Common Fixed Point Theorems in Metric spaces with Applications

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Abstract. In this paper, we investigate the existence and uniqueness of common fixed point theorems for certain contractive type of mappings. As an application the existence and uniqueness of common solutions for a system of functional equations arising in dynamic programming are discuss by using our results.

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1. Introduction

Bellman and Lee [3] first introduced the basic form of the functional equations in dynamic programming is as follows:

$$f(x) = \text{opt}_{y \in D} H(x, y, f(T(x, y))) \forall x \in S$$

(1)

where opt represent sup. or inf., $x$, $y$ denote the state and decision vectors respectively, $T$ stands for the transformation of the process and $f(x)$ represents the optimal return function with the initial state $x$. Afterwards, the existence and uniqueness of fixed point solutions for several classes of contractive mappings and functional equations studied by many investigators such as Bhakta and Mitra [5], Liu [15], Liu and Ume [20], Pathak and Fisher [21], Baskaran and Subhramanyam [1] and others.

Ray [22] proved two common fixed point theorems for three self mappings $f, g$ and $h$ in the complete metric space using the following contractive condition:

$$d(fx, gy) \leq d(hx, hy) - w(d(hx, hy)), \forall x, y \in X$$

(2)
Further Liu[15] established common fixed point theorem and introduced a class of mappings in a complete metric space as follows:

\[ d(fx, gy) \leq \max\{d(hx, hy), d(hx, fx), d(hy, gy)\} - w(\max\{d(hx, hy), d(hx, fx), d(hy, gy)\}). \] (3)

Recall that the notion of orbitally complete metric space and orbitally continuous mapping were introduced by Ciric [9]. These definitions were extended to the case of two or three mappings by Sastry et al.[12]. Some common fixed point results in this situation were obtained in [12]. We give now respective definitions for pairs of mappings given in literature.

2. Preliminaries

Definition 1 (6). A self map \( f \) on a metric space \( (X, d) \) is said to be asymptotically regular at a point \( x \) in \( X \) if \( \lim_{n \to \infty} d(f^n(x), f^{n+1}(x)) = 0 \). Where \( f^n(x) \) denotes the \( n \)th iterate of \( f \) at \( x \).

Definition 2 (6). Let \( f \) and \( g \) be two self mappings of \( X \) and \( \{x_n\} \) a sequence in \( X \), then \( \{x_n\} \) is said to be asymptotically \( g \)-regular with respect to \( f \) if \( \lim_{n \to \infty} d(fx_n, gx_n) = 0 \).

Definition 3 (9). Let \( \{x_n\} \) is a sequence which is asymptotically \( g \)-regular with respect to \( f \), then \( O(f, x_n) = \{fx_1, fx_2, fx_3, ...fx_n, ...\} \) is called asymptotic orbit of \( f \).

Definition 4 (12). \( X \) is said to be \( f \)-asymptotically complete if every Cauchy sequence of the form \( \{fx_n\} \) converges in \( X \).

Definition 5 (12). A self map \( f \) is said to be asymptotically continuous if it is continuous on closure of \( O(f, x_n) \).

Definition 6. Two self maps \( g \) and \( h \) of \( X \) are said to be weakly commuting if \( d(ghx, hgx) \leq d(gx, hx) \) \( \forall x \in X \).

Definition 7 (12). Let \( f, g \) and \( h \) be three self maps on a metric space \( X \).

(i) If for a point \( x_0 \in X \),there exists a sequence \( \{x_n\} \) in \( X \) such that \( fx_{2n} = hx_{2n+1} \) and \( gx_{2n+1} = hx_{2n+2} \), \( n = 0, 1, 2, ... \). Then the set \( O(x_0, f, g, h) = \{Tx_n|n = 0, 1, 2,...\} \) is called the orbit of \( (f, g, h) \) at \( x_0 \).

(ii) The space \( (X, d) \) is said to be \( (f, g, h) \)-orbitally complete if every Cauchy sequence in \( O(x_0, f, g, h) \) converges in \( X \).
(iii) The map $h$ is said to be $(f, g, h)$-orbitally continuous at $x_0$ if it is continuous on $O(x_0, f, g, h)$.

(iv) The pair $(f, g)$ is said to be asymptotically regular w.r.t. $h$ at $x_0$ if there exists a sequence $\{x_n\}$ in $X$ such that $f x_{2n} = h x_{2n+1}, \ g x_{2n+1} = h x_{2n+2}$; $n = 0, 1, 2, \ldots$ and $d(h x_n, h_{n+1}) \to 0$ as $n \to \infty$.

Throughout in this paper, we assume that $R^+ = [0, +\infty), R = (-\infty, +\infty), w$ and $N$ denote the set of all non-negative and positive integers respectively.

$$W = \{w : w : R^+ \to R^+ \text{is continuous mappings with } 0 < w(t) < t \ \forall \ t > 0\} \tag{4}$$

Let $\Phi = \{\phi : \phi : [0, \infty) \to [0, \infty)\}$ satisfying the following conditions:

(i) $\phi$ is continuous and non-decreasing
(ii) $\phi(t) < t \ \forall t \in [0, \infty)$
(iii) $\lim_{n \to \infty} \phi(t_n) = 0 \iff \lim_{n \to \infty} t_n = 0$.

Let $\Psi = \{\psi : \psi : [0, \infty) \to [0, \infty)\}$ satisfying the following conditions:

(i) $\psi$ is non-decreasing
(ii) $\phi(t) < \psi(t) \ \forall t > 0$
(iii) $\psi(a + b) \leq \psi(a) + \psi(b) \ \forall a, b \in [0, \infty)$
(iv) $\psi(t) < t \ \forall t \in [0, \infty)$

The aim of this paper is to provide the sufficient conditions for the existence and uniqueness of common fixed point for the following type of contractive mappings metric space $(X, d)$.

$$\psi(d(f x, g y)) \leq \max\{\phi(d(h x, h y)), \phi(d(h x, f x)), \phi(\frac{1}{2}(d(h x, h y) + d(f x, g y))), \phi(d(h y, g y))\}$$
$$-w(\max\{\phi(d(h x, h y)), \phi(d(h x, f x)), \phi(d(h y, g y)), \phi(\frac{1}{2}(d(h x, h y) + d(f x, g y)))\}). \tag{5}$$

for all $x, y \in X$. Where $\psi$ and $\phi$ are defined above.

As an applications, we discuss the existence and uniqueness of common solutions of the following functional equations arising in dynamic programming.

$$f(x) = \operatorname{opt}_{y \in D}\{u(x, y) + H(x, y, f(T(x, y)))\} \forall x \in S \tag{6}$$

and

$$f_i(x) = \operatorname{opt}_{y \in D}\{u(x, y) + H_i(x, y, f_i(T(x, y)))\} \forall x \in S \& i \in \{1, 2, 3\} \tag{7}$$
3. Main Results

**Theorem 1.** Let \( f, g \) and \( h \) be three self maps on a metric space \( X \) satisfying:

(i) either \( f \) commute with \( h \) or \( g \) commute with \( h \).

(ii) there exists \( w \in W \) such that (5) hold for all \( x, y \in X \).

(iii) The pair \( (f, g) \) is asymptotically regular with respect to \( h \) at \( x_0 \).

(iv) The space \( X \) is \((f, g, h)\)-orbitally complete at \( x_0 \) and \( h \) is orbitally continuous at \( x_0 \).

Then \( f, g \) and \( h \) have a unique common fixed point in \( X \).

**Proof** Since \((f, g)\) is asymptotically respect to \( h \) at \( x_0 \), there exists a sequence \( \{x_n\} \) in \( X \) such that \( fx_{2n} = hx_{2n+1} \) and \( gx_{2n+1} = hx_{2n+2}, n = 0, 1, 2, \ldots \) and \( d(hx_n, hx_{n+1}) \to zero \) as \( n \to \infty \).

Now we show that \( hx_n \) is Cauchy. On contrary suppose that \( hx_n \) is not Cauchy, then there exists an \( \epsilon > 0 \) and positive integers \( m_k \) and \( n_k \) with \( m_k < n_k \) such that \( d(hx_{m_k}, hx_{n_k}) \geq \epsilon \) and \( d(hx_{m_k}, hx_{n_k-1}) \leq \epsilon \) for all \( k = 0, 1, 2, \ldots \). Since \( d(hx_{m_k}, hx_{n_k}) \leq d(hx_{m_k}, hx_{n_k-1}) + d(hx_{n_k-1}, hx_{n_k}) \). Then we obtain \( d(hx_{m_k}, hx_{n_k}) \to \epsilon \) as \( k \to \infty \).

Now there are four cases: (i) \( m_k \) is even and \( n_k \) is odd (ii) \( m_k \) is even and \( n_k \) is even (iii) \( m_k \) is odd and \( n_k \) is even (iv) \( m_k \) is odd and \( n_k \) is odd.

Suppose \( m_k \) is even and \( n_k \) is odd, we have

\[
\psi(d(hx_{m_k}, hx_{n_k})) \leq \psi(d(hx_{m_k}, hx_{n_k+1}))+\psi(d(hx_{m_k+1}, hx_{n_k})) + \psi(d(hx_{n_k+1}, hx_{n_k}))
\]

\[
\leq \psi(d(hx_{m_k}, hx_{m_k+1}))+\max\{\phi(d(hx_{m_k}, hx_{k})), \phi(d(fx_{m_k}, hx_{m_k})),
\phi(d(gx_{n_k}, hx_{n_k})), \phi(\frac{1}{2}[d(hx_{m_k}, hx_{n_k})+d(fx_{m_k}, gx_{n_k})])
\] \[
- w(\max\{\phi(d(hx_{m_k}, hx_{k})), \phi(d(fx_{m_k}, hx_{m_k})), \phi(d(gx_{n_k}, hx_{n_k}))
\phi(\frac{1}{2}[d(hx_{m_k}, hx_{n_k})+d(fx_{m_k}, gx_{n_k})])\}) + \psi(d(hx_{n_k+1}, hx_{n_k}))
\]

Letting \( k \to \infty \), we obtain

\[
\psi(\epsilon) \leq \phi(\epsilon) - w(\phi(\epsilon)) < \phi(\epsilon)
\]

a contradiction. In the remaining cases we have a similar situation. Hence \( \{hx_n\} \) is Cauchy. Since \( X \) is \((f, g, h)\)-orbitally complete at \( x_0 \), it follows that there exist \( z \in X \) s.t. \( hx_n \to z \) as \( n \to \infty \). Now, again

\[
\psi(d(fx_{2n}, gz)) \leq \max\{\phi(d(fx_{2n}, hz)), \phi(d(fx_{2n}, hx_{2n})), \phi(d(gz, hz)),
\phi(\frac{1}{2}[d(hx_{2n}, hz)+d(fx_{2n}, gz)])\} - w(\max\{\phi(d(hx_{2n}, hz)), \phi(d(fx_{2n}, hx_{2n})),
\phi(d(gz, hz)), \phi(\frac{1}{2}[d(hx_{2n}, hz)+d(fx_{2n}, gz)])\})
\]

and

\[
\psi(d(fz, gx_{2n+1})) \leq \max\{\phi(d(hz, hx_{2n+1})), \phi(d(fz, hz)), \phi(d(gx_{2n+1}, hx_{2n+1})),
\phi(\frac{1}{2}[d(hx_{2n+1}, hz)+d(fz,gz)])\})
\]
\[
\phi\left(\frac{1}{2}[d(hz, hx_{2n+1}) + d(fz, gx_{2n+1})]\right) - w\left(\max\{\phi(d(hz, hx_{2n+1})), \phi(d(fz, hz))\},
\phi(d(gx_{2n+1}, hx_{2n+1})), \phi\left(\frac{1}{2}[d(hz, hx_{2n+1}) + d(fz, gx_{2n+1})]\right)\right)
\]

Taking \(k \to \infty\), we obtain

\[
\psi(d(z, gz)) \leq \max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi\left(\frac{1}{2}[d(hz, z) + d(z, gz)]\right)\}
- w\left(\max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi\left(\frac{1}{2}[d(hz, z) + d(z, gz)]\right)\}\right) (8)
\]

and

\[
\psi(d(fz, z)) \leq \max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi\left(\frac{1}{2}[d(hz, z) + d(fz, z)]\right)\}
- w\left(\max\{\phi(d(z, hz)), \phi(d(z, z)), \phi(gz, hz), \phi\left(\frac{1}{2}[d(hz, z) + d(fz, z)]\right)\}\right) (9)
\]

Since \(h\) is orbitally continuous at \(x_0\) and \(fh = hf\), we infer that \(fhx_{2n+1} = hf_{2n} \to Tz\) as \(n \to \infty\). Similarly \(ghx_{2n+1} = gh_{2n+1} \to hz\) as \(n \to \infty\).

Again,

\[
\psi(d(fh_{2n+1}, gx_{2n+1})) \leq \max\{\phi(d(hhx_{2n}, hx_{2n+1})), \phi(d(fh_{2n+1}, hhx_{2n})), \phi(gx_{2n+1}, hx_{2n+1}), \phi\left(\frac{1}{2}[d(hhx_{2n}, hx_{2n+1}) + d(fh_{2n+1}, gx_{2n+1})]\right)\}
- w\left(\max\{\phi(d(hhx_{2n}, hx_{2n+1})), \phi(d(fh_{2n+1}, hhx_{2n})), \phi(gx_{2n+1}, hx_{2n+1}), \phi\left(\frac{1}{2}[d(hhx_{2n}, hx_{2n+1}) + d(fh_{2n+1}, gx_{2n+1})]\right)\}\right)
\]

Taking \(k \to \infty\), we obtain

\[
\psi(d(hz, z)) \leq \max\{\phi(d(hz, z)), \phi(d(hz, hz)), \phi(d(z, z)), \phi\left(\frac{1}{2}[d(hz, z) + d(hz, z)]\right)\}
- w\left(\max\{\phi(d(hz, z)), \phi(d(hz, hz)), \phi(d(z, z)), \phi\left(\frac{1}{2}[d(hz, z) + d(hz, z)]\right)\}\right)
\]

implies that

\[
\psi(d(hz, z)) \leq \phi(d(z, hz)) - w(\phi(d(z, hz))) < \phi(d(z, hz))
\]

a contradiction. Hence \(hz = z\). Using (8) and (9) together with \(Tz = z\), we infer that \(fz = gz = hz = z\). Further uniqueness of common fixed point can easily prove.

Taking \(\psi(t) = t\) and \(\phi(t) = ht\) where \(< h < 1\), we state the following.
Corollary 1. Let $A, B$ and $T$ be self maps on a metric space $(X, d)$ such that $T$ commutes with both $A$ and $B$ and the pair $(A, B)$ is asymptotically regular w.r.to $T$ at $x_0 \in X$, $X$ is orbitally complete and $T$ is orbitally continuous at $x_0$ and

$$d(Ax, By) \leq \phi(\max\{d(Tx, Ty), d(Ax, Tx), d(By, Ty), \frac{1}{2}[d(Tx, Ty) + d(Ax, By)]\}) - w(\max\{d(Tx, Ty), d(Ax, Tx), d(By, Ty), \frac{1}{2}[d(Tx, Ty) + d(Ax, By)]\})$$

for all $x, y \in X$. Then $A, B$ and $T$ have unique common fixed point in $X$.

Theorem 2. Let $(X, d)$ be a metric space and $f, g$ and $h$ be self mappings on $X$ such that $f(X) \cup g(X) \subseteq h(X)$ If there exists a $w \in W$ satisfying (5). Then the pair $(f, h)$ and $(g, h)$ have a coincidence point in $X$, provided that (i) $X$ is $h$-asymptotically complete, (ii) $h$ is asymptotically continuous and (iii) $h$ is weakly commute with $f$ and $g$. Further $f$, $g$ and $h$ have a unique common fixed point in $X$.

Proof Let $x_0 \in X$ be any point in $X$. Since $f(X) \cup g(X) \subseteq h(X)$. We choose sequence \{\{x_n\} \in X such that $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$ for all $n \in w$.

By (5), we conclude that

$$\psi(d(hx_{2n+1}, hx_{2n+2})) = \psi(d(fx_{2n}, gx_{2n+1}))$$

$$\leq \max\{\phi(d(h_{2n}, hx_{2n+1})), \phi(d(hx_{2n}, fx_{2n})), \phi(d(hx_{2n+1}, gx_{2n+1})), \phi(\frac{1}{2}(d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1})))\} - w(\max\{\phi(d(h_{2n}, hx_{2n+1})), \phi(d(hx_{2n}, fx_{2n})), \phi(d(hx_{2n+1}, gx_{2n+1})), \phi(\frac{1}{2}(d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1})))\})$$

This yields

$$\psi(d_{2n+1}) \leq \max\{\phi(d_{2n}), \phi(d_{2n}), \phi(d_{2n+1}), \phi(\frac{1}{2}(d_{2n} + d_{2n+1})))\} - w(\max\{\phi(d_{2n}), \phi(d_{2n}), \phi(d_{2n+1}), \phi(\frac{1}{2}(d_{2n} + d_{2n+1})))\})$$

Suppose $d_{2n+1} > d_{2n}$, then $\phi(d_{2n+1}) > \phi(d_{2n})$. Using (5), we have

$$\psi(d_{2n+1}) \leq \phi(d_{2n+1}) - w(\phi(d_{2n+1})) < \phi(d_{2n+1})$$

a contradiction. Consequently, we have $d_{2n+1} \leq d_{2n}$, from (5) we have

$$\psi(d_{2n+1}) \leq \phi(d_{2n}) - w(\phi(d_{2n})) < \phi(d_{2n})$$

for any $n \in w$. Similarly, we have $\psi(d_{2n}) \leq \phi(d_{2n}) - w(\phi(d_{2n})) < \phi(d_{2n})$ for all $n \in N$. It follows that

$$\psi(d_n) \leq \phi(d_{n-1}) - w(\phi(d_{n-1})) \quad \forall n \in N \quad (10)$$
From (10), we have
\[ \sum_{i=0}^{n} w(\phi(d_i)) \leq \phi(d_0) - \psi(d_n) < \phi(d_0) \quad \forall n \in N \]
Thus the sequence \( \{d_n\} \) is decreasing sequence whereas the series \( \sum_{n=0}^{\infty} w(\phi(d_n)) \) and \( \{\phi(d_n)\} \) are convergent. It is clear that \( \lim_{n \to \infty} w(\phi(d_n)) = 0 \). Since sequence \( \{d_n\} \) is decreasing so there exists \( p \in R^+ \) such that \( \lim_{n \to \infty} d_n = p \). By continuity of \( \phi \) and \( w \) we have \( \lim_{n \to \infty} w(\phi(d_n)) = w(\phi(p)) = 0 \). Thus \( p = 0 \). Therefore \( \lim_{n \to \infty} d(hx_{2n}, hx_{2n+1}) = 0 \) implies that \( \lim_{n \to \infty} d(hx_{2n}, fx_{2n}) = 0 \) and \( \lim_{n \to \infty} d(hx_{2n+1}, gx_{2n+1}) = 0 \). I.e. the sequence \( \{x_n\} \) is asymptotically \( h \)-regular with respect to \( f \) and \( g \).

Next we show that \( \{hx_n\} \) is Cauchy sequence in \( X \). We need only to show that \( \{hx_{2n}\} \) is Cauchy sequence. On contrary suppose \( \{hx_n\} \) is not Cauchy. Then there exists some \( \epsilon > 0 \) such that for any even integers \( 2m(k) \) and \( 2n(k) \) with \( 2m(k) > 2n(k) > 2k \) and \( d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon \). Further, let \( 2m(k) \) denote the least even positive integer exceeding \( 2n(k) \) which satisfies that \( 2m(k) > 2n(k) > 2k \)

\[ d(hx_{2m(k) - 2}, hx_{2n(k)}) \leq \epsilon \quad \text{and} \quad d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon. \] (11)

Note that for any \( k \in N \)

\[ d(hx_{2m(k)}, hx_{2n(k)}) \leq d_{2m(k)-1} + d_{2m(k)-2} + d(hx_{2m(k)-2}, hx_{2n(k)}). \]

\[ |d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)})| \leq d_{2n(k)}. \]

\[ |d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1})| \leq d_{2m(k)}. \]

\[ |d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})| \leq d_{2n(k)+1}. \]

From above inequalities, we infer that

\[ \epsilon = \lim_{n \to \infty} d(hx_{2m(k)}, hx_{2n(k)}) = \lim_{n \to \infty} d(hx_{2m(k)}, hx_{2n(k)+1}) = \lim_{n \to \infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) = \lim_{n \to \infty} d(hx_{2m(k)+1}, hx_{2n(k)+2}). \]
Again from (5), we have

\[
\psi(d(fx_{2m(k)}, gx_{2n(k)+1})) \leq \max\{\phi(d(hx_{2m(k)}, hx_{2n(k)+1})), \phi(d_{2m(k)}), \phi(d_{2n(k)+1}), \phi(2[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1])]) - w(\max\{\phi(d(hx_{2m(k)}, hx_{2n(k)+1})), \phi(d_{2m(k)}), \\
\phi(d_{2n(k)+1}), \phi(2[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1])])\}\}
\]

Taking \( k \to \infty \), we deduce that

\[
\psi(\epsilon) \leq \max\{\phi(\epsilon), \phi(0), \phi(0), \phi(2[\epsilon + \epsilon])\} - w(\max\{\phi(\epsilon), \phi(0), \phi(0), \phi(2[\epsilon + \epsilon])\})
\]

\[
\psi(\epsilon) \leq \phi(\epsilon) - w(\phi(\epsilon)) < \phi(\epsilon)
\]
a contradiction and hence \( \{hx_n\} \) is Cauchy sequence.Since \( X \) is asymptotically \( h \)-complete implies that the sequence \( \{hx_n\} \) converges to a point \( z \in X \).We infer that \( \{hx_{2n}\} \) and \( \{hx_{2n+1}\} \) also converges to \( z \). \( h \) is asymptotically continuous implies that

\[
hx_{2n} \to hz, hhx_{2n+1} \to hz, hx_{2n} \to hz, hgx_{2n+1} \to hz \text{ as } n \to \infty.
\]

Now using weakly commutativity and sub additivity of \( \psi \), we have

\[
\psi(d(hx_{2n}, hz)) \leq \psi(d(hx_{2n}, hx_{2n})) + \psi(d(hx_{2n}, hz)) \leq \psi(d(hx_{2n}, hx_{2n})) + \psi(d(hx_{2n}, hz))
\]

Taking \( n \to \infty \) implies that \( fx_{2n} \to hz \). Similarly \( ghx_{2n+1} \to hz \). Suppose \( d(hz, gz) > 0 \). Then

\[
\psi(d(hz, gz)) \leq \psi(d(hz, fx_{2n})) + \psi(d(hx_{2n}, gz)) \leq \psi(d(hz, fx_{2n})) + \max\{\phi(d(hhx_{2n}, hz)), \phi(d(hhx_{2n}, fx_{2n})), \phi(d(hz, gz)), \\
\phi(2[d(hhx_{2n}, hz) + d(fx_{2n}, gx_{2n})])\} - w(\max\{\phi(d(hhx_{2n}, hz)), \phi(d(hhx_{2n}, fx_{2n})), \\
\phi(d(hz, gz)), \phi(2[d(hhx_{2n}, hz) + d(fx_{2n}, gx_{2n})])\})
\]

Taking \( n \to \infty \), we infer that

\[
\psi(d(hz, gz)) \leq \phi(d(hz, gz))
\]
a contradiction.Hence \( hz = gz \). Similarly we can easily prove that \( hz = fz \).Next we have to show that \( z \) is fixed point of \( h \). Suppose that \( d(hz, z) > 0 \). From (5), we have

\[
\psi(d(fx_{2n}, ghx_{2n})) \leq \max\{\phi(d(hx_{2n}, hhx_{2n})), \phi(d(hx_{2n}, fx_{2n})), \phi(d(hhx_{2n}, ghx_{2n}))\}
\]
\[
\begin{align*}
\phi(\frac{1}{2}[d(hx_{2n}, hhx_{2n}) + d(fx_{2n}, ghx_{2n})]) - w(\max\{\phi(d(hx_{2n}, hhx_{2n})), \phi(d(hx_{2n}, f x_{2n}))\} \\
\phi(d(hhx_{2n}, ghx_{2n})), \phi(\frac{1}{2}[d(hx_{2n}, hhx_{2n}) + d(fx_{2n}, ghx_{2n})])
\end{align*}
\]

Taking \(n \to \infty\), we infer that

\[
\psi(d(z, hz)) \leq \max\{\phi(d(z, hz)), \phi(0), \phi(0), \phi(\frac{1}{2}[d(z, hz) + d(z, hz)])\} \]

\[
- w(\max\{\phi(d(z, hz)), \phi(0), \phi(0), \phi(\frac{1}{2}[d(z, hz) + d(z, hz)])\})
\]

implies that

\[
\psi(d(z, hz)) \leq \phi(d(z, hz)) - w(\phi(d(z, hz))) < \phi(d(z, hz))
\]

a contradiction. Hence \(hz = z\). Therefore \(fz = gz = hz = z\). i.e. \(z\) is common fixed point of \(f\), \(g\) and \(h\).

Finally, we show that \(z\) is unique fixed point of \(f\), \(g\)\& \(h\). Suppose \(z'\) be another fixed point. Then from (5), we obtain

\[
\psi(d(fz, gz')) \leq \max\{\phi(d(hz, hz')), \phi(d(hz, fz)), \phi(d(hz, gz')), \phi(\frac{1}{2}[d(hz, hz') + d(fz, gz')])\} \\
- w(\max\{\phi(d(hz, hz')), \phi(d(hz, fz)), \phi(d(hz, gz')), \phi(\frac{1}{2}[d(hz, hz') + d(fz, gz')])\})
\]

we infer that

\[
\psi(d(hz, hz')) \leq \phi(d(hz, hz')) - w(\phi(d(hz, hz'))) < \phi(d(hz, hz'))
\]

a contradiction. Hence \(z = z'\).

**Corollary 2.** Let \((X, d)\) be a complete metric space and \(f\), \(g\) and \(h\) be self mappings on \(X\) such that \(f(X) \cup g(X) \subseteq h(X)\). If there exists a \(w \in W\) satisfying (5). Then the pair \((f, h)\) and \((g, h)\) have a coincidence point in \(X\), provided that (i) \(h\) is continuous and (iii) \(h\) commutes with both \(f\) and \(g\). Further \(f\), \(g\) and \(h\) have a unique common fixed point in \(X\).

Taking \(\psi(t) = t = \phi(t)\), in cor.2 we obtain the following
Corollary 3. Let $(X, d)$ be a complete metric space. Let $f, g$ and $h$ be self maps on $X$ such that $f(X) \cup g(X) \subseteq h(X)$ and $h$ is continuous and commute with both $f$ and $g$. If there exists a $w \in W$ satisfying the following condition:

\[
d(f, g) \leq \max\{d(h, h), d(h, f), d(h, g), \frac{1}{2}[d(h, h) + d(f, g)]\} - w(\max\{d(h, h), d(h, f), d(h, g), \frac{1}{2}[d(h, h) + d(f, g)]\})
\]

for all $x, y \in X$. Then the pair $(f, h)$ and $(g, h)$ have coincidence point, further $f, g$ and $h$ have unique common fixed point in $X$.

Taking $h = I$, in cor.3 we gain the following

Corollary 4. Let $(X, d)$ be a complete metric space. Let $f$ and $g$ be self maps on $X$. If there exists a $w \in W$ satisfying the following condition:

\[
d(f, g) \leq \max\{d(f, g), d(f, f), d(g, g), \frac{1}{2}[d(f, f) + d(g, g)]\} - w(\max\{d(f, g), d(f, f), d(g, g), \frac{1}{2}[d(f, f) + d(g, g)]\})
\]

for all $x, y \in X$. Then $f$ and $g$ have unique common fixed point in $X$.

4. An Application

Throughout in this section, let $X$ and $Y$ be Banach spaces $S \subseteq X$ be the state space and $D \subseteq Y$ be decision space. $B(S)$ denotes the set of all real-valued bounded functions on $S$. Put $d(a, b) = \sup_{x \in S}|a(x) - b(x)|, \forall a, b \in B(S)$. It is obvious that $(B(S), d)$ is a complete metric space. Define $u : S \times D \to R, T : S \times D \to S$ and $H_i : S \times D \times R \to R$ for $i = \{1, 2, 3\}$.

Now we study those conditions which guarantee the existence and uniqueness of common solutions of functional equations (7).

Theorem 3. If the following conditions are satisfied

$(C_1)$ $u$ and $H_i$ are bounded for $i = \{1, 2, 3\}$
$(C_2)$ there exist $\phi \in \Phi$, $\psi \in \Psi$ and $w \in W$ satisfying

\[
|\psi(H_1(x, y, a(t)) - H_2(x, y, b(t))| \\
\leq \max\{\phi(d(h, h)b), \phi(d(h, f)a), \phi(d(h, gb)), \phi(\frac{1}{2}[d(h, h) + d(fa, gb)])\} - w(\max\{\phi(d(h, h)b), \phi(d(h, f)a), \phi(d(h, gb)), \phi(\frac{1}{2}[d(h, h) + d(fa, gb)])\})
\]
for all \((x, y) \in S \times D\); \(a, b \in B(S)\) and \(t \in S\). Where \(f\), \(g\) and \(h\) are defined as follows:
for all \(x \in S\), \(a_i \in B(S)\) and \(i = \{1, 2, 3\}\)

\[
\begin{align*}
f(a_1(x)) &= \text{opt}_{y \in D}\{u(x, y) + H_1(x, y, a_1(T(x, y)))\} \\
g(a_2(x)) &= \text{opt}_{y \in D}\{u(x, y) + H_2(x, y, a_2(T(x, y)))\} \\
h(a_3(x)) &= \text{opt}_{y \in D}\{u(x, y) + H_3(x, y, a_3(T(x, y)))\}
\end{align*}
\]

\((C_3)\) \(f(B(S)) \cup g(B(S)) \subseteq h(B(S))\) and \(h\) is asymptotically continuous and weakly commute with both \(f\) and \(g\).

Then the system of functional equations possess a unique common solution in \(B(S)\).

**Proof** From \((C_1)\) and \((C_2)\), \(f, g\) and \(h\) be self maps on \(B(S)\). Let \(a, b \in B(S)\) and \(x \in S\). For any \(\epsilon > 0\) there exist \(y, z \in D\) satisfying

\[
\begin{align*}
f(a(x)) &< u(x, y) + H_1(x, y, a(T(x, y))) + \epsilon \\
g(b(x)) &< u(x, z) + H_2(x, z, b(T(x, z))) + \epsilon \\
f(a(x)) &\geq u(x, z) + H_1(x, z, a(T(x, z))) + \epsilon \\
g(b(x)) &\geq u(x, y) + H_2(x, y, b(T(x, y))) + \epsilon
\end{align*}
\]

Combining above inequalities with \((C_2)\), we obtain the following:

\[
\begin{align*}
\psi(|f(a(x)) - g(b(x))|) &\leq \psi(\epsilon) + \psi(\max\{|H_1(x, y, a(T(x, y))) - H_2(x, y, b(T(x, y)))|\}, \\
&\quad \{|H_1(x, z, a(T(x, z))) - H_2(x, z, b(T(x, z)))|\}) \\
&\leq \psi(\epsilon) + \max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\} \\
&\quad - w(\max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\})
\end{align*}
\]

Letting \(\epsilon \to \infty\) we get

\[
\begin{align*}
\psi(|f(a(x)) - g(b(x))|) &\leq \max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\} \\
&\quad - w(\max\{\phi(d(ha, hb)), \phi(d(ha, fa)), \phi(d(hb, gb)), \phi(\frac{1}{2}[d(ha, hb) + d(fa, gb)])\})
\end{align*}
\]

Therefore, Theorem 3.1 ensures that \(f, g\) and \(h\) have a unique common fixed point in \(B(S)\). That is, the system of functional equations \((7)\) possesses a unique common solution \(B(S)\). Similarly we can change the condition \((C_2)\) in Theorem 3.1 and get the common solution using corollaries. Taking \(h = I\) in Theorem, we conclude that
Theorem 4. Let the following condition hold: (C₄) u and $H_i$ are bounded for $i \in \{1, 2\}$. (C₅) there exist a $w \in \{W\}$ satisfying

$$|H_1(x, y, a(t)) - H_2(x, y, b(t))| \leq \max\{d(a, b), d(a, f(a), d(b, g(b)), \frac{1}{2}[d(a, b) + d(f(a), g(b))]\}$$

$$- w(\max\{d(a, b), d(a, f(a), d(b, g(b)), \frac{1}{2}[d(a, b) + d(f(a), g(b))]\})$$

for all $(x, y) \in S \times D$; $a, b \in B(S)$ and $t \in S$. Where $f$ and $g$ are defined as follows:

for all $x \in S$, $a_i \in B(S)$ and $i = \{1, 2, 3\}$

$$f(a_1(x)) = \text{opt}_{y \in D}\{u(x, y) + H_1(x, y, a_1(T(x, y)))\}$$

$$g(a_2(x)) = \text{opt}_{y \in D}\{u(x, y) + H_2(x, y, a_2(T(x, y)))\}$$

for all $x \in S$, $a_1, a_2 \in B(S)$. Then the system of functional equations

$$f(x) = \text{opt}_{y \in D}\{u(x, y) + H_1(x, y, f(T(x, y)))\}$$

$$g(x) = \text{opt}_{y \in D}\{u(x, y) + H_2(x, y, g(T(x, y)))\}$$

possesses a unique common solution in $B(S)$.

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