M(x,y,z) &= \max\left\{\begin{array}{c} \max\{(\frac{x}{32})^2, (\frac{y}{2})^2, (\frac{z}{3})^2\}, \max\{(\frac{y}{36})^2, (\frac{4y}{9})^2\} \\ \max\{(\frac{x}{32})^2, (\frac{z}{48})^2\}, \frac{\max\{(\frac{z}{3})^2, (\frac{z}{48})^2\} + \max\{(\frac{x}{32})^2, (\frac{x}{2})^2\}}{4} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \max\{(\frac{x}{32})^2, (\frac{y}{2})^2, (\frac{z}{3})^2\}, \max\{(\frac{y}{36})^2, (\frac{4y}{9})^2\}, \\ \max\{(\frac{x}{32})^2, (\frac{z}{48})^2\}, \frac{(\frac{z}{3})^2 + (\frac{x}{2})^2}{4}. \end{array}\right\} \\
&= \max\left\{ \max\{(\frac{x}{32})^2, (\frac{y}{2})^2, (\frac{z}{3})^2\}, (\frac{4y}{9})^2, \frac{(\frac{z}{3})^2 + (\frac{x}{2})^2}{4}. \right\} \\
&= \max\left\{ (\frac{x}{32})^2, (\frac{z}{3})^2, (\frac{y}{2})^2, \frac{(\frac{z}{3})^2 + (\frac{x}{2})^2}{4}. \right\} \\
&= \max\left\{ (\frac{y}{2})^2, (\frac{z}{3})^2, \frac{(\frac{z}{3})^2 + (\frac{x}{2})^2}{4} \right\}
\end{aligned}$$

$$= \max\Big\{(\frac{y}{2})^2, (\frac{z}{3})^2, (\frac{z}{6})^2 + (\frac{x}{4})^2\Big\},$$

and so,

$$\begin{aligned}\psi(M(x,y,z)) - \phi(M(x,y,z)) &= 4\max\Big\{\frac{y}{2}, \frac{z}{3}, \sqrt{(\frac{z}{6})^2 + (\frac{x}{4})^2}\Big\} \\ &- \frac{1}{3}\max\Big\{\frac{y}{2}, \frac{z}{3}, \sqrt{(\frac{z}{6})^2 + (\frac{x}{4})^2}\Big\} \\ &= \max\Big\{\frac{11y}{6}, \frac{11z}{9}, \frac{11}{3}\sqrt{(\frac{z}{6})^2 + (\frac{x}{4})^2}\Big\}. \qquad (37)\end{aligned}$$

Combining (36) and (37), it is clear to see that

$$\begin{aligned}\psi(s^2G_b(fx, gy, hz)) &= \max\{\frac{x}{4}, \frac{2y}{9}, \frac{z}{6}\} \\ &\leq \max\Big\{\frac{11y}{6}, \frac{11z}{9}, \frac{11}{3}\sqrt{(\frac{z}{6})^2 + (\frac{x}{4})^2}\Big\} \\ &= \psi(M(x,y,z)) - \phi(M(x,y,z)).\end{aligned}$$

Therefore, all conditions of Theorem 1 are satisfied, and $x = 0$ is the unique common fixed point of f, g, h, R, S and T.

4. Conclusion

As it known well, a G-metric space satisfies all conditions of the notion of a G_b-metric space when $s = 1$. But, if $s > 1$, the converse need not be true. Hence, the observed common fixed point results for six mappings of this paper, can be re-stated in the setting of G-metric spaces by taking $s = 1$.

References

[1] M. Aamri, D. Elmoutawakil, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.* 270 (2002), 181-188.

[2] A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, Filomat, 8 (6) (2014), 1087-1101.

[3] A.E. Al-Mazrooei, J Ahmad, Fixed Point Results for Multivalued Mappings in Gb-cone Metric Spaces, *J. Nonlinear Sci. Appl.* 10 (9) (2017), 4866-4875.

[4] A. Al-Rawashdeh, H. Aydi, A. Felhi, S. Sahmim, W. Shatanawi, On common fixed points for $\alpha - F$-contractions and applications, *J. Nonlinear Sci. Appl.* 9 (5) (2016), 3445–3458

[5] H. Aydi, On common fixed point theorems for (ψ, φ)-generalized f-weakly contractive mappings, *Miskolc Mathematical Notes*, 14 (1) (2013), 19-30.

[6] H. Aydi, B. Damjanovic, B. Samet, W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, *Mathematical and Computer Modelling*, 54 (2011), 2443-2450.

[7] H. Aydi, A. Felhi, S. Sahmim, Related fixed point results for cyclic contractions on G-metric spaces and applications, *Filomat*, 31 (3) (2017), 853–869.

[8] H. Aydi, E. Karapinar, W. Shatanawi, Coupled fixed point results for (ψ, ϕ)-weakly contractive condition in ordered partial metric spaces, *Comput. Math. Appl.* 62 (2011), 4449-4460.

[9] H. Aydi, E. Karapinar, P. Salimi, Some fixed point results in GP-metric spaces, *Journal of Applied Mathematics*, Volume 2012, Article ID 891713, 15 pages.

[10] H. Aydi, W. Shatanawi, M. Postolache, Coupled fixed point results for (ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces, *Comput. Math. Appl.* 63 (2012), 298-309.

[11] H. Aydi, W. Shatanawi, C. Vetro, On generalized weakly G-contraction mapping in G-metric spaces, *Comput. Math. Appl.* 62 (2011), 4222-4229.

[12] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations integrals, *Fund. Math.* 3 (1922), 133-181.

[13] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, 1 (1993), 5-11.

[14] M.M.M. Jaradat, Z. Mustafa, A. H. Ansari, P. S. Kumari, D. Dolicanin-Djekic and H.M. Jaradat, Some fixed point results for $F_{\alpha-\omega\varphi}$-generalized cyclic contractions on metric-like space with applications to graphs and integral equations, *J. Math. Analysis*, **8**(1) (2017) 28–45.

[15] M.M.M. Jaradat, Z. Mustafa, A. H. Ansari, S. Chandok, C. Dolicanin, Some approximate xed point results and application on graph theory for partial (hF)-generalized convex contraction mappings with special class of functions on complete metric space, *J. Nonlinear Sci. Appl.* **10** (4) (2017) 1695-1708.

[16] M.M.M. Jaradat, Z. Mustafa, M. Arshad, S. Ullah Khan, J. Ahmad, Some fixed point results on G-metric and G_b-metric spaces, *Demonstratio Mathematica*, **50** (2017) 190–207.

[17] G. Jungk, Commuting maps and fixed points, *Am. Math. Monthly,* 83 (1976), 261-263.

[18] G. Jungk, Compatible mappings and common fixed points, *Int.J. Math. Sci.,* 9(4)(1986), 771-779.

[19] G. Jungk, Common Fixed points for commuting and Compatible maps on compacta, *Pro. Am. Math. Soc.* 103 (1988), 977-983.

[20] G. Jungk, Common fixed points for noncontinuous nonself maps on nonmetric spaces, *Far East J. Math. Sci.* 4 (1996), 199-215.

[21] G. Jungk, N. Hussain, Compatible maps and invariant approximation, *J. Math. Anal. Appl.* 325(2)(2007), 1003-1012.

[22] M.S. Khan, M. Swalesh, S. Sessa, Fixed points theorems by altering distances between the points, *Bull. Aust. Math. Soc.* 30 (1984), 1-9.

[23] D. Lateef, J. Ahmad, A.E. Al-Mazrooei, Common fixed point theorems for generalized contractions, *Journal of Mathematical Analysis*, 8 (3) (2017), 157-166.

[24] Z. Mustafa, H. Aydi, E. Karapinar, On common fixed points in G-metric spaces using $(E.A)$ property, *Comput. Math. Appl.* 6 (6) (2012), 1944-1956.

[25] Z. Mustafa, H. Aydi, E. Karapinar, Generalized Meir-Keeler type contractions on G-metric spaces, *Appl. Math. Comput.* 219 (2013), 10441-10447.

[26] Z. Mustafa, J.R. Roshan, V. Parvaneh, Coupled coincidence point results for (ψ,ϕ) -weakly contractive mappings in partially ordered G_b-metric spaces, *Fixed Point Theory Appl.* 2013:206, (2013).

[27] Z. Mustafa, J.R. Roshan, V. Parvaneh, Existence of a tripled coincidence point in ordered G_b-metric spaces and applications to a system of integral equations,*Journal of Inequalities and Applications*, 2013, 2013:453

[28] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear and Convex Analysis*, 7 (2006), 289-297.

[29] Z. Mustafa, H. Obiedat, F. Awawdeh, Some common fixed point theorems for mapping on complete G-metric spaces, *Fixed Point Theory Appl*, 2008, Article ID 189870.

[30] Z. Mustafa, J. R. Roshan, V. Parvaneh and Z. Kadelburg. Some common fixed point results in ordered partial b-metric space, *J. Inequalities and Applications* **2013**:562 (2013) 26 pages.

[31] Z. Mustafa, T. V. An and N. V. Dung, Two Fixed Point Theorems for Maps on Incomplete G-Metric Spaces, *Applied Mathematical Sciences*, **7** 46 (2013) 2271 2281.

[32] Z. Mustafa, J. R. Roshan, V. Parvaneh and Z. Kadelburg. Fixed point theorems for Weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces, *J. of Inequalities and Applications* (2014) 46..

[33] Z. Mustafa, M. Jaradat, A. Ansari, B. Z. Popovi and H. Jaradat, C-class functions with new approach on coincidence point results for generalized (ψ,φ) -weakly contractions in ordered $b$metric spaces, *SpringerPlus* (2016) **5**:802, 18 pages.

[34] Z. Mustafa, M. M. M. Jaradatat, H. M. Jaradat, Some common fixed point results of graphs on $b-$ metric space, *J. of Nonlinear Sci. and Appl.*, **9**(6) (2016) 4838-4851.

[35] Z. Mustafa, M. M.M. Jaradat and H.M. Jaradat, A remarks on the paper " some fixed point theorems for generalized contractive mappings in complete metric spaces". *J. of Mathematical Analysis*, **8**(2) (2017) 17-22.

[36] Z. Mustafa, M.M.M. Jaradat, E. Karapnar, A new xed point result via property P with an application, *J. of Nonlinear Sci. and Appl.*, **10** (2017) 2066-2078.

[37] Z. Mustafa, M. Arshad, S. U. Khan, J. Ahmad, M.M.M. Jaradat, Common Fixed Points for Multivalued Mappings in G-Metric Spaces with Applications, *J. Nonlinear Sci. and Appl.*, **10** (2017) 2550-2564.

[38] Abdullah, M. Sarwar, Z. Mustafa, and M.M.M. Jaradat, Common Fixed points of (ϕ, ψ)-contraction on G- metric space using E.A property, *J. of Mathematical Analysis* , **8**(4) (2017) 136–146.

[39] Z. Mustafa, H. Aydi and E. Karapinar, On Common Fixed Points in G-Metric Spaces Using (E.A) Property, *Computer and mathematics with apllication.* **64** (2012) 1944–1956.

[40] Z. Mustafa, Common Fixed Points of Weakly Compatible Mappings in G-Metric Spaces, *Appl. Mathematical Sci.*, **6**(92) (2012) 4589–4600.

[41] B. Moeini, A.H. Ansari, H. Aydi, Some common fixed point theorems without orbital continuity via C-class functions and an application, *Journal of Mathematical Analysis*, 8 (4) (2017), 46–55.

[42] H.K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, *Nonlinear Anal.* 74 (2011), 2201-2209.

[43] R.P. Pant, Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.* 188 (1994), 436-440.

[44] R.P. Pant, Common fixed point of contractive maps, *J. Math. Anal. Appl.* 226 (1998), 251-258.

[45] J.R. Roshan, N. Shobkolaei, S. Sedghi, V. Parvaneh, S. Radenović, Common fixed point theorems for three maps in discontinuous G_b-metric spaces, *Acta Mathematica Scientia*, 34 (5) (2015), 1643-1654.

[46] M. Sarwar, S. Abdullah, I.A. Shah, Fixed point theorem satisfying some rational type contraction in G_b-metric spaces, *J. Adv. Math. Stud.* 9 (2) (2016), 320-329.

[47] S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, *Publ. Inst. Math. Soc.* 32 (1982), 149-153.

[48] F. Yan, Y. Su, Q. Feng, A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations. *Fixed Point Theory Appl.* 2012, 2012:152.