EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 4, 2018, 1177-1190
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Common Fixed Point Theorems in Metric spaces with Applications

Pushpendra Semwal[1,*], Komal[1]

[1] *Department of Mathematics, SoPS, Doon University, Dehradun, India*

Abstract. In this paper, we investigate the existence and uniqueness of common fixed point theorems for certain contractive type of mappings. As an application the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming are discuss by using the our results.

2010 Mathematics Subject Classifications: 49L20, 49L99, 54H25, 90C39

Key Words and Phrases: Common fixed point, asymptotically continuous, weakly compatible mapping.

1. Introduction

Bellman and Lee [3] first introduced the basic form of the functional equations in dynamic programming is as follows:

$$f(x) = opt_{y \in D} H(x, y, f(T(x, y))) \forall x \in S \tag{1}$$

where opt represent sup. or inf., x, y denote the state and decicion vectors respectively, T stands for the transformation of the process and $f(x)$ represents the optimal return function with the initial state x.Afterwards, the existence and uniqueness of fixed point solutions for several classes of contractive mappings and functional equations studied by many investigators such as Bhakta and Mitra [5], Liu [15], Liu and ume [20], Pathak and Fisher [21], Baskaran and Subhramanyam [1] and others.

Ray [22] proved two common fixed point theorems for three self mappings f,g and h in the complete metric space using the following contractive condition:

$$d(fx, gy) \leq d(hx, hy) - w(d(hx, hy)), \forall x, y \in X \tag{2}$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v11i4.3298

Email addresses: psrsdm@gmail.com (P. Semwal)

Further Liu[15] established common fixed point theorem and introduced a class of mappings in a complete metric space as follows:

$$d(fx, gy) \leq max\{d(hx, hy), d(hx, fx), d(hy, gy)\} - w(max\{d(hx, hy), d(hx, fx), d(hy, gy)\}). \tag{3}$$

Recall that the notion of orbitally complete metric space and orbitally continuous mapping were introduced by Ciric [9] . These definitions were extended to the case of two or three mappings by Sastry et al.[12]. Some common fixed point results in this situation were obtained in [12] . We give now respective definitions for pairs of mappings given in literature.

2. Priliminaries

Definition 1 (6). *A self map f on a metric space (X, d) is said to be asymptotically regular at a point x in X if $lim_{n\to\infty} d(f^n(x), f^{n+1}(x)) = 0$. Where $f^n(x)$ denotes the n^{th} iterate of f at x.*

Definition 2 (6). *Let f and g be two self mappings of X and $\{x_n\}$ a sequence in X, then $\{x_n\}$ is said to be asymptotically g- regular with respect to f if $lim_{n\to\infty} d(fx_n, gx_n) = 0$.*

Definition 3 (9). *Let $\{x_n\}$ is a sequence which is asymptotically g- regular with respect to f, then $O(f, x_n) = \{fx_1, fx_2, fx_3, ...fx_n, ...\}$ is called asymptotic orbit of f.*

Definition 4 (12). *X is said to be f-asymptotically complete if every Cauchy sequence of the form $\{fx_n\}$ converges in X.*

Definition 5 (12). *A self map f is said to be asymptotically continuous if it is continuous on closure of $O(f, x_n)$.*

Definition 6. *Two self maps g and h of X are said to be weakly commuting if $d(ghx, hgx) \leq d(gx, hx)$ $\forall x \in X$*

Definition 7 (12). *Let f, g and h be three self maps on a metric space X.*
(i) If for a point $x_0 \in X$,there exists a sequence $\{x_n\}$ in X such that $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$, $n = 0, 1, 2....$ Then the set $O(x_0, f, g, h) = \{Tx_n | n = 0, 1, 2...\}$ is called the orbit of (f, g, h) at x_0.

(ii) The space (X, d) is said to be $(f, g, h)-$ orbitally complete if every Cauchy sequence in $O(x_0, f, g, h)$ converges in X.

(iii) The map h is said to be (f, g, h)- orbitally continuous at x_0 if it is continuous on $O(x_0, f, g, h)$.

(iv) The pair (f, g) is said to be asymptotically regular w.r.to h at x_0 if there exists a sequence $\{x_n\}$ in X such that $fx_{2n} = hx_{2n+1}$, $gx_{2n+1} = hx_{2n+2}$; $n = 0, 1, 2, ...$ and $d(hx_n, h_{n+1}) \to 0$ as $n \to \infty$.

Throughout in this paper, we assume that $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$, w and N denote the set of all non-negative and positive integers respectively.

$$W = \{w : w : R^+ \to R^+ is\ continuous\ mappings\ with\ 0 < w(t) < t \ \forall\ t > 0\} \tag{4}$$

Let $\Phi = \{\phi : \phi : [0, \infty) \to [0, \infty)\}$ satisfying the following conditions:
(i) ϕ is continuous and non-decreasing
(ii) $\phi(t) < t \ \forall t \in [0, \infty)$
(iii) $lim_{n\to\infty}\phi(t_n) = 0 \iff lim_{n\to\infty}t_n = 0$.

Let Ψ= $\{\psi : \psi : [0, \infty) \to [0, \infty)\}$ satisfying the following conditions:
(i) ψ is non-decreasing
(ii) $\phi(t) < \psi(t) \ \forall t > 0$
(iii)$\psi(a + b) \leq \psi(a) + \psi(b) \ \forall a, b \in [0, \infty)$
(iv) $\psi(t) < t \ \forall t \in [0, \infty)$

The aim of this paper is to provide the sufficient conditions for the existence and uniqueness of common fixed point for the following type of contractive mappings metric space (X, d).

$$\begin{aligned}\psi(d(fx, gy)) \leq max\{\phi(d(hx, hy)), \phi(d(hx, fx), \phi(\frac{1}{2}([d(hx, hy) + d(fx, gy)]))), \phi(d(hy, gy))\} \\ -w(max\{\phi(d(hx, hy)), \phi(d(hx, fx)), \phi(d(hy, gy)), \phi(\frac{1}{2}([d(hx, hy) + d(fx, gy)]))\}).\end{aligned} \tag{5}$$

for all $x, y \in X$.Where ψ and ϕ are defined above.

As an applications, we discuss the existence and uniqueness of common solutions of the following functional equations arising in dynamic programming.

$$f(x) = opt_{y\in D}\{u(x, y) + H(x, y, f(T(x, y)))\}\forall x \in S \tag{6}$$

and

$$f_i(x) = opt_{y\in D}\{u(x, y) + H_i(x, y, f_i(T(x, y)))\}\forall x \in S\&i \in \{1, 2, 3\} \tag{7}$$

3. Main Results

Theorem 1. *Let f, g and h be three self maps on a metric space X satisfying:*
(i) either f commute with h or g commute with h.
(ii) there exists $w \in W$ such that (5) hold for all $x, y \in X$.
(iii) The pair (f, g) is asymptotically regular with respect to h at x_0.
(iv)The space X is (f, g, h)-orbitally complete at x_0 and h is orbitally continuous at x_0.
Then f, g and h have a unique common fixed point in X.

Proof Since (f, g) is asymptotically respect to h at x_0, there exists a sequence $\{x_n\}$ in X such that $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$, $n = 0, 1, 2, ...$ and $d(hx_n, hx_{n+1}) \to zero$ as $n \to \infty$.
Now we show that hx_n is Cauchy.On contrary suppose that hx_n is not Cauchy, then there exists an $\epsilon > 0$ and positive integers m_k and n_k with $m_k < n_k$ such that $d(hx_{m_k}, hx_{n_k}) \geq \epsilon$ and $d(hx_{m_k}, hx_{n_k-1}) \leq \epsilon$ for all $k = 0, 1, 2, ...$.Since $d(hx_{m_k}, hx_{n_k}) \leq d(hx_{m_k}, hx_{n_k-1}) + d(hx_{n_k-1}, hx_{n_k})$.Then we obtain $d(hx_{m_k}, hx_{n_k}) \to \epsilon$ as $k \to \infty$.
Now there are four cases: (i) m_k is even and n_k is odd (ii) m_k is even and n_k is even (iii) m_k is odd and n_k is even (iv) m_k is odd and n_k is odd.
Suppose m_k is even and n_k is odd, we have

$$\begin{aligned}
\psi(d(hx_{m_k}, hx_{n_k})) &\leq \psi(d(hx_{m_k}, hx_{m_k+1})) + \psi(d(hx_{m_k+1}, hx_{n_k+1})) + \psi(d(hx_{n_k+1}, hx_{n_k})) \\
&\leq \psi(d(hx_{m_k}, hx_{m_k+1})) + max\{\phi(d(hx_{m_k}, hx_k)), \phi(d(fx_{m_k}, hx_{m_k})), \\
&\quad \phi(d(gx_{n_k}, hx_{n_k})), \phi(\frac{1}{2}[d(hx_{m_k}, hx_{n_k}) + d(fx_{m_k}, gx_{n_k})])\} \\
&- w(max\{\phi(d(hx_{m_k}, hx_k)), \phi(d(fx_{m_k}, hx_{m_k})), \phi(d(gx_{n_k}, hx_{n_k})), \\
&\quad \phi(\frac{1}{2}[d(hx_{m_k}, hx_{n_k}) + d(fx_{m_k}, gx_{n_k})])\}) + \psi(d(hx_{n_k+1}, hx_{n_k}))
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\psi(\epsilon) \leq \phi(\epsilon) - w(\phi(\epsilon)) < \phi(\epsilon)$$

a contradiction.In the remaining cases we have a similar situation.Hence $\{hx_n\}$ is Cauchy.Since X is (f, g, h)orbitally complete at x_0, it follows that there exist $z \in X$ s.t. $hx_n \to z$ as $n \to \infty$. Now, again

$$\begin{aligned}
\psi(d(fx_{2n}, gz)) &\leq max\{\phi(d(hx_{2n}, hz)), \phi(d(fx_{2n}, hx_{2n})), \phi(d(g_z, h_z)), \\
&\quad \phi(\frac{1}{2}[d(hx_{2n}, hz) + d(fx_{2n}, gz)])\} - w((max\{\phi(d(hx_{2n}, hz)), \phi(d(fx_{2n}, hx_{2n})), \\
&\quad \phi(d(gz, hz)), \phi(\frac{1}{2}[d(hx_{2n}, hz) + d(fx_{2n}, gz)])\}))
\end{aligned}$$

and

$$\psi(d(fz, gx_{2n+1})) \leq max\{\phi(d(hz, hx_{2n+1})), \phi(d(fz, hz)), \phi(d(g_{x_{2n+1}}, h_{x_{2n+1}})),$$

$$\phi(\frac{1}{2}[d(hz, hx_{2n+1}) + d(fz, gx_{2n+1})])\} - w((max\{\phi(d(hz, hx_{2n+1})), \phi(d(fz, hz)),$$
$$- \quad \phi(d(gx_{2n+1}, hx_{2n+1})), \phi(\frac{1}{2}[d(hz, hx_{2n+1}) + d(fz, gx_{2n+1})])\}))$$

Taking $k \to \infty$, we obtain

$$\begin{aligned}\psi(d(z, gz)) &\leq max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi(\frac{1}{2}[d(hz, z) + d(z, gz)])\} \\ &- w((max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(d(gz, hz)), \phi(\frac{1}{2}[d(hz, z) + d(z, gz)])\})) \quad (8)\end{aligned}$$

and

$$\begin{aligned}\psi(d(fz, z)) &\leq max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(d(gz, hz)), \phi(\frac{1}{2}[d(hz, z) + d(fz, z)])\} \\ &- w((max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(d(gz, hz)), \phi(\frac{1}{2}[d(hz, z) + d(fz, z)])\})) \quad (9)\end{aligned}$$

Since h is orbitally continuous at x_0 and $fh = hf$, we infer that $fhx_{2n} = hf_{2n} \to Tz$ as $n \to \infty$.Similarly $ghx_{2n+1} = hgx_{2n+1} \to hz$ as $n \to \infty$.
Again,

$$\begin{aligned}\psi(d(fhx_{2n}, gx_{2n+1})) &\leq max\{\phi(d(hhx_{2n}, hx_{2n+1})), \phi(d(fhx_{2n}, hhx_{2n})), \\ &\phi(d(gx_{2n+1}, hx_{2n+1})), \phi(\frac{1}{2}[d(hhx_{2n}, hx_{2n+1}) + d(fhx_{2n}, gx_{2n+1})])\} \\ &- w(max\{\phi(d(hhx_{2n}, hx_{2n+1})), \phi(d(fhx_{2n}, hhx_{2n})), \phi(d(gx_{2n+1}, hx_{2n+1})), \\ &\phi(\frac{1}{2}[d(hhx_{2n}, hx_{2n+1}) + d(fhx_{2n}, gx_{2n+1})])\})\end{aligned}$$

Taking $k \to \infty$, we obtain

$$\begin{aligned}\psi(d(hz, z)) &\leq max\{\phi(d(hz, z)), \phi(d(hz, hz)), \phi(d(z, z)), \phi(\frac{1}{2}[d(hz, z) + d(hz, z)])\} \\ &- w((max\{\phi(d(hz, z)), \phi(d(hz, hz)), \phi(d(z, z)), \phi(\frac{1}{2}[d(hz, z) + d(hz, z)])\}))\end{aligned}$$

implies that

$$\psi(d(hz, z)) \leq \phi(d(z, hz)) - w(\phi(d(z, hz)) < \phi(d(z, hz))$$

a contradiction.Hence $hz = z$.Using (8) and (9) together with $Tz = z$, we infer that $fz = gz = hz = z$.Further uniqueness of common fixed point can easily prove.

Taking $\psi(t) = t$ and $\phi(t) = ht$ where $< h < 1$, we state the following

Corollary 1. *Let A, B and T be self maps on a metric space (X, d) such that T commutes with both A and B and the pair (A, B) is asymptotically regular w.r.to T at $x_0 \in X$, X is orbitally complete and T is orbitally continuous at x_0 and*

$$\begin{aligned} d(Ax, By) &\leq \phi\{max\{d(Tx, Ty), d(Ax, Tx)), d(By, Ty), \frac{1}{2}[d(Tx, Ty) + d(Ax, By)]\}\} \\ &- w((max\{d(Tx, Ty), d(Ax, Tx)), d(By, Ty), \frac{1}{2}[d(Tx, Ty) + d(Ax, By)]\}\})) \end{aligned}$$

for all $x, y \in X$. Then A, B and T have unique common fixed point in X.

Theorem 2. *Let (X, d) be a metric space and f, g and h be self mappings on X such that $f(X) \cup g(X) \subseteq h(X)$.If there exists a $w \in W$ satisfying (5).Then the pair (f, h) and (g, h) have a coincidence point in X, provided that (i) X is h-asymptotically complete, (ii) h is asymptotically continuous and (iii) h is weakly commute with f and g.Further f, g and h have a unique common fixed point in X.*

Proof Let $x_0 \in X$ be any point in X. Since $f(X) \cup g(X) \subseteq h(X)$.We choose sequence $\{x_n\} \in X$ such that $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$ for all $n \in w$.
By (5), we conclude that

$$\begin{aligned} \psi(d(hx_{2n+1}, hx_{2n+2})) &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq max\{\phi(d(h_{2n}, hx_{2n+1})), \phi(d(hx_{2n}, fx_{2n})), \phi(d(hx_{2n+1}, gx_{2n+1})), \\ &\quad \phi(\frac{1}{2}([d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1})]))\} - w(max\{\phi(d(h_{2n}, hx_{2n+1})), \\ &\quad \phi(d(hx_{2n}, fx_{2n})), \phi(d(hx_{2n+1}, gx_{2n+1})), \phi(\frac{1}{2}([d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1})]))\} \end{aligned}$$

This yields

$$\begin{aligned} \psi(d_{2n+1}) &\leq max\{\phi(d_{2n}), \phi(d_{2n}), \phi(d_{2n+1}), \phi(\frac{1}{2}(d_{2n} + d_{2n+1}))\} \\ &\quad -w(max\{\phi(d_{2n}), \phi(d_{2n}), \phi(d_{2n+1}), \phi(\frac{1}{2}(d_{2n} + d_{2n+1}))\}). \end{aligned}$$

Suppose $d_{2n+1} > d_{2n}$, then $\phi(d_{2n+1}) > \phi(d_{2n})$.Using (5), we have

$$\psi(d_{2n+1}) \leq \phi(d_{2n+1}) - w(\phi(d_{2n+1})) < \phi(d_{2n+1})$$

a contradiction.Consequently, we have $d_{2n+1} \leq d_{2n}$, from (5) we have

$$\psi(d_{2n+1}) \leq \phi(d_{2n}) - w(\phi(d_{2n})) < \phi(d_{2n})$$

for any $n \in w$. Similarly, we have $\psi(d_{2n}) \leq \phi(d_{2n-1}) - w(\phi(d_{2n-1})) < \phi(d_{2n-1})$ for all $n \in N$.It follows that

$$\psi(d_n) \leq \phi(d_{n-1}) - w(\phi(d_{n-1})) \; \forall n \in N \tag{10}$$

From (10), we have

$$\sum_{i=0}^{n} w(\phi(d_i)) \leq \phi(d_0) - \psi(d_n) < \phi(d_0) \ \forall n \in N$$

Thus the sequence $\{d_n\}$ is decreasing sequence whereas the series $\sum_{n=0}^{\infty} w(\phi(d_n))$ and $\{\phi(d_n)\}$ are convergent.It is clear that $lim_{n\to\infty} w(\phi(d_n)) = 0$. Since sequence $\{d_n\}$ is decreasing so there exists $p \in R^+$ such that $lim_{n\to\infty} d_n = p$. By continuity of ϕ and w we have $lim_{n\to\infty} w(\phi(d_n)) = w(\phi(p)) = 0$.Thus $p = 0$. Therefore $lim_{n\to\infty} d(hx_n, hx_{n+1}) = 0$ implies that $lim_{n\to\infty} d(hx_{2n}, hx_{2n+1}) = 0$ means that $lim_{n\to\infty} d(hx_{2n}, fx_{2n}) = 0$ and $lim_{n\to\infty} d(hx_{2n+1}, gx_{2n+1}) = 0$. i.e. the sequence $\{x_n\}$ is asymptotically h-regular with respect to f and g.

Next we show that $\{hx_n\}$ is Cauchy sequence in X.We need only to show that $\{hx_{2n}\}$ is Cauchy sequence.On contrary suppose $\{hx_n\}$ is not Cauchy.Then there exists some $\epsilon > 0$ such that for any even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(hx_{2m(k)}, h_{2n(k)}) > \epsilon$. Further, let $2m(k)$ denote the least even positive integer exceeding $2n(k)$ which satisfies that $2m(k) > 2n(k) > 2k$

$$d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \epsilon \ and \ d(hx_{2m_k}, hx_{2n(k)}) > \epsilon. \tag{11}$$

Note that for any $k \in N$

$$d(hx_{2m(k)}, hx_{2n(k)}) \leq d_{2m(k)-1} + d_{2m(k)-2} + d(hx_{2m(k)-2}, hx_{2n(k)}).$$

$$|d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)})| \leq d_{2n(k)}.$$

$$|d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1})| \leq d_{2m(k)}.$$

$$|d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})| \leq d_{2n(k)+1}.$$

From above inequalities, we infer that

$$\begin{aligned} \epsilon &= lim_{n\to\infty} d(hx_{2m(k)}, hx_{2n(k)}) \\ &= lim_{n\to\infty} d(hx_{2m(k)}, hx_{2n(k)+1}) \\ &= lim_{n\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) \\ &= lim_{n\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+2}). \end{aligned}$$

Again from (5), we have

$$\begin{aligned}\psi(d(fx_{2m(k)}, gx_{2n(k)+1})) &\leq max\{\phi(d(hx_{2m(k)}, hx_{2n(k)+1})), \phi(d_{2m(k)}), \phi(d_{2n(k)+1}), \phi(\frac{1}{2}[d(hx_{2m(k)}, hx_{2n(k)+1}) \\ &+d(fx_{2m(k)}, gx_{2n(k)+1})])\} - w(max\{\phi(d(hx_{2m(k)}, hx_{2n(k)+1})), \phi(d_{2m(k)}), \\ &\phi(d_{2n(k)+1}), \phi(\frac{1}{2}[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1})])\})\end{aligned}$$

Taking $k \to \infty$, we deduce that

$$\psi(\epsilon) \leq max\{\phi(\epsilon), \phi(0), \phi(0), \phi(\frac{1}{2}[\epsilon + \epsilon])\} - w(max\{\phi(\epsilon), \phi(0), \phi(0), \phi(\frac{1}{2}[\epsilon + \epsilon])\})$$

$$\psi(\epsilon) \leq \phi(\epsilon) - w(\phi(\epsilon)) < \phi(\epsilon)$$

a contradiction and hence $\{hx_n\}$ is Cauchy sequence.Since X is asymptotically h-complete implies that the sequence $\{hx_n\}$ converges to a point $z \in X$.We infer that $\{hx_{2n}\}$ and $\{hx_{2n+1}\}$ also converges to z. h is asymptotically continuous implies that

$$hhx_{2n} \to hz, hhx_{2n+1} \to hz, hfx_{2n} \to hz, hgx_{2n+1} \to hz \quad as\ n \to \infty.$$

Now using weakly commutativity and sub additivity of ψ, we have

$$\begin{aligned}\psi(d(fhx_{2n}, hz)) &\leq \psi(d(fhx_{2n}, hfx_{2n})) + \psi(d(hfx_{2n}, hz)) \\ &\leq \psi(d(fx_{2n}, hx_{2n})) + \psi(d(hfx_{2n}, hz))\end{aligned}$$

Taking $n \to \infty$ implies that $fhx_{2n} \to hz$. Similarly $ghx_{2n+1} \to hz$. Suppose $d(hz, gz) > 0$. Then

$$\begin{aligned}\psi(d(hz, gz)) &\leq \psi(d(hz, fhx_{2n})) + \psi(d(fhx_{2n}, gz)) \\ &\leq \psi(d(hz, fhx_{2n})) + max\{\phi(d(hhx_{2n}, hz)), \phi(d(hhx_{2n}, fhx_{2n})), \phi(d(hz, gz)), \\ &\phi(\frac{1}{2}[d(hhx_{2n}, hz) + d(fh_{2n}, gz)])\} - w(max\{\phi(d(hhx_{2n}, hz)), \phi(d(hhx_{2n}, fhx_{2n})), \\ &\phi(d(hz, gz)), \phi(\frac{1}{2}[d(hhx_{2n}, hz) + d(fh_{2n}, gz)])\})\end{aligned}$$

Taking $n \to \infty$, we infer that

$$\psi(d(hz, gz)) \leq \phi(d(hz, gz))$$

a contradiction.Hence $hz = gz$. Similarly we can easily prove that $hz = fz$.Next we have to show that z is fixed point of h. Suppose that $d(hz, z) > 0$.From (5), we have

$$\psi(d(fx_{2n}, ghx_{2n})) \leq max\{\phi(d(hx_{2n}, hhx_{2n})), \phi(d(hx_{2n}, fx_{2n})), \phi(d(hhx_{2n}, ghx_{2n})),$$

$$\phi(\frac{1}{2}[d(hx_{2n}, hhx_{2n}) + d(fx_{2n}, ghx_{2n})])\} - w(max\{\phi(d(hx_{2n}, hhx_{2n})), \phi(d(hx_{2n}, fx_{2n}))$$
$$\phi(d(hhx_{2n}, ghx_{2n})), \phi(\frac{1}{2}[d(hx_{2n}, hhx_{2n}) + d(fx_{2n}, ghx_{2n})])\})$$

Taking $n \to \infty$, we infer that

$$\begin{aligned}\psi(d(z,hz)) &\leq max\{\phi(d(z,hz)), \phi(0), \phi(0), \phi(\frac{1}{2}[d(z,hz) + d(z,hz)])\} \\ &\quad -w(max\{\phi(d(z,hz)), \phi(0), \phi(0), \phi(\frac{1}{2}[d(z,hz) + d(z,hz)])\})\end{aligned}$$

implies that

$$\psi(d(z,hz)) \leq \phi(d(z,hz)) - w(\phi(d(z,hz))) < \phi(d(z,hz))$$

a contradiction.Hence $hz = z$. Therefore $fz = gz = hz = z$. i.e. z is common fixed point of f, g and h.

Finally, we show that z is unique fixed point of $f, g\&h$.Suppose z' be another fixed point. Then from (5), we obtain

$$\begin{aligned}\psi(d(fz,gz')) &\leq max\{\phi(d(hz,hz')), \phi(d(hz,fz)), \phi(d(hz',gz')), \phi(\frac{1}{2}[d(hz,hz') + d(fz,gz')])\} \\ &\quad -w(max\{\phi(d(hz,hz')), \phi(d(hz,fz)), \phi(d(hz',gz')), \phi(\frac{1}{2}[d(hz,hz') + d(fz,gz')])\})\end{aligned}$$

we infer that

$$\psi(d(hz,hz')) \leq \phi(d(hz,hz')) - w(\phi(d(hz,hz'))) < \phi(d(hz,hz'))$$

a contradiction. Hence $z = z'$.

Corollary 2. *Let (X,d) be a complete metric space and f, g and h be self mappings on X such that $f(X) \cup g(X) \subseteq h(X)$.If there exists a $w \in W$ satisfying (5).Then the pair (f,h) and (g,h) have a coincidence point in X, provided that (i) h is continuous and (iii) h commutes with both f and g.Further f, g and h have a unique common fixed point in X.*

Taking $\psi(t) = t = \phi(t)$,in cor.2 we obtain the following

Corollary 3. *Let (X, d) be a complete metric space.Let f, g and h be self maps on X such that $f(X) \cup g(X) \subseteq h(X)$ and h is continuous and commute with both f and g. If there exists a $w \in W$ satisfying the following condition:*

$$\begin{aligned} d(fx, gy) &\leq max\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)]\} \\ &- w(max\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)]\}) \end{aligned}$$

for all $x, y \in x$. Then the pair (f, h) and (g, h) have coincidence point, further f, g and h have unique common fixed point in X.

Taking $h = I$, in cor.3 we gain the following

Corollary 4. *Let (X, d) be a complete metric space.Let f and g be self maps on X. If there exists a $w \in W$ satisfying the following condition:*

$$\begin{aligned} d(fx, gy) &\leq max\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, y) + d(fx, gy)]\} \\ &- w(max\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, y) + d(fx, gy)]\}) \end{aligned}$$

for all $x, y \in x$. Then f and g have unique common fixed point in X.

4. An Application

Throughout in this section,let X and Y be Banach spaces $S \subseteq X$ be the state space and $D \subseteq Y$ be decision space. B(S) denotes the set of all real-valued bounded functions on S.Put $d(a, b) = sup_{x \in S}|a(x) - b(x)|, \forall a, b \in B(S)$.It is obvious that $(B(S), d)$ is a complete metric space. Define $u : S \times D \to R$, $T : S \times D \to S$ and $H_i : S \times D \times R \to R$ for $i = \{1, 2, 3\}$.

Now we study those conditions which guarantee the existence and uniqueness of common solutions of functional equations (7).

Theorem 3. *If the following conditions are satisfied*
(C_1) u and H_i are bounded for $i = \{1, 2, 3\}$
(C_2) there exist $\phi \in \Phi$, $\psi \in \Psi$ and $w \in W$ satisfying

$$\begin{aligned} \psi|H_1(x, y, a(t)) &- H_2(x, y, b(t))| \\ &\leq max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\} \\ &- w(max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\}) \end{aligned}$$

for all $(x, y) \in S \times D$; $a, b \in B(S)$ and $t \in S$. Where f, g and h are defined as follows: for all $x \in S$, $a_i \in B(S)$ and $i = \{1, 2, 3\}$

$$\begin{aligned}
f(a_1(x)) &= opt_{y\in D}\{u(x,y) + H_1(x,y,a_1(T(x,y)))\}\\
g(a_2(x)) &= opt_{y\in D}\{u(x,y) + H_2(x,y,a_2(T(x,y)))\}\\
h(a_3(x)) &= opt_{y\in D}\{u(x,y) + H_3(x,y,a_3(T(x,y)))\}
\end{aligned}$$

(C_3) $f(B(S)) \cup g(B(S)) \subseteq h(B(S))$ and h is asymptotically continuous and weakly commute with both f and g.
Then the system of functional equations possess a unique common solution in $B(S)$.

Proof From (C_1) and (C_2), f, g and h be self maps on $B(S)$. Let $a, b \in B(S)$ and $x \in S$.For any $\epsilon > 0$ there exist $y, z \in D$ satisfying

$$\begin{aligned}
f(a(x)) &< u(x,y) + H_1(x,y,a(T(x,y))) + \epsilon\\
g(b(x)) &< u(x,z) + H_2(x,z,b(T(x,z))) + \epsilon\\
f(a(x)) &\geq u(x,z) + H_1(x,z,a(T(x,z))) + \epsilon\\
g(b(x)) &\geq u(x,y) + H_2(x,y,b(T(x,y))) + \epsilon
\end{aligned}$$

Combining above inequalities with (C_2), we obtain the following:

$$\begin{aligned}
\psi(|f(a(x)) - g(b(x))|) &\leq \psi(\epsilon) + \psi(max\{|H_1(x,y,a(T(x,y))) - H_2(x,y,b(T(x,y)))|,\\
&\quad |H_1(x,z,a(T(x,z))) - H_2(x,z,b(T(x,z)))|\})\\
&\leq \psi(\epsilon) + max\{\phi(d(ha,hb)), \phi(d(ha,fa)), \phi(d(hb,gb)), \phi(\frac{1}{2}[d(ha,hb) + d(fa,gb)])\}\\
&- w(max\{\phi(d(ha,hb)), \phi(d(ha,fa)), \phi(d(hb,gb)), \phi(\frac{1}{2}[d(ha,hb) + d(fa,gb)])\})
\end{aligned}$$

Letting $\epsilon \to \infty$ we get

$$\begin{aligned}
\psi(|f(a(x)) - g(b(x))|) &\leq max\{\phi(d(ha,hb)), \phi(d(ha,fa)), \phi(d(hb,gb)), \phi(\frac{1}{2})[d(ha,hb) + d(fa,gb)]\}\\
&- w(max\{\phi(d(ha,hb)), \phi(d(ha,fa)), \phi(d(hb,gb)), \phi(\frac{1}{2}[d(ha,hb) + d(fa,gb)])\})
\end{aligned}$$

Therefore, Theorem 3.1 ensures that f, g and h have a unique common fixed point in B(S). That is, the system of functional equations (7) possesses a unique common solution B(S). Similarly we can change the condition (C_2) in Theorem 3.1 and get the common solution using corollaries. Taking $h = I$ in Theorem, we conclude that

Theorem 4. *Let the following condition hold:* (C_4) *u and* H_i *are bounded for* $i \in \{1, 2\}$. (C_5) *there exist a* $w \in \{W\}$ *satisfying*

$$\begin{aligned} |H_1(x, y, a(t)) - H_2(x, y, b(t))| \leq \; & max\{d(a, b), d(a, fa), d(b, gb), \frac{1}{2}[d(a, b) + d(fa, gb)])\} \\ & -w(max\{d(a, b), d(a, fa), d(b, gb), \frac{1}{2}[d(a, b) + d(fa, gb)])\}) \end{aligned}$$

for all $(x, y) \in S \times D$; $a, b \in B(S)$ *and* $t \in S$.*Where f and g are defined as follows: for all* $x \in S$, $a_i \in B(S)$ *and* $i = \{1, 2, 3\}$

$$f(a_1(x)) = opt_{y \in D}\{u(x, y) + H_1(x, y, a_1(T(x, y)))\}$$

$$g(a_2(x)) = opt_{y \in D}\{u(x, y) + H_2(x, y, a_2(T(x, y)))\}$$

for all $x \in S$, $a_1, a_2 \in B(S)$. *Then the system of functional equations*

$$f(x) = opt_{y \in D}\{u(x, y) + H_1(x, y, f(T(x, y)))\}$$

$$g(x) = opt_{y \in D}\{u(x, y) + H_2(x, y, g(T(x, y)))\}$$

possesses a unique common solution in B(S).

Acknowledgements

The first author is thankful to the National Board for Higher Mathematics(NBHM),India for providing financial assistance during this research study.

References

[1] R. Baskaran and P. V. Subrahmanyam. A note on the solution of a class of functional equations principle. Appl. Anal,volume 22 (3-4); pages 235–241, 1986.

[2] R. Bellman. Dynamic Programming, Princeton University Press. Princeton, New Jersey, 1957.

[3] R. Bellman and E. S. Lee. Functional equations in dynamic programming. Aequationes Math. 17(1):1–18, 1978.

[4] P. C. Bhakta and S. R. Choudhury. Some existence theorems for functional equations arising in dynamic programming. II J. Math. Anal. Appl.,131(1);217–231, 1988.

[5] P. C. Bhakta and S. Mitra. Some existence theorems for functional equations arising in dynamic programming J. Math. Anal. Appl.,98(2):348–362, 1984.

[6] F.E Brawder, W.V. Petryshyn. Contraction of fixed points of nonlinear mappings in Hilbert space J. Math. Anal. Appl, 20:197–228,1967.

[7] S. S. Chang and Y. H. Ma. Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming J. Math. Anal. Appl, 160(2):468–479,1991.

[8] P.Chumki and A.P. Baisnab. Asymptotically regularity and fixed point theorems The math. student, 46:54–59,1978.

[9] Lj.B. Ciric. A generalization of Banachs contraction principle Proc. Am. Math. Soc.,45:267–273, 1974.

[10] B. Fisher. Results on common fixed points on bounded metric spaces Math. sem. Notes,7:73–80, 1979.

[11] M. D. Guay, K. L. Singh. Fixed points of asymptotically regular mappings Math. vesnik,35:101–106, 1983.

[12] K.P.R. Sastry, S.R. Naidu, I.H.N. Rao and K.P.R. Rao. Common fixed point for asymptotically regular mappings Int.J.Pure Appl. Math.,15:849–854, 1984.

[13] S.L.Singh and S.P. Singh. A fixed point theorems Indian J. pure appl. Math.,11:1584–1586, 1980.

[14] P.L. Sharma and A.K. Yud. Fixed point theorems under asymptotic regularity at a point II Jnanabha,11:127–131, 1981.

[15] Z. Liu. A note on unique common fixed point Bull. Calcutta Math. Soc.,85(5):469–472, 1993.

[16] Z. Liu. Existence theorems of solutions for certain classes of functional equations arising in dynamic programming, J. Math. Anal. Appl. 262 (2): 529-553, 2001.

[17] Z. Liu. Coincidence theorems for expansion mappings with applications to the solutions of functional equations arising in dynamic programming, Acta Sci. Math. (Szeged) 65 (1): 359-369,1999.

[18] Z. Liu. Compatible mappings and fixed points, Acta Sci. Math. (Szeged) 65(2), 371-383,1999.

[19] Z. Liu, R. P. Agarwal, and S. M. Kang On solvability of functional equations and system of functional equations arising in dynamic programming, J. Math. Anal. Appl. 297 (1), 111-130,2004.

[20] Z. Liu and J. S. Ume On properties of solutions for a class of functional equations arising in dynamic programming, J. Optim. Theory Appl. 117 (3), 533-551,2003.

[21] H. K. Pathak and B. Fisher Common fixed point theorems with applications in dynamic programming, Glas. Mat. Ser. III 31(51)(2), 321-328,1996.

[22] B. N. Ray On common fixed points in metric spaces, Indian J. Pure Appl. Math. 19 (10), 960-962,1988.

[23] B. E. Rhoades, S. Sessa, M. S. Khan and M. Swaleh On fixed point of asymptotically regular mappings, J. Austral. Math. Soc. (Series A) 43, 328–346,1987.

[24] S. S. Zhang Some existence theorems of common and coincidence solutions for a class of systems of functional equations arising in dynamic programming, Appl. Math. Mech. 12 (1), 31-37,1991.