Abstract. A topological space $X$ is called $C$-Tychonoff if there exist a one-to-one function $f$ from $X$ onto a Tychonoff space $Y$ such that the restriction $f|_K : K \rightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. We discuss this property and illustrate the relationships between $C$-Tychonoffness and some other properties like submetrizability, local compactness, $L$-Tychonoffness, $C$-normality, $C$-regularity, epinormality, $\sigma$-compactness, pseudocompactness and zero-dimensional.

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1. Introduction

We define a new topological property called $C$-Tychonoff. Unlike $C$-normality[2], we prove that $C$-Tychonoffness is a topological property which is multiplicative and hereditary. We show that $C$-Tychonoff and $C$-normal are independent. Also we investigate the function witnesses the $C$-Tychonoffness when it is continuous and when it is not. We introduce the notion of $L$-Tychonoffness. Throughout this paper, we denoted of the set of positive integers by $\mathbb{N}$, and an order pair by $\langle x, y \rangle$. An ordinal $\gamma$ is the set of all ordinal $\alpha$, with $\alpha < \gamma$, we denoted the first infinite ordinal by $\omega_0$ and the first uncountable ordinal by $\omega_1$. A $T_3$ space is a $T_1$ regular space, a Tychonoff ($T_{3_1}$) space is a $T_1$ completely regular space, and a $T_4$ space is a $T_1$ normal space. For a subset $B$ of a space $X$, $\text{int}B$ denote the interior of $B$ and $\overline{B}$ denote the closure of $B$. A space $X$ is locally compact if for each $y \in X$ and each open neighborhood $U$ of $y$ there exists an open neighborhood $V$ of $y$ such that $y \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact, we do not assume $T_2$ in the definition of local compactness.
2. **C-Tychonoffness**

**Definition 1.** A topological space $X$ is called $C$-Tychonoff if there exist a one-to-one function $f$ from $X$ onto a Tychonoff space $Y$ such that the restriction $f|_K : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$.

Recall that a topological space $(X, \tau)$ is called submetrizable if there exists a metric $d$ on $X$ such that the topology $\tau_d$ on $X$ generated by $d$ is coarser than $\tau$, i.e., $\tau_d \subseteq \tau$, see [10].

**Theorem 1.** Every submetrizable space is $C$-Tychonoff.

**Proof.** Let $\tau'$ be a metrizable topology on $X$ such that $\tau' \subseteq \tau$. Then $(X, \tau')$ is Tychonoff and the identity function $id_X : (X, \tau) \longrightarrow (X, \tau')$ is a bijective and continuous. If $K$ is any compact subspace of $(X, \tau)$, then $id_X(K)$ is Hausdorff being a subspace of the metrizable space $(X, \tau')$, and the restriction of the identity function on $K$ onto $id_X(K)$ is a homeomorphism by [8, 3.1.13].

Since any Hausdorff locally compact space is Tychonoff, then we have the following theorem.

**Theorem 2.** Every Hausdorff locally compact space is $C$-Tychonoff.

The converse of Theorem 1 is not true in general. For example, the Tychonoff Plank $((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{(\omega_1, \omega_0)\}$ is $C$-Tychonoff being Hausdorff locally compact, but it is not submetrizabl, because if it was, then $((\omega_1 + 1) \times \{0\}) \subseteq (\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{(\omega_1, \omega_0)\}$ is submetrizabl, because submetrizability is hereditary, but $((\omega_1 + 1) \times \{0\}) \cong \omega_1 + 1$ and $\omega_1 + 1$ is not submetrizabl.

The converse of Theorem 2 is not true in general as the Dieudonné Plank [16] is Tychonoff, hence $C$-Tychonoff but not locally compact. Hausdorffness is essential in Theorem 2. Here is an example of a locally compact space which is neither $C$-Tychonoff nor Hausdorff.

**Example 1.** The particular point topology $\tau_{\sqrt{2}}$ on $\mathbb{R}$, see [16], is not $C$-Tychonoff. It is well-known that $(\mathbb{R}, \tau_{\sqrt{2}})$ is neither $T_1$ nor Tychonoff. If $B \subseteq \mathbb{R}$, then $\{x, \sqrt{2} : x \in B\}$ is an open cover for $B$, thus a subset $B$ of $\mathbb{R}$ is compact if and only if it is finite. To show that $(\mathbb{R}, \tau_{\sqrt{2}})$ is not $C$-Tychonoff, suppose that $(\mathbb{R}, \tau_{\sqrt{2}})$ is $C$-Tychonoff. Let $Z$ be a Tychonoff space and $f : \mathbb{R} \longrightarrow Z$ be a bijective function such that the restriction $f|_K : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K$ of $(\mathbb{R}, \tau_{\sqrt{2}})$. Take $K = \{x, \sqrt{2}\}$, such that $x \neq \sqrt{2}$, hence $K$ is a compact subspace of $(\mathbb{R}, \tau_{\sqrt{2}})$. By assumption $f|_K : K \longrightarrow f(K) = \{f(x), f(\sqrt{2})\}$ is a homeomorphism. Because $f(K)$ is a finite subspace of $Z$ and $Z$ is $T_1$, then $f(K)$ is discrete subspace of $Z$. Therefore, we obtain that $f|_K$ is not continuous and this a contradiction as $f|_K$ is a homeomorphism. Thus $(\mathbb{R}, \tau_{\sqrt{2}})$ is not $C$-Tychonoff. ■
By the definition, it is clear that a compact $C$-Tychonoff space must be Tychonoff see Theorem 3 below. Obviously, any Tychonoff space is $C$-Tychonoff, just by taking $Y = X$ and $f$ to be the identity function, but the converse is not true in general. For example, the Half-Disc space [16] is $C$-Tychonoff which is not Tychonoff. It is $C$-Tychonoff because it is submetrizable. $C$-Tychonoffness does not imply Tychonoffness even with first countability. For example, Smirnov’s deleted sequence topology [16] is first countable and $C$-Tychonoff being submetrizable but not Tychonoff.

**Theorem 3.** If $X$ is a compact non-Tychonoff space, then $X$ cannot be $C$-Tychonoff.

We conclude that from the above theorem, $\mathbb{R}$ with the finite complement topology is not $C$-Tychonoff.

**Theorem 4.** If $X$ is a $T_1$-space such that the only compact subspace are the finite subspace, then $X$ is $C$-Tychonoff.

**Proof.** Let $Y = X$ and consider $Y$ with the discrete topology. Then the identity function from $X$ onto $Y$ is a bijective function. If $K$ is any compact subspace of $(X, \tau)$, then by assumption $K$ is a finite subspace. Because any finite set in a $T_1$-space is discrete, hence the restriction of the identity function on $K$ onto $K$ is a homeomorphism since both of the domain and the codomain are discrete and have the same cardinality.

If $X$ is $C$-Tychonoff and $f : X \to Y$ is a witness of the $C$-Tychonoffness of $X$, then $f$ may not be continuous. Here is an example.

**Example 2.** Consider $\mathbb{R}$ with the countable complement topology $CC$ [16]. Since the only compact subspace are the finite subspaces and $(\mathbb{R}, CC)$ is $T_1$, then the compact subspace are discrete. Hence $\mathbb{R}$ with the discrete topology and the identity function will give the $C$-Tychonoffness, see Theorem 4. Observe that the identity function in this case is not continuous. ■

Recall that a space $X$ is Fréchet if for any subset $B$ of $X$ and any $x \in \overline{B}$ there exist a sequence $(b_n)_{n \in \mathbb{N}}$ of points of $B$ such that $b_n \to x$, see [8].

**Theorem 5.** If $X$ is $C$-Tychonoff and Fréchet, then any function witnesses its $C$-Tychonoffness is continuous.

**Proof.** Let $X$ be $C$-Tychonoff and Fréchet. Let $f : X \to Y$ be a witness of the $C$-Tychonoffness of $X$. Take $B \subseteq X$ and pick $y \in f(\overline{B})$. There is a unique $x \in X$ such that $f(x) = y$, thus $x \in \overline{B}$. Since $X$ is Fréchet, then there exists a sequence $(b_n) \subseteq B$ such that $b_n \to x$. The sequence $K = \{x\} \cup \{b_n : n \in \mathbb{N}\}$ of $X$ is compact since it is a convergent sequence with its limit, thus $f|_K : K \to f(K)$ is a homeomorphism. Let
$W \subseteq Y$ be any open neighborhood of $y$. Then $W \cap f(K)$ is open in the subspace $f(K)$ containing $y$. Since $f((b_n : n \in \mathbb{N})) \subseteq f(K) \cap f(B)$ and $W \cap f(K) \neq \emptyset$, then we have $W \cap f(B) \neq \emptyset$. Hence $y \in \overline{f(B)}$ and $f(B) \subseteq \overline{f(B)}$. Thus $f$ is continuous.

Since any first countable space is Fréchet [8], we conclude the following corollary:

**Corollary 1.** If $X$ is $C$-Tychonoff first countable and $f : X \rightarrow Y$ witnessing the $C$-Tychonoffness of $X$, then $f$ is continuous.

**Corollary 2.** Any $C$-Tychonoff Fréchet space is Urysohn.

**Proof.** Let $(X, \tau)$ be any $C$-Tychonoff Fréchet space. We may assume that $X$ has more than one element. Pick a Tychonoff space $(Y, \tau')$ and a bijection function $f : (X, \tau) \rightarrow (Y, \tau')$ such that $f_{|A} : A \rightarrow f(A)$ is a homeomorphism for each compact subspace $A$ of $X$. Since $X$ is Fréchet, then $f$ is continuous. Define a topology $\tau^*$ on $X$ as follows: $\tau^* = \{ f^{-1}(U) : U \in \tau' \}$. It clear that $\tau^*$ is a topology on $X$ coarser that $\tau$ such that $f : (X, \tau^*) \rightarrow (Y, \tau')$ is continuous. If $W \in \tau^*$, then $W$ is of the form $W = f^{-1}(U)$ where $U \in \tau'$. So, $f(W) = f(f^{-1}(U)) = U$ which gives that $f$ is open, hence homeomorphism. Thus $(X, \tau^*)$ is Tychonoff. Pick distinct $a, b \in X$. Using $T_2$ of $(X, \tau^*)$, choose $G, H \in \tau^*$ such that $a \in G, b \in H$, and $G \cap H = \emptyset$. Using regularity of $(X, \tau^*)$, choose $U, V \in \tau^*$ such that $a \in U \subseteq U^\tau \subseteq G$ and $b \in V \subseteq V^\tau \subseteq H$. We have that $U, V \in \tau$ and since $\overline{B^\tau} \subseteq \overline{B^\tau}$ for any $B \subseteq X$, we get $\overline{U^\tau} \cap \overline{V^\tau} = \emptyset$. Therefore, $(X, \tau)$ is Urysohn.

So, we conclude that any first countable $C$-Tychonoff space is Hausdorff.

Recall that a space $X$ is a $k$-space if $X$ is $T_2$ and it is a quotient image of a locally compact space [8]. By the theorem: “a function $f$ from a $k$-space $X$ into a space $Y$ is continuous if and only if $f_{|Z} : Z \rightarrow Y$ is continuous for each compact subspace $Z$ of $X$”, [8, 3.3.21]. We conclude the following:

**Corollary 3.** If $X$ is a $C$-Tychonoff $k$-space and $f : X \rightarrow Y$ witnessing the $C$-Tychonoffness of $X$, then $f$ is continuous.

Recall that a topological space $X$ is called $C$-normal if there exist a one-to-one function $f$ from $X$ onto a normal space $Y$ such that the restriction $f_{|K} : K \rightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$[2].

**Theorem 6.** Every $C$-Tychonoff Fréchet Lindelöf space is $C$-normal.
Proof. Let $X$ be any $C$-Tychonoff Fréchet Lindelöf space. Pick a Tychonoff space $Y$ and a bijective function $f : X \to Y$ such that the restriction $f_K : K \to f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. By Theorem 5, $f$ is continuous. Since the continuous image of a Lindelöf space is Lindelöf [8, 3.8.7], we conclude that $Y$ is Lindelöf, hence normal as any regular Lindelöf space is normal [8, 3.8.2]. Therefore, $X$ is $C$-normal.

$C$-normality and $C$-Tychonoffness are independent from each other. Here is an example of a $C$-normal which is not $C$-Tychonoff.

Example 3. Consider $\mathbb{R}$ with its right ray topology $\mathcal{R}$ [16]. So, $\mathcal{R} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. Since any two non-empty closed sets must intersect, then $(\mathbb{R}, \mathcal{R})$ is normal, hence $C$-normal [2]. Now, suppose that $(\mathbb{R}, \mathcal{R})$ is $C$-Tychonoff. Pick a Tychonoff space $Y$ and a bijective function $f : \mathbb{R} \to Y$ such that the restriction $f_K : K \to f(K)$ is a homeomorphism for each compact subspace $K \subseteq \mathbb{R}$. It is well-known that a subspace $K$ of $(\mathbb{R}, \mathcal{R})$ is compact if and only if $K$ has a minimal element. Thus $[2, \infty)$ is compact, hence $f_{[2, \infty)} : [2, \infty) \to f([2, \infty)) \subseteq Y$ is a homeomorphism. i.e. $f([2, \infty))$ as a subspace of $(\mathbb{R}, \mathcal{R})$ is regular which is a contradiction as $[2, 3]$ is closed in $[2, \infty)$ and $5 \notin [2, 3]$ and any non-empty open sets in $[2, \infty)$ must intersect. Therefore, $(\mathbb{R}, \mathcal{R})$ cannot be $C$-Tychonoff.

Here is an example of a $C$-Tychonoff space which is not $C$-normal.

Example 4. Consider the infinite Tychonoff product space $G = D^{\omega_1} = \prod_{\alpha \in \omega_1} D$, where $D = \{0, 1\}$ considered with the discrete topology. Let $H$ be the subspace of $G$ consisting of all points of $G$ with at most countably many non-zero coordinates. Put $M = G \times H$. Raushan Buzyakova proved that $M$ cannot be mapped onto a normal space $Z$ by a bijective continuous function [7]. Using Buzyakova’s result and the fact that $M$ is a $k$-space, we conclude that $M$ is a Tychonoff space which is not $C$-normal [13]. Since $M$ is Tychonoff, then it is $C$-Tychonoff.

Theorem 7. $C$-Tychonoffness is a topological property.

Proof. Let $X$ be a $C$-Tychonoff space and $X \cong Y$. Let $Z$ be a Tychonoff space and let $f : X \to Z$ be a bijective function such that the restriction $f_K : K \to f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. Let $h : Y \to X$ be a homeomorphism. Then $Z$ and $f \circ h : Y \to Z$ satisfies the requirement.

Theorem 8. $C$-Tychonoffness is an additive property.
Corollary 4. Since any compact subspace of a Tychonoff space is Tychonoff, for each compact subspace $K$ of $X_s$, the function $f : X_s \rightarrow Y_s$ such that $f|_{K_s} : K_s \rightarrow f_s(K_s)$ is a homeomorphism for each compact subspace $K$ of $X_s$. Because $Y_s$ is Tychonoff for each $s \in S$, then the sum $\bigoplus_{s \in S} Y_s$ is Tychonoff, by the Tychonoff theorem. Hence $f(x) = f_s(x)$ if $x \in X_s$, $s \in S$. A subspace $K \subseteq \bigoplus_{s \in S} X_s$ is compact if and only if the set $S_0 = \{s \in S : K \cap X_s \neq \emptyset\}$ is finite and $K \cap X_s$ is compact in $X_s$ for each $s \in S_0$. If $K \subseteq \bigoplus_{s \in S} X_s$ is compact, then $\bigoplus_{s \in S} f_s|_K$ is a homeomorphism since $f_s|_{K \cap X_s}$ is a homeomorphism for each $s \in S_0$.

Theorem 9. C-Tychonoffness is a multiplicative property.

Proof. Let $X_s$ be a C-Tychonoff space for each $s \in S$. Pick a Tychonoff space $Y_s$ and a bijective function $f_s : X_s \rightarrow Y_s$ such that $f|_{K_s} : K_s \rightarrow f_s(K_s)$ is a homeomorphism for each compact subspace $K_s$ of $X_s$. Since $Y_s$ is Tychonoff for each $s \in S$, then the Cartesian product $\prod_{s \in S} Y_s$ is Tychonoff [8, 2.2.7]. Consider the function sum $f = \bigoplus_{s \in S} f_s : \bigoplus_{s \in S} X_s \rightarrow \bigoplus_{s \in S} Y_s$ defined by $f(x) = f_s(x)$ if $x \in X_s$, $s \in S$. A subspace $K \subseteq \bigoplus_{s \in S} X_s$ is compact if and only if the set $S_0 = \{s \in S : K \cap X_s \neq \emptyset\}$ is finite and $K \cap X_s$ is compact in $X_s$ for each $s \in S_0$. If $K \subseteq \bigoplus_{s \in S} X_s$ is compact, then $\bigoplus_{s \in S} f_s|_K$ is a homeomorphism. Thus $f|_K$ is a homeomorphism, because the restriction of a homeomorphism is a homeomorphism.

Theorem 10. C-Tychonoffness is a hereditary property.

Proof. Let $A$ be any non empty subspace of C-Tychonoff space $X$. Pick a bijective function $f$ from $X$ onto a Tychonoff space $Y$ such that $f|_K : K \rightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. Let $B = f(A) \subseteq Y$. Then $B$ is Tychonoff being a subspace of a Tychonoff space $Y$. Now, we have $f|_A : A \rightarrow B$ is a bijective function. Since any compact subspace of $A$ is compact in $X$ and $f|_{A|_K} = f|_K$, we conclude that $A$ is C-Tychonoff.

From Theorem 9 and Theorem 10, we conclude the following corollary.

Corollary 4. $\prod_{s \in S} X_s$ is C-Tychonoff if and only if $X_s$ is C-Tychonoff $\forall s \in S$.

3. $L$-Tychonoffness and Other Properties

We introduce another new topological property called $L$-Tychonoff.
Definition 2. A topological space \( X \) is called \( L \)-Tychonoff if there exist a one-to-one function \( f \) from \( X \) onto a Tychonoff space \( Y \) such that the restriction \( f|_L : L \rightarrow f(L) \) is a homeomorphism for each Lindelöf subspace \( L \subseteq X \).

By the definition it is clear that a Lindelöf \( L \)-Tychonoff space must be Tychonoff. Since any compact space is Lindelöf, then any \( L \)-Tychonoff space is \( C \)-Tychonoff. The converse is not true in general. Obviously, no Lindelöf non-Tychonoff space is \( L \)-Tychonoff. So, no countable complement topology on uncountable set \( X \) is \( L \)-Tychonoff, but it is \( C \)-Tychonoff, see Example 2. An example of an \( L \)-Tychonoff space which is not Tychonoff.

Example 5. Consider \( \omega_2 \), the successor cardinal number of the cardinal number \( \omega_1 \). Let \( X = \omega_2 \cup \{i, j\} \) where \( \{i, j\} \cap \omega_2 = \emptyset \) so \( i \notin \omega_2 \) and \( j \notin \omega_2 \). Generate a topology on \( X \) as follows: Each \( \alpha \in \omega_2 \) is isolated. A basic open neighborhood of \( i \) is of the form \( U = \{i\} \cup (\omega_2 \setminus E) \) where \( E \subseteq \omega_2 \) with \( |E| = \omega_1 \). Similarly, a basic open neighborhood of \( j \) is of the form \( V = \{j\} \cup (\omega_2 \setminus F) \) where \( F \subseteq \omega_2 \) with \( |F| = \omega_1 \). Then \( X \) is not \( T_2 \) as \( i \) and \( j \) cannot be separated by disjoint open sets. \( X \) is not Lindelöf as the open cover \( \{\{i\} \cup (\omega_2 \setminus \omega_1), \{j\} \cup (\omega_2 \setminus \omega_1)\}, \{\alpha\} : \alpha \in \omega_1 \} \) of \( X \) has no countable subcover. Also, if \( C \) is any countable subspace of \( X \), then \( C \) is discrete as a subspace because if \( i \in C \), then \( U = \{i\} \cup (\omega_2 \setminus (\omega_1 \cup (C \setminus \{j\})) \) is an open neighborhood of \( i \) in \( X \) such that \( U \cap C = \{i\} \). Similarly, if \( j \in C \). It is clear that if \( C \) is countable, then \( C \) is Lindelöf. Assume that \( C \) is uncountable. Then \( |C| \geq \omega_1 \). Suppose that \( \{i, j\} \subseteq C \). Partition \( C \) into three partitions \( C_1, C_2, \) and \( C_3 \) such that \( i \in C_1 \) with \( |C_1| = \omega_1 \), \( j \in C_2 \) with \( |C_2| = \omega_1 \), and \( C_3 \geq \omega_1 \). The open cover \( \{\{i\} \cup (\omega_2 \setminus (C_1 \cup C_2) \setminus \{i, j\}), \{j\} \cup (\omega_2 \setminus (C_1 \cup C_2) \setminus \{i, j\})\}, \{\alpha\} : \alpha \in C_1 \cup C_2 \} \) of \( C \) has no countable subcover. If \( C \) contains either \( i \) or \( j \), we do the same idea but for just two partitions. Thus a subspace \( C \) of \( X \) is Lindelöf if and only if \( C \) is countable. Thus \( X \) is \( L \)-Tychonoff which is not Tychonoff.

A function \( f : X \rightarrow Y \) witnessing the \( L \)-Tychonoffness of \( X \) need not be continuous. But it will be if \( X \) is of countable tightness. Recall that a space \( X \) is of countable tightness if for each subset \( B \) of \( X \) and each \( x \in \overline{B} \), there exists a countable subset \( B_0 \) of \( B \) such that \( x \in \overline{B_0} \) [8].

Theorem 11. If \( X \) is \( L \)-Tychonoff and of countable tightness and \( f : X \rightarrow Y \) is a witness of the \( L \)-Tychonoffness of \( X \), then \( f \) is continuous.

Proof. Let \( A \) be any non-empty subset of \( X \). Let \( y \in f(\overline{A}) \) be arbitrary. Let \( x \in X \) be the unique element such that \( f(x) = y \). Then \( x \in \overline{A} \). Pick a countable subset \( A_0 \subseteq A \) such that \( x \in \overline{A_0} \). Let \( B = \{x\} \cup A_0 \); then \( B \) is a Lindelöf subspace of \( X \) and hence \( f|_B : B \rightarrow f(B) \) is a homeomorphism. Now, let \( V \subseteq Y \) be any open neighborhood of \( y \); then \( V \cap f(B) \) is open in the subspace \( f(B) \) containing \( y \). Thus \( f^{-1}(V) \cap B \) is open in the subspace \( B \) containing \( x \). Thus \( (f^{-1}(V) \cap B) \cap A_0 \neq \emptyset \). So \( (f^{-1}(V) \cap B) \cap A \neq \emptyset \). Hence \( \emptyset \neq f((f^{-1}(V) \cap B) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A) \). Thus \( y \in f(A) \). Therefore, \( f \) is continuous.
Recall that if \((x_n)_{n \in \mathbb{N}}\) is a sequence in a topological space \(X\), then the \textit{convergency set of} \((x_n)\) is defined by \(C(x_n) = \{ x \in X : x_n \rightarrow x \}\) and a topological space \(X\) is \textit{sequential} if for any \(A \subseteq X\) we have that \(A\) is closed if and only if \(C(x_n) \subseteq A\) for any sequence \((x_n) \subseteq A\), see [8]. We have the following implications, see [8, 1.6.14, 1.7.13].

First countability \(\Rightarrow\) Fréchet \(\Rightarrow\) Sequential \(\Rightarrow\) Countable tightness.

**Corollary 5.** If \(X\) is \(L\)-Tychonoff and first countable (Fréchet, Sequential) and \(f : X \rightarrow Y\) is a witness of the \(L\)-Tychonoffness of \(X\), then \(f\) is continuous.

**Theorem 12.** \(L\)-Tychonoffness is a topological property.

**Theorem 13.** \(L\)-Tychonoffness is an additive property.

**Theorem 14.** \(L\)-Tychonoffness is a multiplicative property.

**Theorem 15.** \(L\)-Tychonoffness is a hereditary property.

**Theorem 16.** If any countable subspace of a space \(X\) is discrete and the only Lindelöf subspaces are the countable subspaces, then \(X\) is \(L\)-Tychonoff.

\textit{Proof.} Let \(Y = X\) and consider \(Y\) with the discrete topology. Then the identity function from \(X\) onto \(Y\) is a bijective function. If \(K\) is any Lindelöf subspace of \(X\), then, by assumption, \(K\) is countable and discrete, hence the restriction of the identity function on \(K\) onto \(K\) is a homeomorphism.

**Theorem 17.** If \(X\) is \(C\)-Tychonoff space such that each Lindelöf subspace is contained in a compact subspace, then \(X\) is \(L\)-Tychonoff.

\textit{Proof.} Assume that \(X\) is \(C\)-Tychonoff and if \(L\) is any Lindelöf subspace of \(X\), then there exists a compact subspace \(K\) with \(L \subseteq K\). Let \(f\) be a bijective function from \(X\) onto a Tychonoff space \(Y\) such that the restriction \(f|_C : C \rightarrow f(C)\) is a homeomorphism for each compact subspace \(C\) of \(X\). Now, let \(L\) be any Lindelöf subspace of \(X\). Pick a compact subspace \(K\) of \(X\) where \(L \subseteq K\), then \(f|_K : K \rightarrow f(K)\) is a homeomorphism, thus \(f|_L : L \rightarrow f(L)\) is a homeomorphism as \((f|_K)|_L = f|_L\).

Now, we study some relationships between \(C\)-Tychonoffness and some other properties.
Recall that a topological space $X$ is called $C$-regular if there exist a one-to-one function $f$ from $X$ onto a regular space $Y$ such that the restriction $f|_K : K \to f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$ [5]. Any $C$-Tychonoff space is $C$-regular space, but the converse is not true in general. For example, any indiscrete space which has more than one element is an example of $C$-regular space which is not $C$-Tychonoff by Theorem 3.

Recall that a topological space $(X, \tau)$ is called epinormal if there is a coarser topology $\tau'$ on $X$ such that $(X, \tau')$ is $T_4$ [3]. By a similar proof as that of Theorem 1 above, we can prove the following corollary:

**Corollary 6.** Any epinormal space is $C$-Tychonoff.

$\mathbb{R}$ with the countable complement topology $\mathbb{C}C$ [16], is an example of $C$-Tychonoff space which is not epinormal because $(\mathbb{R}, \mathbb{C}C)$ is not $T_2$ and any epinormal space is $T_2$ [3].

Let $X$ be any Hausdorff non-$k$-space. Let $kX = X$. Define a topology on $kX$ as follows: a subset of $kX$ is open if and only if its intersection with any compact subspace $C$ of the space $X$ is open in $C$. $kX$ with this topology is Hausdorff and $k$-space such that $X$ and $kX$ have the same compact subspace and the same topology on these subspace [6], we conclude the following:

**Theorem 18.** If $X$ is Hausdorff but not $k$-space, then $X$ is $C$-Tychonoff if and only if $kX$ is $C$-Tychonoff.

$C$-Tychonoffness and $\sigma$-compactness are independent from each other. For example the rational sequence space [16] is $C$-Tychonoff being Tychonoff, but not $\sigma$-compact. $\mathbb{R}$ with the finite complement topology is not $C$-Tychonoff by Theorem 3, but it is $\sigma$-compact being compact. Any pseudocompact is $C$-Tychonoff being Tychonoff, but the converse is not true, for example Sorgenfrey line square topology [16], it is $C$-Tychonoff being Tychonoff but not pseudocompact. Also any zero-dimensional space is $C$-Tychonoff, but the converse is not true, for example Niemytzki’s tangent disc topology [16], it is $C$-Tychonoff being Tychonoff but not zero-dimensional because it is connected.

Let $X$ be any topological space. Let $X' = X \times \{a\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, a \rangle$ in $X'$ by $x'$ and for a subset $E \subseteq X$ let $E' = \{x' : x \in E\} = E \times \{a\} \subseteq X'$. For each $x' \in X'$, let $B(x') = \{\{x'\}\}$. For each $x \in X$, let $B(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in $X$ with $x \in U\}$. Let $T$ denote the unique topology on $A(X)$ which has $\{B(x) : x \in X\} \cup \{B(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate* of $X$. Similar proof as in [2], we get the following theorem.
Theorem 19. If $X$ is $C$-Tychonoff, then its Alexandroff Duplicate $A(X)$ is also $C$-Tychonoff.

Also a similar proof as in [15], we get the following theorem.

Theorem 20. If $X$ is $L$-Tychonoff, then its Alexandroff Duplicate $A(X)$ is also $L$-Tychonoff.

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