Fixed point results for geraghty type generalized F-contraction for weak $\alpha-$admissible mappings in metric-like spaces

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Abstract. In this paper, we establish the existence of some fixed point results for generalized (α, β, F)-Geraghty contraction in metric-like spaces. We provide an example in order to support our results where some consequence applications of such result will be considered in this article. The obtained results improve and extend some well-known common fixed point results in the literature.

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1. Introduction and Preliminaries

During the last decades, issues related to "Fixed Point Theory" in order to semantics domain with a notion of distance that has been extensively researched in different spaces. Recently, different generalizations of metric spaces have been introduced (for example see [12],[10],[22],[2],[28],[8],[10],[7],[6],[23],[27],[29],[32]). In 1994, Matthews [19] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the contraction mapping principle [9] can be generalized to the partial metric context for applications in program verifications. Later on, there have been several recent extensive researches on (common) fixed points for different contractions on partial metric spaces, see [[10],[1],[17],[1],[30],[24],[16],21,[3],[5], [11],[13],[25],[15],[20],[4]].

In this section, we recall some basic definitions and concepts.

Definition 1. *[19] Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is called a partial metric space if for all $x, y, z \in X$, the following conditions are satisfied:*

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called the notion of a partial metric space(PMS). The sequence $\{x_n\}$ in X converges to a point $x \in X$ if $\lim_{n \to \infty} p(x_n, x) = p(x, x)$. Also the sequence $\{x_n\}$ is called $p-$Cauchy if the $\lim_{n,m \to \infty} p(x_n, y_m)$ exists. The partial metric space (X, p) is called complete if for every p-Cauchy sequence $\{x_n\}_{\infty}^n$, there is some $x \in X$ such that

$$p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m).$$

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Harandi [14] introduced a new generalization of partial metric space, called a metric-like space. He established the existence and uniqueness of fixed points in a metric-like space as well as in a partially ordered metric-like space.

Definition 2. *[14] Let X be a nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a metric like space on X if for any $x, y, z \in X$, the following conditions hold:*

(σ_1) $\sigma(x, y) = 0 \Rightarrow x = y$,

(σ_2) $\sigma(x, y) = \sigma(y, x)$,

(σ_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is called a metric-like space.

It is clear that every partial metric space is a metric-like space but the converse is not true.

Example 1. *[14] Let $X = \{0, 1\}$ and*

$$\sigma(x, y) = \begin{cases} 2, & \text{if } x = y = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, σ) is a metric-like space but it is not a partial metric space. Note that $\sigma(0, 0) \nleq \sigma(0, 1)$.

Moreover, each metric-like space σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x, \epsilon) = \{y \in X :| \sigma(x,y) - \sigma(x,x) |< \epsilon\}, \ for \ all \ x \in X \ and \ \epsilon > 0.$$

Let (X, σ) and (Y, σ) be metric-like spaces, and let $f : X \to Y$ be a continuous mapping. Then

$$\lim_{n\to\infty} x_n = x \ \Rightarrow \ \lim_{n\to\infty} fx_n = fx.$$

A sequence $\{x_n\}_\infty^{n=0}$ of elements of X is called σ-Cauchy if the limit $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists. The metric-like space (X, σ) is called complete if for each σ-Cauchy sequence $\{x_n\}_n^\infty$, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

Remark 1. *[16] Let $X = \{0, 1\}$, and $\sigma(x, y) = 1$ for each $x, y \in X$. Consider the sequence $\{x_n\}$ such that $x_n = 1$ for each $n \in \mathbb{N}$. Then it is easy to see that $x_n \to 0$ and $x_n \to 1$, therefore the limit of a convergent sequence is not necessarily unique.*

Lemma 1. *[16] Let (X, σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X that converges to $x \in X$ such that, $\sigma(x, x) = 0$. Then, for all $y \in X$, we have $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$.*

Example 2. *Let $X = \mathbb{R}$ and $\sigma : X \times X \to [0, +\infty)$ be defined by*

$$\sigma(x, y) = \begin{cases} 2k, & if \ x = y = 0; \\ \\ k, & otherwise. \end{cases}$$

Then (X, σ) is a metric-like space, but for $k > 0$, it is not a partial metric space, as $\sigma(0, 0) \nleq \sigma(0, 1)$.

Now let $\mathcal{F}$ be the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition

$\lim_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

In 2015, Karapinar et al.[18] proved the following particular result(it corresponds to $S = 1$ and $\psi(t) = t$).

Theorem 1. *[18] Let (X, σ) be a complete metric-like space and $f : X \to X$ be a mapping. Suppose that there exists $\beta \in \mathcal{F}$ such that*

$$\sigma(fx, fy) \leq \beta(\sigma(x, y))\sigma(x, y), \tag{1}$$

for all $x, y \in X$. Then f has a unique fixed point.

In 2012, Samet et al. [26] introduced the concept of α-admissible mappings as the following.

Definition 3. *[26] Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then f is called α-admissible if for all $x, y \in X$ with $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$.*

Sintunavarat [30] presented the notion of weak α-admissible mappings as follows:

Definition 4. *[30] Let X be a nonempty set and let $\alpha : X \times X \to [0, \infty)$ be a given mapping. A mapping $f : X \to X$ is said to be a weak α-admissible mappings if the following condition holds:*

$$x \in X \text{ with } \alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, f^2 x) \geq 1.$$

Remark 2. *[30] It is customary to write $\mathcal{A}(X, \alpha)$ and $\mathcal{WA}(X, \alpha)$ to denote the collection of all α-admissible mappings on X and the collection of all weak α-admissible mappings on X. One can verify that $\mathcal{A}(X, \alpha) \subseteq \mathcal{WA}(X, \alpha)$.*

On the other hand, the concept of F-contraction was introduced by Wardowski in [31].

Definition 5. *[31] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following:*

(F_1) *F be a strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$,*

(F_2) *For each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$,*

(F_3) *There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.*

Recently, Piri and Kumam [21] investigated some fixed point theorems concerning F-contraction in complete metric spaces by replacing the condition (F_3) with the condition: (F_3) F is continuous on $(0, \infty)$.

Definition 6. *[31] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(Tx, Ty) > 0 \Rightarrow (\tau + F(d(Tx, Ty)) \leq \ F(d(x, y)),$$

for all $x, y \in X$.

2. Main Result

In this section, we shall state and prove our main results. We firstly recall the following classes of functions. Let $F : \mathbb{R}^+ \to \mathbb{R}$ is strictly increasing contraction function. Let $\mathcal{F}$ be the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition

$\lim_{n \to \infty} \beta(t_n) = 1$ implies $\lim_{n \to \infty} t_n = 0$.

Definition 7. *Let (X, σ) be a metric-like space and $\alpha : X \times X \to [0, \infty)$. A mapping $f : X \to X$ is said to be an (α, β, F)-Geraghty contraction mapping if there exist $\beta \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $\sigma(fx, fy) > 0$ and $\alpha(x, y) \geq 1$,*

$$\alpha(x, y)(\tau + F(\sigma(fx, fy)) \leq \beta(M_{x,y})F(M_{x,y}), \tag{2}$$

where

$$M_{x,y} = \max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy), \frac{\sigma(fx, y) + \sigma(x, fy)}{4}, \frac{[1 + \sigma(x, fx)]\sigma(y, fy)}{\sigma(x, y) + 1}\}.$$

Remark 3. *Since the functions belonging to $\mathcal{F}$ are strictly smaller than 1, the expression $\beta(M_{x,y})$ in 20 can be estimated from above as follows:*

$$\beta(M_{x,y}) < 1,$$

for all $x, y \in X$ with $\sigma(fx, fy) > 0$.

Lemma 2. *Let (X, σ) be a metric-like space, and let $f : X \to X$ is said to be an $(\alpha, \beta, F)-$Geraghty contraction mapping. Define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ is non-decreasing and $\lim_{n \to \infty} \sigma(x_n, x_{n+1}) = 0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Suppose that the sequence $\{x_n\}$ is not a Cauchy, then there exists $\epsilon > 0$ and two subsequences $\{x_{p_n}\}$ and $\{x_{q_n}\}$ of the sequence $\{x_n\}$ such that $p_n > q_n > n, \sigma(x_{p_n-1}, x_{q_n}) < \epsilon$ and $\sigma(x_{p_n}, x_{q_n}) \leq \epsilon$. This implies that

$$
\begin{aligned}
\epsilon &\leq \sigma(x_{p_n}, x_{q_n}) \\
&\leq \sigma(x_{p_n}, x_{q_n-1}) + \sigma(x_{q_n-1}, x_{q_n}) \\
&\leq \sigma(x_{p_n}, x_{p_n-1}) + \sigma(x_{p_n-1}, x_{q_n-1}) + \sigma(x_{q_n-1}, x_{q_n}) \\
&\leq \sigma(x_{p_n}, x_{p_n-1}) + \sigma(x_{p_n-1}, x_{q_n}) + 2\sigma(x_{q_n-1}, x_{q_n}) \\
&< \sigma(x_{p_n}, x_{p_n-1}) + \epsilon + 2\sigma(x_{q_n-1}, x_{q_n}).
\end{aligned}
$$

Since $\sigma(x_n, x_{n+1}) \neq 0$, we have

$$\lim_{n \to \infty} \sigma(x_{p_n}, x_{q_n}) = \lim_{n \to \infty} \sigma(x_{p_n}, x_{q_n-1}) \tag{3}$$

$$= \lim_{n \to \infty} \sigma(x_{p_n-1}, x_{q_n-1}) \tag{4}$$

$$= \lim_{n \to \infty} \sigma(x_{p_n-1}, x_{q_n}) \tag{5}$$

$$= \epsilon.$$

Since f is an (α, β, F)-Geraghty contraction mapping and $\alpha(x, y) \geq 1$, we have

$$(\tau + F(\sigma(x_{p_n-1}, x_{q_n-1}))) \leq \alpha(x_{p_n-1}, x_{q_n-1})(\tau + F(\sigma(x_{p_n-1}, x_{q_n-1}))) \leq \beta(M_{x_{p_n-1}, x_{q_n-1}})F(M_{x_{p_n-1}, x_{q_n-1}}),$$

where

$$M_{x_{p_n-1}, x_{q_n-1}} = \max\{\sigma(x_{p_n-1}, x_{q_n-1}), \sigma(x_{p_n-1}, fx_{p_n-1}), \sigma(x_{q_n-1}, fx_{q_n-1}),$$

$$\frac{\sigma(fx_{p_n-1}, x_{q_n-1}) + \sigma(x_{p_n-1}, fx_{q_n-1})}{4},$$

$$\frac{[1 + \sigma(x_{p_n-1}, fx_{p_n-1})]\sigma(x_{q_n-1}, fx_{q_n-1})}{\sigma(x_{p_n-1}, x_{q_n-1}) + 1}\}$$

$$= \max\{\sigma(x_{p_n-1}, x_{q_n-1}), \sigma(x_{p_n-1}, x_{p_n}), \sigma(x_{q_n-1}, x_{q_n}),$$

$$\frac{\sigma(x_{p_n}, x_{q_n-1}) + \sigma(x_{p_n-1}, x_{q_n})}{4},$$

$$\frac{[1 + \sigma(x_{p_n-1}, x_{p_n})]\sigma(x_{q_n-1}, x_{q_n})}{\sigma(x_{p_n-1}, x_{q_n-1}) + 1}\}.$$

Letting $n \to \infty$ in the above inequalities and using (2.2), (2.3) and (2.4), we obtain

$$\lim_{n\to\infty} M_{(x_{p_n-1}, x_{q_n-1})} = \epsilon. \tag{6}$$

Since $\lim_{n\to\infty} \beta(M_{(x_{p_n-1}}, x_{q_n-1}) \le 1$, we conclude that

$$\begin{aligned} \tau + F(\epsilon) &\le \beta(\epsilon)F(\epsilon) \\ &\le F(\epsilon), \end{aligned} \tag{7}$$

a contradiction since $\tau > 0$. Hence

$$\lim_{n\to\infty} \sigma(x_n, x_m) = 0.$$

We denote with $\Xi(X, \alpha, \beta, F)$ the collection of all almost generalized $(\alpha, \beta, F)-$contractive mappings.

Theorem 2. *Let (X, σ) be a metric-like space and $\alpha : X \times X \to [0, \infty)$. A mapping $f : X \to X$ be an $(\alpha, \beta, F)-$Geraghty contraction mapping. Assume that the following conditions are satisfied:*

(i) $f \in \Xi(X, \alpha, \beta, F) \cap \mathcal{WA}(X, \alpha)$.

(ii) There exists $x_0 \in X$ such that $\sigma(x_0, fx_0) \ge 1$.

(iii) f is $\sigma-$continuous.

Then f has a unique fixed point $z \in X$ with $\sigma(z, z) = 0$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \ge 1$. We define a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. If $\sigma(x_n, x_{n+1}) = 0$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of f and it is done. Now, suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Since $f \in \mathcal{WA}(X, \alpha, \beta)$ and $\alpha(x_0, fx_0) \ge 1$, we have

$$\alpha(x_1, x_2) = \alpha(fx_0, ffx_0) \ge 1, \; \alpha(x_2, x_3) = \alpha(fx_1, ffx_1) \ge 1.$$

Using this process again, we get $\alpha(x_n, x_{n+1}) \geq 1$.

Since $f : X \to X$ is (α, β, F)-Geraghty contraction mapping with $\alpha(fx_{n-1}, ffx_{n-1}) = \alpha(x_n, x_{n+1}) \geq 1$, we have

$$
\begin{aligned}
0 < \tau + F(\sigma(x_n, x_{n+1})) & \leq \alpha(x_n, x_{n+1})(\tau + F(\sigma(fx_{n-1}, fx_n))) \\
& \leq \beta(M_{x_{n-1}, x_n})F(M_{x_{n-1}, x_n}),
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
M_{x_{n-1}, x_n} & = \max\{\sigma(x_{n-1}, x_n), \sigma(x_{n-1}, fx_{n-1}), \sigma(x_n, fx_n), \\
& \quad \frac{\sigma(x_{n-1}, fx_n) + \sigma(fx_{n-1}, x_n)}{4}, \frac{[1 + \sigma(x_{n-1}, fx_{n-1})]\sigma(x_n, fx_n)}{\sigma(x_{n-1}, x_n) + 1}\} \\
& = \max\{\sigma(x_{n-1}, x_n), \sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \\
& \quad \frac{\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)}{4}, \frac{[1 + \sigma(x_{n-1}, x_n)]\sigma(x_n, x_{n+1})}{\sigma(x_{n-1}, x_n) + 1}\} \\
& = \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{\sigma(x_{n-1}, x_{n+1})}{4}, \sigma(x_n, x_{n+1})\} \\
& < \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})}{4}\} \\
& = \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})}{4}\} \\
& = \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}.
\end{aligned} \tag{9}
$$

If $\max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} = \sigma(x_{n-1}, x_n)$, then

$$
\begin{aligned}
F(\sigma(x_{n-1}, x_n)) & \leq \beta(\sigma(x_{n-1}, x_n))F(\sigma(x_{n-1}, x_n) - \tau \\
& \leq F(\sigma(x_{n-1}, x_n)),
\end{aligned}
$$

which is a contradiction. Thus,we conclude that

$$
\max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} = \sigma(x_n, x_{n+1})
$$

, for all $n \in \mathbb{N}$. Then

$$
F(\sigma(x_n, x_{n+1})) \leq F(\sigma(x_n, x_{n+1})) - \tau
$$

, for all $n \in \mathbb{N}$. Repeating this process, we obtain

$$
F(\sigma(x_n, x_{n+1})) \leq F(\sigma(x_0, x_1)) - n\tau \tag{10}
$$

By taking $n \to \infty$ in (2.11) that shows $\lim_{n\to\infty} F(\sigma(x_n, x_{n+1})) = -\infty$, hence

$$
\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{11}
$$

Now, by Lemma 2, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that

$$
\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{12}
$$

Since f is continuous, we claim $z = fz$. Assume the contrary, that is $z \neq fz$. In this case, there exists a sequence $\{x_n\}$ for $n_0 \in \mathbb{N}$ such that $\sigma(fx_n, fz) > 0$ for all $n \geq n_0$. Then from our assumption (with $n \geq n_0$), we have

$$
\begin{aligned}
\tau + F(\sigma(x_{n+1}, fz)) &= \tau + F(\sigma(fx_n, fz)) \\
&\leq \alpha(x_n, z)(\tau + F(\sigma(x_n, fz))) \\
&\leq \beta(M_{x_n,z})F(M_{x_n,z}),
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
M_{x_n,z} &= \max\{\sigma(x_n, z), \sigma(x_n, fx_n), \sigma(z, fz), \frac{\sigma(fx_n, z) + \sigma(x_n, fz)}{4}, \\
&\qquad \frac{[1 + \sigma(x_n, fx_n)]\sigma(z, fz)}{\sigma(x_n, z) + 1}\} \\
&= \max\{\sigma(x_n, z), \sigma(x_n, x_{n+1}), \sigma(z, fz), \frac{\sigma(x_{n+1}, z) + \sigma(x_n, fz)}{4}, \\
&\qquad \frac{[1 + \sigma(x_n, x_{n+1})]\sigma(z, fz)}{\sigma(x_n, z) + 1}\}.
\end{aligned} \tag{14}
$$

By taking $n \to \infty$, we get

$$
\begin{aligned}
\lim_{n \to \infty} M_{x_n,z} &= \max\{\sigma(z, z), \sigma(z, fz), \sigma(z, fz), \frac{\sigma(fz, z) + \sigma(z, fz)}{4}, \\
&\qquad \frac{[1 + \sigma(z, fz)]\sigma(z, fz)}{\sigma(z, z) + 1}\} \\
&= \max\{\sigma(z, fz), \frac{\sigma(z, fz)}{4}\} \\
&= \sigma(z, fz).
\end{aligned} \tag{15}
$$

Therefore, by taking the limits as $n \to \infty$ in (2.12), we get

$$
\begin{aligned}
F(\sigma(z, fz)) &\leq \beta(\sigma(z, fz)))F(\sigma(z, fz)) - \tau \\
&\leq F(\sigma(z, fz)) - \tau,
\end{aligned} \tag{16}
$$

which gives a contradiction. Hence, we conclude z is a fixed point of f.

Further, suppose that $z, \acute{z}$ are two fixed points of f such that $z \neq \acute{z}$ and $\alpha(fz, ff\acute{z}) = \alpha(z, \acute{z}) \geq 1$ and $\sigma(fz, f\acute{z}) = \sigma(z, \acute{z}) \geq 0$. From (2.1), we have

$$
\begin{aligned}
\tau + F(\sigma(z, \acute{z})) &= \tau + F(\sigma(fz, f\acute{z})) \\
&\leq \alpha(z, \acute{z})(\tau + F(\sigma(fz, f\acute{z}))) \\
&\leq \beta(M_{z,\acute{z}})F(M_{z,\acute{z}}),
\end{aligned}
$$

where

$$
M_{z,\acute{z}} = \max\{\sigma(z, \acute{z}), \sigma(z, f\acute{z}), \sigma(\acute{z}, f\acute{z}), \frac{\sigma(fz, \acute{z}) + \sigma(z, f\acute{z})}{4},
$$

$$\frac{[1 + \sigma(z, fz)]\sigma(\acute{z}, f\acute{z})}{\sigma(z, \acute{z}) + 1}\}$$

$$= \max\{\sigma(z, \acute{z}), \sigma(z, \acute{z}), \sigma(\acute{z}, \acute{z}), \frac{\sigma(z, \acute{z})}{2}, \sigma(\acute{z}, f\acute{z})$$

$$= \max\{\sigma(z, \acute{z}), \frac{\sigma(z, \acute{z})}{2}$$

$$= \sigma(z, \acute{z}).$$

Hence

$$\tau + F(\sigma(z, \acute{z})) \le \beta(\sigma(z, \acute{z}))F(\sigma(z, \acute{z}))$$
$$\le F(\sigma(z, \acute{z})),$$

which is a contradiction. Hence $\sigma(z, \acute{z}) = 0$, that is $z = \acute{z}$. Thus, we conclude that the fixed point of f is unique.

Next, we will prove that $\sigma(z, z) = 0$. If $\sigma(fz, fz) = \sigma(z, z) > 0$ and $\alpha(fz, ffz) = \alpha(z, z) \ge 1$, then from (2.1)and applying the routine calculation as mentioned above, we get

$$\tau + F(\sigma(z, z)) = \tau + F(\sigma(fz, fz))$$
$$\le \alpha(z, z)(\tau + F(\sigma(fz, fz)))$$
$$\le \beta(M_{z,z})F(M_{z,z}),$$

where

$$M_{z,z} = \max\{\sigma(z, z), \sigma(z, fz), \sigma(z, fz), \frac{\sigma(fz, z) + \sigma(z, fz)}{4}, \frac{[1 + \sigma(z, fz)]\sigma(z, fz)}{\sigma(z, z) + 1}\}$$
$$= \sigma(z, z).$$

Hence

$$\tau + F(\sigma(z, z)) < \beta(\sigma(z, z))F(\sigma(z, z))$$
$$\le F(\sigma(z, z)),$$

is a contradiction, thus, $\sigma(z, z) = 0$.

The following two corollaries are direct results of Theorem 2.

Corollary 1. *Let (X, σ) be a complete metric-like space, $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be two given mapping satisfying the following conditions:*

(i) $f \in \Xi(X, \alpha, \beta, F) \cap \mathcal{WA}(X, \alpha)$.

(ii) There exists $x_0 \in X$ such that $\sigma(x_0, fx_0) \ge 1$.

(iii) f is $\sigma-$continuous.

Then f has a unique fixed point $z \in X$ such that $\sigma(z,z) = 0$.
   Proof. It follows from Theorem 2 by putting

$$M_{x,y} = \max\{\sigma(x,y), \sigma(x, fx), \sigma(y, fy)\}.$$

Corollary 2. *Let (X, σ) be a complete metric-like space, $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be two given mapping satisfying the following conditions:*

   (i) $f \in \Xi(X, \alpha, \beta, F) \cap \mathcal{WA}(X, \alpha)$.

   (ii) There exists $x_0 \in X$ such that $\sigma(x_0, fx_0) \geq 1$.

   (iii) f is $\sigma-$continuous.

Then f has a unique fixed point $z \in X$ such that $\sigma(z,z) = 0$.
   Proof. It follows from Theorem 2 by putting $M_{x,y} = a\sigma(x,y) + b\sigma(x, fx) + c\sigma(y, fy) + e[\frac{\sigma(fx,y)+\sigma(x,fy)}{4}] + e[\frac{[1+\sigma(x,fx)]\sigma(y,fy)}{\sigma(x,y)+1}]$.
   For all $x, y \in X$, we have

$$
\begin{aligned}
M_{x,y} &= a\sigma(x,y) + b\sigma(x, fx) + c\sigma(y, fy) + e[\frac{\sigma(fx,y) + \sigma(x,fy)}{4}] \\
&\leq (a+b+c+2e)\max\{\sigma(x,y), \sigma(x,fx), \sigma(y,fy), \frac{\sigma(fx,y)+\sigma(x,fy)}{4}, \frac{[1+\sigma(x,fx)]\sigma(y,fy)}{\sigma(x,y)+1} \\
&\quad \frac{[1+\sigma(x,fx)]\sigma(y,fy)}{\sigma(x,y)+1}\} \\
&\leq \max\{\sigma(x,y), \sigma(x,fx), \sigma(y,fy), \frac{\sigma(fx,y)+\sigma(x,fy)}{4}, \frac{[1+\sigma(x,fx)]\sigma(y,fy)}{\sigma(x,y)+1}\}.
\end{aligned}
$$

Then, we see that (2.1) is a consequence of (2.14), then the corollary is proved.

Example 3. *Let $X = \{0, 1, 2\}$. Let $\sigma : X \times X \to \mathbb{R}$ be a metric like function define by*
$\sigma(0,0) = \sigma(1,1) = \sigma(2,2) = 0$,
$\sigma(1,2) = \sigma(2,1) = 3$,
$\sigma(2,0) = \sigma(0,2) = 2$,
$\sigma(0,1) = \sigma(1,0) = \frac{3}{2}$.
It is easy to see that (X, σ) is a complete metric-like space. Also, define $f : X \to X$ be given by $f0 = 0 = f1$ and $f2 = 1$.
   Define $\alpha : [0, +\infty) \to [0, 1)$ by

$$
\alpha(x,y) = \begin{cases} 1 & \text{if } x \in \{0,1,2\} \\ 0 & \text{if } \ otherwise. \end{cases}
$$

   Define $\beta : [0, \infty) \to [0, 1)$ by

$$
\beta(t) = \begin{cases} \dfrac{1}{1 + \frac{1}{7}t} & \text{if } t > 0 \\[2mm] \dfrac{1}{2} & \text{if } \ t = 0. \end{cases}
$$

Suppose that $F(t) = e^t$ and $\tau = \frac{1}{4}$. The function (f) satisfies the inequality (20). For that, given $x, y \in X$. Then we have the following cases:

Case 1: $x = 0$ *and* $x = 1$.
Then $\alpha(0, 1) = 1$ *and* $M_{0,1} = \max\{0, 0, 0, 0, 1\} = 1$.
$\sigma(f0, f1) = \sigma(0, 0) = 0$. *Now*

$$
\begin{aligned}
0 < \alpha(0, 1)(\tau + F(\sigma(f0, f1))) &= \tau + F(\sigma(0, 0)) \\
&= (\tau + F(0) \\
&= \tau \\
&\leq \beta(M_{0,1})F(M_{0,1}) \\
&= \beta(1)F(1) \\
&= e
\end{aligned}
\tag{17}
$$

Case 2: $x = 0$ *and* $y = 2$.
Then $\alpha(0, 2) = 1$ *and* $M_{0,2} = \max\{2, 0, 3, \frac{13}{16}, 1\} = 3$.
$\sigma(f0, f2) = \sigma(0, 1) = \frac{3}{2}$. *Now*

$$
\begin{aligned}
0 < \alpha(0, 2)(\tau + F(\sigma(f0, f2))) &= \tau + F(\frac{3}{2}) \\
&= \tau + \frac{3}{2} \\
&\leq \beta(M_{0,2})F(M_{0,2}) \\
&= \beta(3)F(3) \\
&= 3e^3,
\end{aligned}
$$

Case 3: $x = 1$ *and* $y = 2$.
Then $\alpha(1, 2) = 1$ *and* $M_{1,2} = \max\{3, \frac{3}{2}, 3, \frac{1}{2}, \frac{15}{8}\} = 3$.
$\sigma(f1, f2) = \sigma(0, 1) = \frac{3}{2}$. *Now*

$$
\begin{aligned}
0 < \alpha(0, 2)(\tau + F(\sigma(f0, f2))) &= \tau + F(\frac{3}{2}) \\
&= \tau + \frac{3}{2} \\
&\leq \beta(M_{0,2})F(M_{0,2}) \\
&= \beta(3)F(3) \\
&= 3e^3,
\end{aligned}
$$

Thus, all the conditions of Theorem 2 are satisfied and hence f has a unique fixed point.

3. Consequences

In this section, we derive the analog of Theorem 2 in the context of partial metric spaces (PMS). In the following theorem we conclude the existence and the uniqueness of

a fixed point of the given mapping.

Theorem 3. *Let (X, p) be a a complete partial metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $f : X \to X$ be an $(\alpha, \beta, F)-$ Geraghty contraction mapping. Suppose there exist $f \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $\sigma(fx, fy) > 0$ and $\alpha(x, y) \geq 1$,*

$$0 < \alpha(x,y)(\tau + F(\sigma(fx, fy)) \leq \beta(M_{x,y})F(M_{x,y}), \tag{18}$$

where

$$M_{x,y} = \max\{\max\{p(x,y), p(x, fx), p(y, fy), \frac{p(fx, y) + p(x, fy)}{4}, \frac{[1 + p(x, fx)]p(y, fy)}{p(x, y) + 1}\}.$$

Then f has a unique fixed point $z \in X$ with $p(z, z) = 0$.

Proof. Since every partial metric space is a metric-like space, we obtain the proof by following the proof in Theorem 2. We now show the uniqueness of the fixed point of f. Suppose there is another fixed point $y^* \in X$ of f, such that $x^* \neq y^*$. Thus from Lemma ??, we have $p(x^*, y^*) > 0$. From (p_2), we have $p(fx^*, fy^*) = p(x^*, y^*) > 0$. Thus

$$\begin{aligned}
0 < \tau + F(p(x^*, y^*)) &\leq \alpha(x^*, y^*)(\tau + F(p(fx^*, fy^*))) \\
&\leq \beta(M_{x^*, y^*})F(M_{x^*, y^*}) \\
&= \beta(p(x^*, y^*))F(p(x^*, y^*)) \\
&\leq F(p(x^*, y^*)),
\end{aligned}$$

where

$$\begin{aligned}
M_{x^*, y^*} &= \max\{p(x^*, y^*), p(x^*, fx^*), p(y^*, fy^*), \frac{p(fx^*, y^*) + p(x^*, fy^*)}{4}, \\
&\quad \frac{[1 + p(x^*, fx^*)]p(y^*, fy^*)}{p(x^*, y^*) + 1}\} \\
&= \max\{p(x^*, y^*), p(x^*, x^*), p(y^*, y^*), \frac{p(x^*, y^*) + p(x^*, y^*)}{4}, \\
&\quad \frac{[1 + p(x^*, y^*)]p(y^*, y^*)}{p(x^*, y^*) + 1}\} \\
&= \max\{p(x^*, y^*), p(x^*, x^*), p(y^*, y^*), \frac{p(x^*, y^*)}{2}, p(y^*, y^*)\} \\
&= p(x^*, y^*).
\end{aligned}$$

This is a contradiction, and hence $x^* = y^*$.

Theorem 4. *Let (X, p) be a a complete partial metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $f : X \to X$ be an $(\alpha, \beta, F)-$ Geraghty contraction mapping. Suppose there exist $f \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $\sigma(fx, fy) > 0$ and $\alpha(x, y) \geq 1$,*

$$0 < \alpha(x,y)(\tau + F(\sigma(fx, fy)) \leq \beta(M_{x,y})F(M_{x,y}), \tag{19}$$

where

$$M_{x,y} = \max\{\max\{p(x,y), p(x, fx), p(y, fy)\}\}.$$

Then f has a unique fixed point $z \in X$ with $p(z, z) = 0$.

Theorem 5. *Let (X, p) be a a complete partial metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $f : X \to X$ be an (α, β, F) Geraghty contraction mapping. Suppose there exist $f \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $\sigma(fx, fy) > 0$ and $\alpha(x, y) \geq 1$,*

$$0 < \alpha(x, y)(\tau + F(\sigma(fx, fy)) \leq \beta(M_{x,y})F(M_{x,y}), \tag{20}$$

where

$$M_{x,y} = p(x, y).$$

Then f has a unique fixed point $z \in X$ with $p(z, z) = 0$.

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