EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 4, 2018, 1143-1176
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Reduction of modern mathematical problems to the classical Riemann–Poincaré–Hilbert problem

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Abstract. Using the example of a complicated problem such as the Cauchy problem for the Navier–Stokes equation, we show how the Poincaré–Riemann–Hilbert boundary-value problem enables us to construct effective estimates of solutions for this case. The apparatus of the three-dimensional inverse problem of quantum scattering theory is developed for this. It is shown that the unitary scattering operator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to go on to study the potential in the Schrödinger equation, which we consider as a velocity component in the Navier–Stokes equation. The same scheme of reduction of Riemann integral equations for the zeta function to the Poincaré–Riemann–Hilbert boundary-value problem allows us to construct effective estimates that describe the behaviour of the zeros of the zeta function very well.

2010 Mathematics Subject Classifications: 11

Key Words and Phrases: Schrödinger's equation, potential; scattering amplitude, Cauchy problem, Navier–Stokes equations, Millennium Prize problems; Dirichlet, Riemann, Hilbert; Poincaré, Riemann hypothesis; zeta function, Hadamard

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1. Introduction

Using the example of a complicated problem such as the Cauchy problem for the Navier–Stokes equation, we show how the Poincaré–Riemann–Hilbert boundary-value problem enables us to construct effective estimates of solutions for this case. The apparatus of the three-dimensional inverse problem of quantum scattering theory is developed for this. It is shown that the unitary scattering operator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to go on to study the potential in the Schrödinger equation, which we consider as a velocity component in the Navier–Stokes equation. The same scheme of reduction of Riemann integral equations for the zeta function to the Poincaré–Riemann–Hilbert boundary-value problem allows us to construct effective estimates that describe the behaviour of the zeros of the zeta function very well.

DOI: https://doi.org/10.29020/nybg.ejpam.v11i4.3328

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2. Results for the one-dimensional case

Let us consider a one-dimensional function f and its Fourier transformation $\tilde{f}$. Using the notions of module and phase, we write the Fourier transformation in the following form: $\tilde{f} = |\tilde{f}| \exp(i\Psi)$, where Ψ is the phase. The Plancherel equality states that $||f||_{L_2} = \mathrm{const}||\tilde{f}||_{L_2}$. Here we can see that the phase does not contribute to determination of the X norm. To estimate the maximum we make a simple estimate as $\max|f|^2 \leq 2||f||_{L_2}||\nabla f||_{L_2}$. Now we have an estimate of the function maximum in which the phase is not involved. Let us consider the behaviour of a progressing wave travelling with a constant velocity of $v = a$ described by the function $F(x,t) = f(x+at)$. Its Fourier transformation with respect to the variable x is $\tilde{F} = \tilde{f}\exp(iatk)$. Again, in this case, we can see that when we study a module of the Fourier transformation, we will not obtain major physical information about the wave, such as its velocity and location of the wave crest because $|\tilde{F}| = |\tilde{f}|$. These two examples show the weaknesses of studying the Fourier transformation. Many researchers focus on the study of functions using the embedding theorem, in which the main object of the study is the module of the function. However, as we have seen in the given examples, the phase is a principal physical characteristic of any process, and as we can see in mathematical studies that use the embedding theorem with energy estimates, the phase disappears. Along with the phase, all reasonable information about the physical process disappears, as demonstrated by Tao [1] and other research studies. In fact, Tao built progressing waves that are not followed by energy estimates . Let us proceed with a more essential analysis of the influence of the phase on the behaviour of functions.

Theorem 1. *There are functions of $W_2^1(R)$ with a constant rate of the norm for a gradient catastrophe for which a phase change of its Fourier transformation is sufficient.*

Proof: To prove this, we consider a sequence of testing functions $\tilde{f}_n = \Delta/(1+k^2)$, $\Delta = (i-k)^n/(i+k)^n$. It is obvious that $|\tilde{f}_n| = 1/(1+k^2)$ and $\max|f_n|^2 \leq 2||f_n||_{L_2}||\nabla f_n||_{L_2} \leq$ const. Calculating the Fourier transformation of these testing functions, we obtain

$$f_n(x) = x(-1)^{(n-1)} 2\pi \exp(-x) L^1_{(n-1)}(2x) \text{if } x > 0, \; f_n(x) = 0 \;\; if \;\; x \leq 0, \tag{1}$$

where $L^1_{(n-1)}(2x)$ is a Laguerre polynomial. Now we see that the functions are equibounded and derivatives of these functions will grow with the growth of n. Thus, we have built an example of a sequence of the bounded functions of $W_2^1(R)$ which have a constant norm $W_2^1(R)$, and this sequence converges to a discontinuous function.

The results show the flaws of the embedding theorems when analyzing the behavior of functions. Therefore, this work is devoted to overcoming them and the basis for solving the formulated problem is the analytical properties of the Fourier transforms of functions on compact sets. Analytical properties and estimates of the Fourier transform of functions are studied using the Poincar Riemann Hilbert boundary value problem

3. Results for the three-dimensional case

Consider Schrödinger's equation:

$$-\Delta_x \Psi + q\Psi = k^2\Psi, \; k \in C. \tag{2}$$

Let $\Psi_+(k,\theta,x)$ be a solution of (2) with the following asymptotic behaviour:

$$\Psi_+(k,\theta,x) = \Psi_0(k,\theta,x) + \frac{e^{ik|x|}}{|x|}A(k,\theta^{'},\theta) + 0\left(\frac{1}{|x|}\right), \; |x| \to \infty, \tag{3}$$

where $A(k,\theta^{'},\theta)$ is the scattering amplitude and $\theta^{'} = \frac{x}{|x|}$, $\theta \in S^2$ for $k \in \bar{C}^+ = \{\mathrm{Im}k \geq 0\}$ $\Psi_0(k,\theta,x) = e^{ik(\theta,x)}$:

$$A(k,\theta^{'},\theta) = -\frac{1}{4\pi}\int_{R^3} q(x)\Psi_+(k,\theta,x)e^{-ik\theta^{'}x}dx.$$

Solutions to (2) and (3) are obtained by solving the integral equation

$$\Psi_+(k,\theta,x) = \Psi_0(k,\theta,x) + \int_{R^3} q(y)\frac{e^{+ik|x-y|}}{|x-y|}\Psi+(k,\theta,y)dy = G(q\Psi_+),$$

which is called the Lippman–Schwinger equation.

Let us introduce

$$\theta,\theta^{'} \in S^2, Df = k\int_{S^2} A(k,\theta^{'},\theta)f(k,\theta^{'})d\theta^{'}.$$

Let us also define the solution $\Psi_-(k,\theta,x)$ for $k \in \bar{C}^- = \{\mathrm{Im}k \leq 0\}$ as

$$\Psi_-(k,\theta,x) = \Psi_+(-k,-\theta,x).$$

As is well known [8],

$$\Psi_+(k,\theta,x) - \Psi_-(k,\theta,x) = -\frac{k}{4\pi}\int_{S^2} A(k,\theta^{'},\theta)\Psi_-(k,\theta^{'},x)d\theta^{'}, \; k \in R. \tag{4}$$

This equation is the key to solving the inverse scattering problem and was first used by Newton [8,9] and Somersalo et al. [10].

Definition 1. *The set of measurable functions* $\mathbf{R}$ *with the norm defined by*

$$||q||_{\mathbf{R}} = \int_{R^6} \frac{q(x)q(y)}{|x-y|^2}dxdy < \infty$$

is recognised as being of Rollnik class.

Equation (4) is equivalent to the following:

$$\Psi_+ = S\Psi_-,$$

where S is a scattering operator with the kernel

$$S(k,l) = \int_{R^3} \Psi_+(k,x)\Psi_-^*(l,x)dx.$$

The following theorem was stated in [9]:

Theorem 2. ***(Energy and momentum conservation laws)*** *Let $q \in \mathbf{R}$. Then, $SS^* = I$ and $S^*S = I$, where I is a unitary operator.*

Corollary 1. *$SS^* = I$ and $S^*S = I$ yield*

$$A(k,\theta',\theta) - A(k,\theta,\theta')^* = \frac{ik}{2\pi}\int_{S^2} A(k,\theta,\theta'')A(k,\theta',\theta'')^* d\theta''.$$

Theorem 3. ***(Birmann–Schwinger estimation)*** *Let $q \in \mathbf{R}$. Then, the number of discrete eigenvalues can be estimated as*

$$N(q) \le \frac{1}{(4\pi)^2}\int_{R^3}\int_{R^3} \frac{q(x)q(y)}{|x-y|^2}dxdy.$$

Lemma 1. *Let $\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right) < \alpha < 1/2$. Then,*

$$\|\Psi_+\|_{L_\infty} \le \frac{\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}{1-\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)} < \frac{\alpha}{1-\alpha},$$

$$\left\|\frac{\partial(\Psi_+ - \Psi_0)}{\partial k}\right\|_{L_\infty} \le \frac{|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}}{1-\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)} < \frac{\alpha}{1-\alpha}.$$

Proof. By the Lippman–Schwinger equation, we have

$$|\Psi_+ - \Psi_0| \le |Gq\Psi_+|,$$

$$|\Psi_+ - \Psi_0|_{L_\infty} \le |\Psi_+ - \Psi_0|_{L_\infty}|Gq| + |Gq|,$$

and, finally,

$$|\Psi_+ - \Psi_0| \le \frac{\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}{1-\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}.$$

By the Lippman–Schwinger equation, we also have

$$\left|\frac{\partial\left(\Psi_+ - \Psi_0\right)}{\partial k}\right| \le \left|\frac{\partial Gq}{\partial k}\Psi_+\right| + \left|Gq\frac{\partial\left(\Psi_+ - \Psi_0\right)}{\partial k}\right| + |Gq|,$$

$$\left|\frac{\partial(\Psi_+ - \Psi_0)}{\partial k}\right| \le \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right),$$

$$\left\|\frac{\partial(\Psi_+ - \Psi_0)}{\partial k}\right\|_{L_\infty} \le \frac{|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)},$$

which completes the proof.

Let us introduce the following notation:

$$Q(k,\theta,\theta') = \int_{R^3} q(x)e^{ik(\theta-\theta')x}dx,\ K(s) = s,\ X(x) = x,$$

$$T_+Q = \int_{-\infty}^{+\infty} \frac{Q(s,\theta,\theta')}{s-t-i0}ds,\ \ T_-Q = \int_{-\infty}^{+\infty} \frac{Q(s,\theta,\theta')}{s-t+i0}ds.$$

Lemma 2. *Let* $q \in \mathbf{R} \cap L_1(R^3)$, $\|q\|_{L_1} + 4\pi|q|_{L_2(R^3)} < \alpha < 1/2$. *Then,*

$$\|A_+\|_{L_\infty} < \alpha + \frac{\alpha}{1-\alpha},$$

$$\left\|\frac{\partial A_+}{\partial k}\right\|_{L_\infty} < \alpha + \frac{\alpha}{1-\alpha}.$$

Proof. Multiplying the Lippman–Schwinger equation by $q(x)\Psi_0(k,\theta,x)$ and then integrating, we have

$$A(k,\theta,\theta') = Q(k,\theta,\theta') + \int_{R^3} q(x)\Psi_0(k,\theta,x)Gq\Psi_+dx.$$

We can estimate this latest equation as

$$|A| \le \alpha + \alpha\frac{\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}.$$

Following a similar procedure for $\left\|\frac{\partial A_+}{\partial k}\right\|$ completes the proof.

We define the operators $T_\pm, T$ for $f \in W_2^1(R)$ as follows:

$$T_+f = \frac{1}{2\pi i}\lim_{\mathrm{Im}z\to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds,\ \mathrm{Im}\ z>0,\ T_-f = \frac{1}{2\pi i}\lim_{\mathrm{Im}z\to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds,\ \mathrm{Im}\ z<0,$$

$$Tf = \frac{1}{2}(T_+ + T_-)f.$$

Consider the Riemann problem of finding a function Φ that is analytic in the complex plane with a cut along the real axis. Values of Φ on the two sides of the cut are denoted as Φ_+ and Φ_-. The following presents the results of [12]:

Lemma 3.

$$TT = \frac{1}{4}I,\ TT_+ = \frac{1}{2}T_+,\ TT_- = -\frac{1}{2}T_-,\ T_+ = T + \frac{1}{2}I,\ T_- = T - \frac{1}{2}I,\ T_-T_- = -T_-.$$

Denote

$$\Phi_+(k,\theta,x) = \Psi_+(k,\theta,x) - \Psi_0(k,\theta,x),\ \ \Phi_-(k,\theta,x) = \Psi_-(k,-\theta,x) - \Psi_0(k,\theta,x),$$

$$g(k,\theta,x) = \Phi_+(k,\theta,x) - \Phi_-(k,\theta,x)/$$

Lemma 4. *Let* $q \in \mathbf{R},\ N(q) < 1,\ g_+ = g(k,\theta,x),$ *and* $g_- = g(k,-\theta,x)$. *Then,*

$$\Phi_+(k,\theta,x) = T_+g_+ + e^{ik\theta x},\ \ \Phi_-(k,\theta,x) = T_-g_+ + e^{ik\theta x}.$$

Proof. The proof of the above follows from the classic results for the Riemann problem.

Lemma 5. *Let* $q \in \mathbf{R},\ N(q) < 1,\ g_+ = g(k,\theta,x),$ *and* $g_- = g(k,-\theta,x),)$. *Then,*

$$\Psi_+(k,\theta,x) = (T_+g_+ + e^{ik\theta x}),\ \ \Psi_-(k,\theta,x) = (T_-g_- + e^{-ik\theta x}).$$

Proof. The proof of the above follows from the definitions of g, $\Phi_\pm$, and $\Psi_\pm$.

Lemma 6. *Let*

$$\sup_k \left| \int\limits_{-\infty}^{\infty} \frac{pA(p,\theta',\theta)}{4\pi(p-k+i0)}dp \right| < \alpha,\ \int_{S_2} \alpha d\theta < 1/2.$$

Then,

$$\prod_{0\le j<n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_jA(k_j,\theta'_{k_j},\theta_{k_j})}{4\pi(k_{j+1}-k_j+i0)}dk_j \right| d\theta_{k_j} \le 2^{-n}.$$

Proof.
Denote

$$\alpha_j = \left| Vp \int_{-\infty}^{\infty} \frac{k_jA(k_j,\theta'_{k_j},\theta_{k_j})}{4\pi(k_{j+1}-k_j+i0)}dk_j \right|,$$

Therefore,

$$\prod_{0\le j<n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_jA(k_j,\theta'_{k_j},\theta_{k_j})}{4\pi(k_{j+1}-k_j+i0)}dk_j \right| d\theta_{k_j} \le \prod_{0\le j<n} \int_{S_2} \alpha_j d\theta_{k_j} < 2^{-n}.$$

This completes the proof.

Lemma 7. *Let*

$$\sup_k \int_{S^2} |T_- QK|\, d\theta \le \alpha < \frac{1}{2C} < 1, \quad \sup_k \int_{S^2} |T_- \tilde{q}K|\, d\theta \le \alpha < \frac{1}{2C} < 1,$$

$$\sup_k \int_{S^2} \left|T_- Q\tilde{q}K^2\right| d\theta \le \alpha < \frac{1}{2C} < 1.$$

Then,

$$\sup_k \int_{S^2} |T_- AK|\, d\theta \le \frac{C \int_{S^2} |T_- QK|\, d\theta}{1 - \sup\limits_k \int_{S^2} |T_- A\tilde{q}K^2|\, d\theta},$$

$$\sup_k \left| \int_{S^2} T_- A\tilde{q}K^2 d\theta \right| \le \frac{C \left|T_- \int_{S^2} Q\tilde{q}K^2 d\theta\right|}{1 - \left|T_- \int_{S^2} \tilde{q}K d\theta\right|}.$$

Proof. By the definition of the amplitude and Lemma 4, we have

$$A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \Psi_+(k, \theta, x) e^{-ik\theta' x} dx$$

$$= -\frac{1}{4\pi} \int_{R^3} q(x) \left[e^{ik\theta' x} + T_+ g(k, \theta, \theta') \right] e^{-ik\theta' x} dx.$$

We can rewrite this as

$$A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \left[e^{ik\theta x} + \sum_{n \ge 0} (-T_- D)^n \Psi_0 \right] e^{-ik\theta' x} dx. \tag{5}$$

Lemma 6 yields

$$\sup_k \int_{S^2} |T_- AK|\, d\theta \le \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- QK \right| d\theta + \frac{\left(\sup\limits_k \int_{S^2} |T_- KA|\, d\theta \right)^2 \int_{S^2} \left|T_- A\tilde{q}K^2\right| d\theta}{\left(1 - \sup\limits_k \int_{S^2} |T_- KA|\, d\theta \right)^2}.$$

Owing to the smallness of the terms on the right-hand side, the following estimate follows:

$$\sup_k \int_{S^2} |T_- AK|\, d\theta \le 2 \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- QK \right| d\theta.$$

Similarly,

$$\sup_k \int_{S^2} \left|T_- A\tilde{q}K^2\right| d\theta \le C \int_{S^2} \left|T_- Q\tilde{q}K^2\right| d\theta + \int_{S^2} \left|T_- A\tilde{q}K^2\right| d\theta \int_{S^2} |T_- \tilde{q}K|\, d\theta,$$

$$\sup_k \int_{S^2} \left|T_- A\tilde{q}K^2\right| d\theta \le \frac{C \int_{S^2} \left|T_- Q\tilde{q}K^2\right| d\theta}{1 - \int_{S^2} |T_- \tilde{q}K|\, d\theta},$$

$$\sup_k \int_{S^2} \left|T_- A\tilde{q}K^2\right| d\theta \leq 2\sup_k \int_{S^2} \left|\frac{1}{4\pi}T_- Q\tilde{q}K^2\right| d\theta.$$

This completes the proof.

To simplify the writing of the following calculations, we introduce the set defined by

$$M_\epsilon(k) = \left(s|\epsilon < |s| + |k-s| < \frac{1}{\epsilon}\right).$$

The Heaviside function is given by

$$\Theta(x) = \{1, \text{ if } x > 0, \quad -1 \text{ if } x < 0 \ \}.$$

Lemma 8. *Let* $q, \nabla q \in \cap L_2(R^3)$, $|A| > 0$. *Then,*

$$\pi i \int_{R^3} \Theta(A) e^{ik|x|A} q(x) dx = \lim_{\epsilon \to 0} \int_{s \in M_\epsilon(k)} \int_{R^3} \frac{e^{is|x|A}}{k-s} q(x) dx ds,$$

$$\pi i \int_{R^3} \Theta(A) k e^{ik|x|A} q(x) dx = \lim_{\epsilon \to 0} \int_{s \in M_\epsilon(k)} \int_{R^3} s\frac{e^{is|x|A}}{k-s} q(x) dx ds.$$

Proof. The lemma can be proved by the conditions of lemma and the lemma of Jordan.

Lemma 9. *Let*

$$l = 2, \;\; I_0 = \Psi_0(x,k)|_{r=r_0}.$$

Then

$$\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} \tilde{q}(k(\theta-\theta'))I_0 k^2 dk d\theta d\theta'\right| \leq \sup_{x\in R^3}|q(x)| + C_0(\frac{1}{r_0}+r_0)\left\|q\right\|_{L_2(R^3)},$$

$$\sup_{\theta\in S^2}\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} QTKQI_0 k^2 d\theta'' d\theta' dk\right| \leq C_0(\frac{1}{r_0}+r_0)\left\|q\right\|^2_{L_2(R^3)}.$$

Proof. By the definition of the Fourier transform, we have

$$\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} \tilde{q}(k(\theta-\theta'))I_0 k^2 dk d\theta d\theta' = \int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\int_0^{+\infty} q(x)e^{ikx(\theta-\theta')}e^{ix_0 k}k^2 dk d\theta d\theta' dr d\gamma,$$

where $x = r\gamma$ The lemma of Jordan completes the proof for the first inequality. The second inequality is proved like the first:

$$\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} QTKQI_0 k^2 d\theta'' d\theta' dk$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\int_{S^2} \frac{(\tilde{q}(s\cos(\theta') - s\cos(\theta''))\tilde{q}(k\cos(\theta) - s\cos(\theta''))\, s}{k-s} I_0 k^2 d\theta' d\theta'' d\theta dk ds.$$

Lemma 8 yields

$$\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\int_{S^2}\left(\tilde{q}(k\cos(\theta')-k\cos(\theta))\tilde{q}(k\cos(\theta)-k\cos(\theta''))\right)I_0k^3\Theta(\cos(\theta''))d\theta'd\theta''d\theta dk-$$

$$\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\int_{S^2}\left(\tilde{q}(k\cos(\theta')-k\cos(\theta))\tilde{q}(k\cos(\theta)-k\cos(\theta''))\right)I_0k^3\Theta(-\cos(\theta''))d\theta'd\theta''d\theta dk.$$

Integrating θ, θ', θ'', and k, we obtain the proof of the second inequality of the lemma.

Lemma 10. *Let*

$$\sup_k|T_-QK|\le\alpha<\frac{1}{2C}<1,\quad \sup_k|T_-\tilde{q}K|\le\alpha<\frac{1}{2C}<1,$$

$$\sup_k\left|T_-Q\tilde{q}K^2\right|\le\alpha<\frac{1}{2C}<1,\ \ l=0,1,2.$$

Then,

$$\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}A(k,\theta',\theta)k^ldkd\theta'd\theta\right|\le\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\tilde{q}(k(\theta-\theta'))k^ldkd\theta'd\theta\right|$$

$$+C\sup_{\theta\in S^2}\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}QTKAk^ld\theta''d\theta'dk\right|,$$

$$\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}A(k,\theta',\theta)k^2dkd\theta'd\theta\right|\le\sup_{x\in R^3}|q|+C_0\left\|q\right\|_{W_2^1(R^3)}\left\|q\right\|_{L_2(R^3)}\left(\left|\int_{S^2}TKAd\theta''\right|+1\right).$$

Proof.

Using the definition of the amplitude, Lemmas 3 and 4, and the lemma of Jordan yields

$$\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}A(k,\theta^{'},\theta)k^ldkd\theta'd\theta=-\int_{-\infty}^{+\infty}\frac{1}{4\pi}\int_{S^2}\int_{S^2}\int_{R^3}q(x)\Psi_+(k,\theta,x)e^{-ik\theta^{'}x}k^ldxdkd\theta'=$$

$$-\frac{1}{4\pi}\int_{S^2}\int_{S^2}\int_{R^3}q(x)\left[e^{ik\theta x}+\sum_{n\ge1}(-T_-D)^n\Psi_0\right]e^{-ik\theta^{'}x}k^ld\theta'dxdk$$

$$=\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\tilde{q}(k(\theta-\theta'))k^ldkd\theta'd\theta+\sum_{n\ge1}W_n,$$

$$W_1=\int_{R^3}\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\frac{sA(s,\theta^{''},\theta)e^{-ik\theta^{'}x}q(x)e^{is\theta''x}}{k-s}k^ldkdxdsd\theta'd\theta'',$$

$$|W_1|\le C\sup_{\theta\in S^2}\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}QTKAk^ld\theta''d\theta'dk\right|.$$

Similarly,

$$|W_n| \le C \sup_{\theta\in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QTKAk^l d\theta'' d\theta' dk \right| \left| \int_{S^2} TKAd\theta'' \right|^n .$$

Finally,

$$\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k,\theta',\theta) dk d\theta' d\theta \right| \le \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta-\theta')) dk d\theta d\theta' \right|$$
$$+C_0 \|q\|^2_{L_2(R^3)} \left(\left| \int_{S^2} TKAd\theta'' \right| + 1 \right),$$

$$\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k,\theta',\theta) k^2 dk d\theta' \right| \le \sup_{x\in R^3} |q| + C_0 \|q\|^2_{L_2(R^3)} \left(\left| \int_{S^2} TKAd\theta'' \right| + 1 \right).$$

This completes the proof.

Lemma 11. *Let*

$$\sup_k \int_{S^2} \left| \int_{-\infty}^{\infty} \frac{pA(p,\theta',\theta)}{4\pi(p-k+i0)} dp \right| d\theta < \alpha < 1/2, \ \sup_k \left| pA(p,\theta',\theta) \right| < \alpha < 1/2.$$

Then,

$$|T_- D\Psi_0| < \frac{\alpha}{1-\alpha}, \quad |T_+ D\Psi_0| < \frac{\alpha}{1-\alpha}, \qquad |D\Psi_0| < \frac{\alpha}{1-\alpha},$$
$$T_- g_- = (I - T_- D)^{-1} T_- D\Psi_0, \qquad \Psi_- = (I - T_- D)^{-1} T_- D\Psi_0 + \Psi_0,$$

and q satisfies the following inequalities:

$$\sup_{x\in R^3} |q(x)| \le \left| \int_{S^2} TKQd\theta \right| C_0 \left(\|q\|^2_{L_2(R^3)} + 1 \right) + C_0 \|q\|_{L_2(R^3)} .$$

Proof. Using the equation

$$\Psi_+(k,\theta,x) - \Psi_-(k,\theta,x) = -\frac{k}{4\pi} \int_{S^2} A(k,\theta',\theta) \Psi_-(k,\theta',x) d\theta', \ k \in R,$$

we can write

$$T_+ g_+ - T_- g_- = D(T_- g_- + \Psi_0).$$

Applying the operator T_- to the last equation, we have

$$T_- g_- = T_- D(T_- g_- + \Psi_0),$$

$$(I - T_- D) T_- g_- = T_- D\Psi_0, \ T_- g_- = \sum_{n\ge 0} (-T_- D)^n \Psi_0.$$

Estimating the terms of the series, we obtain using Lemma 4

$$|(T_-D)^n\Psi_0| \leq \sum_{n\geq 0}\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Psi_0\prod_{0\leq j<n}\frac{\int_{S^2}k_jA(k_j,\theta'_{k_j},\theta_{k_j})d\theta'_{k_j}}{4\pi(k_{j+1})-k_j+i0)}dk_1\dots d_{k_n}\right|$$

$$\leq \sum_{n>0}2^n\alpha^n = \frac{2\alpha}{1-2\alpha}.$$

Denoting

$$\Lambda = \frac{\partial}{\partial k},\ r=\sqrt{x_1^2+x_2^2+x_3^2},$$

we have

$$\Lambda\int_{S^2}\Psi_0 d\theta = \Lambda\frac{\sin(kr)}{ikr} = \frac{\cos(kr)}{ik} - \frac{\sin(kr)}{ik^2r},$$

$$\Lambda\int_{S^2}H_0\Psi_0 d\theta = \Lambda k^2\frac{\sin(kr)}{ikr} = k\frac{\cos(kr)}{i} + \frac{\sin(kr)}{ik^2r},$$

$$\left|\Lambda\int_{S^2}\Psi d\theta\right| = \left|\Lambda\int_{S^2}\Psi_0 d\theta + \Lambda\int_{S^2}\sum_{n\geq 0}\left(-T_-D\right)^n\Psi_0 d\theta\right| > \left(\frac{1}{k}-\frac{\alpha}{1-\alpha}\right),\ \text{as } kr=\pi,$$

and

$$\Lambda\frac{1}{k-t} = -\frac{1}{(k-t)^2}$$

Equation (2) yields

$$q = \frac{\Lambda\left(H_0\int_{S^2}\Psi d\theta + k^2\int_{S^2}\Psi d\theta\right)}{\Lambda\int_{S^2}\Psi d\theta}$$

$$= \frac{2k\int_{S^2}T_-g_-d\theta + k^2\int_{S^2}\Lambda T_-g_-d\theta + H_0\Lambda\int_{S^2}T_-g_-d\theta}{\Lambda\int_{S^2}\Psi d\theta}$$

$$= \frac{2k\int_{S^2}T_-g_-d\theta + \Lambda\int_{S^2}\sum_{n\geq 1}\left(-T_-D\right)^n(K^2-k^2)\Psi_0 d\theta}{\Lambda\int_{S^2}\Psi d\theta}$$

$$= \frac{W_0+\sum_{n\geq 1}\int_{S^2}W_n}{\Lambda\int_{S^2}\Psi d\theta}.$$

Denoting

$$Z(k,s) = s+2k+\frac{2k^2}{k-s},$$

we then have

$$|W_1| \leq \left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}A(s,\theta,\theta')s\frac{s^2-k^2}{(k-s)^2}\Psi_0\sin(\theta)dsd\theta\right|_{k=k_0}$$

$$\leq \left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}Z(k,)\tilde{q}(k(\theta-\theta'))\Psi_0 dkd\theta\right| + C_0\left|\int_{S^2}TKQd\theta\right|.$$

For calculating W_n, as $n \geq 1$, take the simple transformation

$$\frac{s_n^3}{s_n - s_{n-1}} = \frac{s_n^3 - s_n^2 s_{n-1}}{s_n - s_{n-1}} + \frac{s_n^2 s_{n-1}}{s_n - s_{n-1}} = s_n^2 + \frac{s_n^2 s_{n-1}}{s_n - s_{n-1}}$$

$$= s_n^2 + \frac{s_n^2 s_{n-1} - s_n s_{n-1}^2}{s_n - s_{n-1}} + \frac{s_n s_{n-1}^2}{s_n - s_{n-1}} = s_n^2 + s_n s_{n-1} + \frac{s_n s_{n-1}^2}{s_n - s_{n-1}}, \tag{6}$$

$$\frac{A s_n^3}{s_n - s_{n-1}} = A s_n^2 + A s_n s_{n-1} + \frac{A s_n s_{n-1}^2}{s_n - s_{n-1}} = V_1 + V_2 + V_3.$$

Using Lemma 10 for estimating V_1 and V_2 and, for V_3, taking again the simple transformation for s_{n-1}^3, which will appear in the integration over s_{n-1}, we finally get

$$|q(x)|_{r=r_0} = \left| \frac{\Lambda \left(H_0 \int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta \right)}{\Lambda \int_{S^2} \Psi d\theta} \right|_{k=k_0, r=\frac{\pi}{k_0}}$$

$$\leq \frac{\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} Z(k,)\tilde{q}(k(\theta - \theta'))\Psi_0 dk d\theta d\theta' \right| + C_0 \left| \int_{S^2} TKQ d\theta \right|}{\left(\frac{1}{k_0} - \frac{\alpha}{(1-\alpha)} \right)} +$$

Finally, we get

$$|q(x)|_{r=r_0} \leq \sup_{x \in R^3} |q(x)| \alpha + C_0 \|q\|_{L_2(R^3)}^2 + C_0 \|q\|_{L_2(R^3)} + \left| \int_{S^2} TKQ d\theta \right|.$$

The invariance of the Schrödinger equations with respect to translations and the arbitrariness of r_0 yield

$$\sup_{x \in R^3} |q(x)| \leq \left| \int_{S^2} TKQ d\theta \right| C_0 \left(\|q\|_{L_2(R^3)}^2 + 1 \right) + C_0 \|q\|_{L_2(R^3)}.$$

4. Discussion of the three-dimensional inverse scattering problem

This study has shown, once again, the outstanding properties of the scattering operator, which, in combination with the analytical properties of the wave function, allows us to obtain almost-explicit formulas for the potential from the scattering amplitude. Furthermore, this appro. The estimations following from this overcome the problem of overdetermination, resulting from the fact that the potential is a function of three variables, whereas the amplitude is a function of five variables. We have shown that it is sufficient to average the scattering amplitude to eliminate the two extra variables.

5. Studying the properties of solutions of the Cauchy problem for the Navier–Stokes equations using analytic functions generated by the Schrödinger equations and related to the Poincaré--Riemann-Hilbert problem

Numerous studies of the Navier–Stokes equations have been devoted to the problem of the smoothness of its solutions. A good overview of these studies is given in Refs. [13–17]. The spatial differentiability of the solutions is an important factor, as it controls their evolution. Obviously, differentiable solutions do not provide an effective description of turbulence. Nevertheless, the global solvability and differentiability of the solutions have not been proven, and therefore the problem of describing turbulence remains open. It is interesting to study the properties of the Fourier transform of solutions of the Navier–Stokes equations. Of particular interest is how they can be used in the description of turbulence and whether they are differentiable. The differentiability of such Fourier transforms appears to be related to the appearance or disappearance of resonance, as this implies the absence of large energy flows from small to large harmonics, which in turn precludes the appearance of turbulence. Therefore, obtaining uniform global estimations of the Fourier transform of solutions of the Navier–Stokes equations means that the principle modelling of complex flows and related calculations will be based on the Fourier transform method. We are continuing to research these issues in relation to a numerical weather prediction model; this paper provides a theoretical justification for this approach.

Consider the Cauchy problem for the Navier–Stokes equations:

$$\frac{\partial \vec{v}}{\partial t} - \nu\Delta\vec{v} + (\vec{v}, \nabla\vec{v}) = -\nabla p + \vec{f}(x,t), \ \text{div}\ \vec{v} = 0, \tag{7}$$

$$\vec{v}|_{t=0} = \vec{v}_0(x) \tag{8}$$

in the domain $Q_T = R^3 \times (0,T)$, where

$$\text{div}\ \vec{v}_0 = 0. \tag{9}$$

The problem defined by (7)–(9) has at least one weak solution $(\vec{v}, p)$ in the so-called Leray–Hopf class [16]. The following results have been proved [15]:

Theorem 4. *If*

$$\vec{v}_0 \in W_2^1(R^3), \vec{f}(x,t) \in L_2(Q_T),$$

there is a single generalised solution of (7)–(9) in the domain Q_{T_1}, $T_1 \in [0,T]$, *satisfying the following conditions:*

$$\vec{v}, \nabla^2\vec{v}, \quad \nabla p \in L_2(Q_T).$$

Note that T_1 depends on $\vec{v}_0$ and $\vec{f}(x,t)$.

Lemma 12. *If we let $\vec{v_0} \in W_2^2(R^3), \vec{f} \in L_2(Q_T)$, then the solution of (7)–(9) satisfies the following inequalities:*

$$\sup_{0\le t\le T} ||\vec{v}||^2_{L_2(R^3)} + \nu \int\limits_0^t ||\nabla \vec{v}||^2_{L_2(R^3)} d\tau \le \ ||\vec{v_0}||^2_{L_2(R^3)} + ||\vec{f}||_{L_2(Q_T)},$$

$$\sup_{0\le t\le T} ||\vec{\nabla} v||^2_{L_2(R^3)} + \nu \int\limits_0^t ||H_0\vec{v}||^2_{L_2(R^3)} d\tau$$

$$\le ||\nabla \vec{v}_0||^2_{L_2(R^3)} + ||\vec{f}||_{L_2(Q_T)} + \int_0^t ||(\vec{v}, \nabla \vec{v})||_{L_2(R^3)} ||H_0\vec{v}||_{L_2(R^3)},$$

$$\nu \int\limits_0^t ||H_0\vec{v}||^2_{L_2(R^3)} d\tau \le C + \frac{1}{\nu} \int_0^t ||(\vec{v}, \nabla \vec{v})||^2_{L_2(R^3)} dt.$$

Lemma 13. *Let $\vec{v_0} \in W_2^2(R^3)$, $\tilde{\vec{v_0}} \in W_2^2(R^3)$, and $\vec{f} \in L_2(Q_T)$. Then, the solution of (7)–(9) satisfies the following:*

$$\widetilde{\vec{v}} = \widetilde{\vec{v}}_0 + \int\limits_0^t e^{-\nu k^2 |(t-\tau)} ([\widetilde{(\vec{v}, \nabla) \vec{v}}] + \widetilde{\vec{F}}) d\tau,$$

where $\vec{F} = -\nabla p + \vec{f}$.

Proof. This follows from the definition of the Fourier transform and the theory of linear differential equations.

Let us introduce the operators F_k and $F_{kk\prime}$ as

$$F_k f = \int_{R^3} e^{i(k,x)} f(x) dx, \ \ F_{kk\prime} f = \int_{R^3} e^{i(k,x) - i(x,k')} f(x) dx,$$

$$\tilde{\vec{v}}(k) = F_k \vec{v}, \ \vec{V}(k,k') = F_{kk\prime} \vec{v} = \int_{R^3} e^{i(k,x) - i(x,k')} \vec{v} dx.$$

Lemma 14. *Let $\vec{v_0} \in W_2^2(R^3)$, $\vec{f} \in L_2(Q_T)$, and $|TKV_0| + |TKV_0| + \left| TK^2 V_0 \tilde{\vec{v_0}} \right| < C$. Then, the solution of (7)–(9) in Theorem 4 satisfies the following inequalities:*

$$|\tilde{v}(k)| < C,$$

$$|TK\tilde{v}(k)| < C_0 ||v||_{L_2(R^3)} + \frac{C_0 t}{\sqrt{\nu}} ||\nabla v||_{L_2(R^3)} ||v||_{L_2(R^3)}.$$

Proof. This follows from

$$\dot{\vec{v}} = -(\vec{v}\nabla)\vec{v} + (\nu\vec{v} + \nabla p) + F,$$

$$\tilde{\vec{v}} = \tilde{\vec{v}}_0 + \int_0^t e^{-\nu k^2(t-\tau)} F_k\left(-\ (\vec{v},\nabla)\vec{v}) + \nabla p + F\right) d\tau.$$

From the last equation we have

$$|\vec{v}| \leq |\vec{v}_0| + C_T.$$

Denote

$$\beta = \sqrt{\nu(t-\tau)},\ \ a = \theta x$$

formula 121 (23) from [11] as $n = 0$: yield

$$|TK\vec{v}| < \left|ke^{-\beta^2k^2}\right| + \sqrt{\pi}\beta^{-1}e^{-\frac{a^2}{8\beta^2}}D_0\left(\frac{a}{\sqrt{2}\beta}\right),$$

$$|TK\vec{v}| \leq |TK\vec{v}_0|$$

$$+\left|TK\int_0^t e^{-\nu k^2(t-\tau)}F_k\left(-(\vec{v},\nabla)\vec{v}] + \nabla p + F\right) dk\right|$$

$$\leq |TK\vec{v}_0| + \int_0^t \left|ke^{-\beta^2k^2}\right| + \left|\sqrt{\pi}\beta^{-1}e^{-\frac{a^2}{8\beta^2}}D_0(\frac{a}{\sqrt{2}\beta})\right| ||\nabla\vec{v}||_{L_2(R^3)}dt$$

$$\leq C_0||v||_{L_2(R^3)} + \frac{C_0t}{\sqrt{\nu}}||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}.$$

Lemma 15. *Let* $\vec{v_0} \in W_2^2(R^3)$, $\vec{f} \in L_2(Q_T)$, *and* $|TKV_0| + |TKV_0| + \left|TK^2V_0\vec{\tilde{v}}_0\right|$. *Then, the solution of (7)–(9) in Theorem 4 satisfies the following inequalities:*

$$|\vec{V}(k,k')| < C,\ \ k|\vec{V}(k,k')| < \frac{C}{\sqrt{(1-\cos(\theta))}},$$

$$|T\vec{V}K| < C_0||v||_{L_2(R^3)} + \frac{C_0t}{\sqrt{\nu(1-\cos(\theta))}}||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}.$$

Proof. This follows from

$$\dot{\vec{V}} = -F_{kk\prime}[(\vec{v},\nabla)\vec{v}] + F_{kk\prime}(\nu\Delta\vec{v} + \nabla p) + F_{kk\prime}F.$$

After the transformations, we obtain

$$\dot{\vec{V}} = -F_{kk\prime}[(\vec{v}\nabla)\vec{v}] + (\nu_k F_{kk\prime}\vec{v} + F_{kk\prime}\nabla p) + F_{kk\prime}F,$$

$$\vec{V} = \vec{V}_0 + \int_0^t e^{-\nu k^2(1-\cos(\theta))(t-\tau)}\left(-F_{kk\prime}[(\vec{v},\nabla)\vec{v}] + F_{kk\prime}\nabla p + F_{kk\prime}F\right).$$

From the last equation, we have

$$|\vec{V}| \leq |\vec{V}_0| + C_0 \int_0^t ||\nabla v||_{L_2(R^3)} ||v||_{L_2(R^3)} d\tau.$$

Denote $\beta = \sqrt{(1-\cos(\theta))(t-\tau)\nu}$, $a = (\theta - \theta')x$ formula 121 (23) from [11] as $n = 0$: yield

$$\left|TK\vec{V}\right| < \left|ke^{-\beta^2 k^2}\right| + \sqrt{\pi}\beta^{-1} e^{-\frac{a^2}{8\beta^2}} D_0\left(\frac{a}{\sqrt{2}\beta}\right),$$

$$\left|TK\vec{V}\right| \leq \left|TK\vec{V}_0\right|$$

$$+\left|TK \int_0^t e^{-\nu k^2(1-\cos(\theta))(t-\tau)} \left(-F_{kk\prime}(\vec{v},\nabla)\vec{v}] + F_{kk\prime}\nabla p + F_{kk\prime}F\right) dk\right|$$

$$\leq \left|TK\vec{V}_0\right| + \int_0^t \left|ke^{-\beta^2 k^2}\right| + \left|\sqrt{\pi}\beta^{-1} e^{-\frac{a^2}{8\beta^2}} D_0\left(\frac{a}{\sqrt{2}\beta}\right)\right| ||\nabla\vec{v}||_{L_2(R^3)} ||\vec{v}||_{L_2(R^3)} dt$$

$$< C_0||v||_{L_2(R^3)} + \frac{C_0 t}{\sqrt{\nu(1-\cos(\theta))}} ||\nabla v||_{L_2(R^3)} ||v||_{L_2(R^3)}.$$

Theorem 5. *Let $\vec{v_0} \in W_2^2(R^3)$, $\vec{f} \in L_2(Q_T)$, $\vec{\tilde{f}} \in W_2^{2,1}(Q_T)$, $|TKV_0| + |TKV_0| + \left|TK^2 V_0 \vec{\tilde{v}}_0\right| < C$, and $\int_0^\infty ||H_0\vec{f}||_{L_2(R^3)} dt < C$. Then, the solution of (7)–(9) in Theorem 4 satisfies the following inequalities:*

$$\sup_{x\in R^3} ||\vec{v}(x)|| < C,$$

$$||\nabla\vec{v}||_{L_2(R^3)} + \nu \int_0^T \int_{R^3} |H_0\vec{v}|^2 dx d\tau \leq \text{const}.$$

Proof.

Consider the Cauchy problem for the Navier–Stokes equations:

$$\frac{\partial\vec{v}}{\partial t} - \nu\Delta\vec{v} + (\vec{v},\nabla\vec{v}) = -\nabla p + \vec{f}(x,t), \text{ div } \vec{v} = 0, \tag{10}$$

$$\vec{v}|_{t=0} = \vec{v}_0(x) \tag{11}$$

in the domain $Q_T = R^3 \times (0,T)$, where

$$\text{div } \vec{v}_0 = 0. \tag{12}$$

We perform the following transformations:

$$\vec{u_\epsilon} = \epsilon\vec{v},\ p_\epsilon = p\epsilon,\ f_\epsilon = f\epsilon^2,\ \nu_\epsilon = \epsilon\nu, s = \frac{t}{\epsilon}.$$

Then,

$$\frac{\partial \vec{u_\epsilon}}{\partial s} - \nu_\epsilon \Delta \vec{u_\epsilon} + (\vec{u_\epsilon}, \nabla \vec{u_\epsilon}) = -\nabla_\epsilon p_\epsilon + \vec{f_\epsilon}(x,t), \text{ div } \vec{u_\epsilon} = 0, \tag{13}$$

$$\vec{u_\epsilon}|_{t=0} = \vec{u_{\epsilon 0}}(x) \tag{14}$$

in the domain $Q_T = R^3 \times (0, T_\epsilon)$, where

$$\text{div } \vec{u_\epsilon}|_{t=0} = 0. \tag{15}$$

Let us return for convenience to the notation $v_i = u_{\epsilon_i}$, using the equation for each $v_i = u_{\epsilon_i}$. This gives us

$$-\Delta_x \Psi + v_i \Psi = k^2 \Psi, \ k \in C.$$

Using Lemmas 12-15, we get estimates for

$$A_i, \ \vec{V_i}, \ TA_i, \ T\vec{V_i}, \ kA_i, k\vec{V_i}, \ TKA_i, \ TK\vec{V_i}, \ TK\tilde{v_i}, \ TK^2 V\tilde{v_i}.$$

The last estimations yield the representation

$$q = \frac{\Lambda \left(H_0 \int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta \right)}{\Lambda \int_{S^2} \Psi d\theta} |_{r=\frac{\pi}{k_0}, k=k_0},$$

and Lemma 11 implies

$$||\nabla \vec{v}||^2_{L_2(R^3)} + \nu_\epsilon \int\limits_0^t ||H_0 \vec{v}||^2_{L_2(R^3)} d\tau \leq \int_0^\infty ||(\vec{v})||_{L_2(R^3)} ||||H_0 \vec{f}||_{L_2(R^3)} d\tau +$$

$$||\nabla \vec{v_0}||^2_{L_2(R^3)} + \frac{C_0}{\nu_\epsilon} \int_0^t \left(\frac{C_1}{\nu_\epsilon} ||(\nabla \vec{v})||^2_{L_2(R^3)} ||(\vec{v})||^2_{L_2(R^3)} + ||\vec{v}||^2_{L_2(R^3)} \right) ||(\nabla \vec{v})||^2_{L_2(R^3)} d\tau.$$

Denote

$$\alpha(s) = \frac{C_0}{\nu_\epsilon} \left(\frac{C_1}{\nu_\epsilon} ||(\nabla \vec{v})||^2_{L_2(R^3)} ||(\vec{v})||^2_{L_2(R^3)} + ||\vec{v}||^2_{L_2(R^3)} \right),$$

$$\int_0^{\frac{T}{T\epsilon\nu}} \alpha(s) ds \leq \int_0^{\frac{1}{\nu\epsilon}} \frac{C_0}{\nu_\epsilon} \left(\frac{C_1}{\nu_\epsilon} ||(\nabla \vec{v})||^2_{L_2(R^3)} ||(\vec{v})||^2_{L_2(R^3)} + ||\vec{v}||^2_{L_2(R^3)} \right) ds$$
$$\leq \frac{C_0 C_1}{\nu_\epsilon^3} \sup_t ||(\vec{v})||^2_{L_2(R^3)} \int_0^\infty \nu_\epsilon ||(\nabla \vec{v})||^2_{L_2(R^3)} ||ds + \frac{C_0}{\nu_\epsilon} \sup_t ||(\vec{v})||^2_{L_2(R^3)}$$
$$\leq \frac{C_0 \epsilon^4}{\epsilon \nu_\epsilon^3} + \frac{C_0 \epsilon^2 \frac{\nu}{\epsilon}}{\nu_\epsilon} \leq 2C_0.$$

As $\epsilon = \nu\epsilon_0$, the Gronwall–Bellman lemma yields

$$||\nabla \vec{v}||^2_{L_2(R^3)} + \nu_\epsilon \int\limits_0^t \int\limits_{R^3} |H_0 \vec{v}|^2 dx d\tau \leq ||\nabla \vec{v_0}||^2_{L_2(R^3)} e^{2C_0}$$

$$+ e^{2C_0} \int_0^\infty ||(\vec{v})||_{L_2(R^3)} ||||H_0 \vec{f}||_{L_2(R^3)} d\tau.$$

Theorem 5 asserts the global solvability and uniqueness of the Cauchy problem for the Navier–Stokes equations.

6. Discussion

As noted in the introduction, the key method of investigating the Cauchy problem for the Navier–Stokes equations is its reduction to the Poincaré–Riemann–Hilbert problem. By studying the wave functions for the Schrdinger equation of the generated velocity components, we obtain unique estimates for the maximum velocity. Uniform global estimations of the Fourier transform of solutions of the Navier–Stokes equations indicate that the principle modelling of complex flows and related calculations can be based on the Fourier transform method. In terms of the Fourier transform, under both smooth initial conditions and right-hand sides, no exacerbations appear in the speed and pressure modes. A loss of smoothness in terms of the Fourier transform can only be expected in the case of singular initial conditions or of unlimited forces in $L_2(Q_T)$. The theory developed by us is supported by numerical calculations performed in Refs. [18–20], where the dependence of the smoothness of the solution on the oscillations of the system is clearly deduced.

7. Reduction of the Riemann hypothesis to the Poincaré–Riemann–Hilbert problem

This study is concerned with the properties of modified zeta functions. Riemann's zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \tag{16}$$

which is absolutely and uniformly convergent in any finite region of the complex s plane for which $\sigma \geq 1+\epsilon$, $\epsilon > 0$. If $\sigma > 1$, then ζ is represented by the following Euler product formula:

$$\zeta(s) = \prod_p \left[1 - \frac{1}{p^s}\right]^{-1}, \tag{17}$$

where p runs over all prime numbers. $\zeta(s)$ was first introduced in 1737 by Euler [21], who also obtained formula (22). Dirichlet and Chebyshev considered this function in their study on the distribution of prime numbers [22]. However, the most profound properties of $\zeta(z)$ were only discovered later, when it was extended to the complex plane. In 1876, Riemann [23] proved that $\zeta(s)$ allows analytical continuation to the entire z plane as follows:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = 1/(s(s-1)) + \int_1^{+\infty} (x^{s/2-1} + x^{-(1+s)/2})\theta(x)dx, \tag{18}$$

where $\Gamma(z)$ is the gamma function and

$$\theta(x) = \sum_{n=1}^{\infty} \exp(-\pi n^2 x).$$

$\zeta(s)$ is a regular function for all values of s, except $s = 1$, where it has a simple pole with residue 1; moreover, it satisfies the following functional equation:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s). \tag{19}$$

This equation is called Riemann's functional equation.

Riemann's zeta function is an important subject of study and has numerous interesting generalisations. The role of the zeta function is highly significant in number theory, where it is connected with various fundamental functions, such as the Möbius function, the Liouville function, the number of divisors, and the number of prime divisors. The detailed theory of zeta functions is presented in Ref. [24]. The zeta function has found application in various other fields, notably in quantum statistical mechanics and quantum field theory [25–27]. Riemann's zeta function is often introduced in quantum statistics formulas. A well-known example is the Stefan–Boltzman law for blackbody radiation. Its ubiquitous use in seemingly unrelated areas demonstrates the necessity for further investigation.

The present study is concerned with the analytical properties of the following generalised zeta functions:

$$P(s) = \sum_{j\geq 1} \frac{1}{p_j^s}, \ \mathrm{Re}(s) > 1+\delta, \ \delta > 0,$$

where $\{p_j : j \geq 1\}$ is an increasing enumeration of all prime numbers. The form of $P(s)$ suggests that it possesses the same properties as the zeta function; however, this is not quite obvious and can be seen by considering

$$\ln(\zeta(s)) = \sum_{n=1}^{\infty} P(ns)/n, \ f(s) = \ln(\zeta(s)) - P(s), \ \mathrm{Re}(s) > 1+\delta, \ \delta > 0. \tag{20}$$

Hadamard was the first to apply $P(s)$ in the study of the zeta function [28]. Chernoff made significant progress in the Riemann hypothesis using $P(s)$ [29]. In the present study, modifications of Chernoff's results are obtained. Specifically, his study on the pseudo zeta function is completed. Chernoff obtained an equivalent formulation of the Riemann hypothesis in terms of a pseudo zeta function as follows.

THEOREM. (Chernoff) Let

$$C(s) = \prod_{n>1} \left[1 - \frac{1}{(n\ln(n))^s}\right]^{-1}.$$

Then, $C(s)$ continues analytically into the critical strip and has no zeros there.

The significance of this theorem is that, if the primes were distributed more regularly (i.e., if $p_n \equiv n\log n$), then the Riemann hypothesis would be trivially true. In an effort to further develop the work of Chernoff and Hadamard, the following question naturally arises: Does the pseudo zeta function $P(s)$ continue analytically into the critical strip? It should be noted that analytic extensions of $P(s)$ were first studied by Landau and Walvis

[30] and Estarmann [31, 32]; however, no satisfactory estimates for $P(s)$ were obtained, and the present study is concerned with this question.

THEOREM. (Landau, Walvis, and Estarmann)
Let $\mu(n)$ be a Möbius function. Then,

$$P(s) = \sum_{n\geq 1} \frac{\mu(n)\ln\zeta(ns)}{n} \text{ as } \mathrm{Re}(s) > 1+\delta,\ \delta > 0,$$

$$\sum_{n\geq 1} \frac{\mu(n)\ln\zeta(ns)}{n} \text{ is a meromorphic function as } \mathrm{Re}(s) > \delta,\ \delta > 0.$$

The section is organised as follows. Intermediate estimates are first obtained for $\ln\zeta(s)$. Subsequently, the sets where the logarithm of the zeta function is uniquely determined are defined. These sets are composed of rectangles in which the zeta function has no roots, and they cover the entire critical strip except for the rectangular regions in which the zeros of the zeta functions are located. In the rectangles in which there are no zeros of the zeta function, the real value of its logarithm can be defined, and in these sets, the mirror-symmetric equation that arises by taking the logarithm on both sides of the Riemann functional equation is investigated. Then, the Fourier transform is applied to it, and it is multiplied by a regulating factor. In this manner, a Riemann–Hilbert boundary-value problem is obtained for $\ln\zeta(s)$ The properties of the solution to the Riemann–Hilbert boundary-value problem are expressed in terms of the Hilbert integral transform. In the rectangles in which the zeta function has no roots, the Hilbert transform can be used to obtain exact lower bounds for the zeta function in the critical strip.

8. Results

As mentioned in the introduction, certain simple intermediate estimates are first obtained.

The rectangles in which the zeta function hasn't zeros are first introduced as follows:

$$D(n) = (s|0.1 < \mathrm{Re}(s) < 0.9,\ \mathrm{Im}(s) \neq \mathrm{Im}(s_n),\ \mathrm{Im}(s_n) - d_n \leq \mathrm{Im}(s) \leq \mathrm{Im}(s_n) + d_n,$$

$$\overline{D(n)} = (s|0.1 < \mathrm{Re}(s) < 0.9,\ \mathrm{Im}(-s) \neq \mathrm{Im}(-s_n),\ -\mathrm{Im}(s_n) - d_n \leq \mathrm{Im}(s) \leq -\mathrm{Im}(s_n) + d_n, ,$$

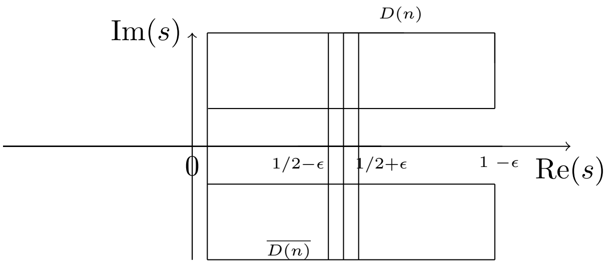
where

$$\zeta(s_{n+1}) = 0,\ \zeta(s_n) = 0,\ \zeta(1-s_n) = 0, \zeta(1-s_{n+1}) = 0,\ \zeta(1-s_n) = 0,$$

$$d_n = (\mathrm{Im}(s_{n+1}) - \mathrm{Im}(s_n))/2.$$

The sets of $D(n)$, are shown in the Figure 1 below.

Figure 1:



Theorem 6. *Let* $s \in D_n \cup \overline{D(n)}$ *and* $F(s) = \frac{s}{2}\ln(\pi) - \ln(\Gamma(s/2)) - \frac{1-s}{2}\ln(\pi) + \ln(\Gamma(1-s)/2))$. *Then,*

$$\sup_{s\in D_n\cup\overline{D(n)}} |F(\tau+i\alpha)| + \sup_{s\in D_n\cup\overline{D(n)}} |\frac{dF(\tau+i\alpha)}{d\tau}| < CC_n.$$

Proof. As $s \in D_n \cup \overline{D(n)}$, this implies that F is holomorphic, which completes the proof.

As mentioned in the introduction, a Riemann–Hilbert boundary-value problem should be obtained. To this end, an equation should be derived that determines the difference between the boundary values of the analytic functions in the upper plane and the lower plane. Denote

$$Q(s) = \mathrm{Re}(\ln(\zeta(s))) = \ln(|\zeta(s)|).$$

Theorem 7. *Let* $s \in D(n)$ *or* $s \in \overline{D(n)}$ *and*

$$F(s) = \frac{s}{2}\ln(\pi) - \ln(\Gamma(s/2)) - \frac{1-s}{2}\ln(\pi) + \ln(\Gamma(1-s)/2)).$$

Then,

$$Q(s) = Q(1-s) + \mathrm{Re}(F(s)).$$

Proof. Can be taking the logarithm of Eq. (19)for $s \in D(n)$ or $s \in \overline{D(n)}$, and an equation for $Q(s)$ may be obtained as an equation for the real part.

Consequently, the following functions are introduced:

$$\begin{cases} \nu_\epsilon(s) = 0, \ \ \mathrm{Re}(s) < 2\epsilon; \\ \nu_\epsilon(s) = 1, \ \ \epsilon < \mathrm{Re}(s) < 1/2 - 2\epsilon; \\ \nu_\epsilon(s) = 0, \ \ 1/2 - 2\epsilon < \mathrm{Re}(s) < 1/2 + 2\epsilon; \\ \nu_\epsilon(s) = 1, \ \ 1/2 + 2\epsilon < \mathrm{Re}(s) < 1 - 2\epsilon; \\ \nu_\epsilon(s) = 0, \ \ \mathrm{Re}(s) > 1 - 2\epsilon; \\ \psi(t) = \sigma e^{\frac{1}{t^2-1}}, \ \ t^2 < 1, \ \frac{1}{\sigma} = \int\limits_{-1}^{1} e^{\frac{1}{t^2-1}} dt; \\ \psi(t) = 0, \ t^2 \geq 1; \\ \mu_\epsilon(x) = \int \psi(s/\epsilon)\nu_\epsilon(x-t)dt/\epsilon = \int \psi(x - t/\epsilon)\nu_\epsilon(t)dt/\epsilon. \end{cases}$$

Lemma 16. *For ν_ϵ and μ_ϵ, we have $\nu_\epsilon(x) = \nu_\epsilon(1-x)$ and $\mu_\epsilon(x) = \mu_\epsilon(1-x)$.*

Proof. We have $\nu_\epsilon(x) = \nu_\epsilon(1-x)$ by definition. Moreover,

$$\mu_\epsilon(x) = \int \psi(s/\epsilon)\nu_\epsilon(x-s)ds/\epsilon = \int \psi(s/\epsilon)\nu_\epsilon(1-x+s)ds/\epsilon = \mu_\epsilon(1-x).$$

Lemma 17. *Let $1/2 + 3\epsilon < x < 1 - 3\epsilon$. Then, $\mu_\epsilon(x) = 1$.*

Proof.

$$\mu_\epsilon(x) = \int \psi(s/\epsilon)\nu_\epsilon(x-s)ds/\epsilon = \int_{-\epsilon}^{\epsilon} \psi(s/\epsilon)\nu_\epsilon(x-s)ds/\epsilon = \int_{-\epsilon}^{\epsilon} \psi(s/\epsilon)ds/\epsilon = 1.$$

To compute the Fourier transform of the equation for Q, the functions $R(k)$, $Q_\epsilon(s)$, $Q_\epsilon(1-s)$, and $\mathrm{Re}(F_\epsilon(s))$, the Fourier transform of $Q_\epsilon(s)$, the Fourier transform of $Q_\epsilon(1-s)$, and the Fourier transform of $F_\epsilon(s)$ are introduced as follows:

$$Q_\epsilon(s) = Q(s)\mu_\epsilon(\mathrm{Re}(s)), \ \ Q_\epsilon(1-s) = Q(1-s)\mu_\epsilon(1-\mathrm{Re}(s)),$$

$$R(k) = \frac{e^{-ik}}{k - ia} + 1, \ F_\epsilon(s) = \mathrm{Re}(F(s))\mu_\epsilon(\mathrm{Re}(s)),$$

$$\frac{1}{\sqrt{2\pi}} \int_\epsilon^{1-\epsilon} Q_\epsilon(1-\tau-i\alpha)e^{-ik\tau}d\tau = \frac{e^{-ik}}{\sqrt{2\pi}} \int_\epsilon^{1-\epsilon} \mu_\epsilon(1-\mathrm{Re}(s))Q_\epsilon(\tau - i\alpha)e^{ik\tau}d\tau,$$

$$\widetilde{Q_\epsilon}(k,\alpha) = \frac{1}{\sqrt{2\pi}} \int_\epsilon^{1-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau}d\tau,$$

$$\widetilde{F_\epsilon}(k,\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0.1}^{0.9} \mathrm{Re}(F_\epsilon(\tau + i\alpha)e^{-ik\tau})ds,$$

$$J_\epsilon(k,\alpha) = \frac{1}{\sqrt{2\pi}} \int_\epsilon^{1-\epsilon} Q_\epsilon(\tau - i\alpha)e^{ik\tau}d\tau \ \ , I_\epsilon(k,\alpha) = \frac{1}{\sqrt{2\pi}} \int_\epsilon^{1-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau}d\tau. \quad (21)$$

To obtain the Riemann–Hilbert boundary-value problem, the following lemma is required.

Lemma 18. *Let* $a > 2$. *Then,* $\operatorname{ind}(R) = 0$.

Proof. By definition,

$$\operatorname{ind}(R) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{R(k)'}{R(k)}dk$$

as

$$\operatorname{Im}(k) < 0,\ |e^{-ik}| < 1 \text{ and } |k - ia| > 2 \text{ yield } \frac{R(k)'}{R(k)}$$

have no pole. This latest statement and lemma of Jordan yield ind $= 0$.

To obtain the necessary asymptotics, the following lemma is required.

Lemma 19. *Let*

$$a > 2.$$

Then, $\ln(R(k))$ *is a single-valued analytical function in the lower half plane.*

Proof.

$$\operatorname{Im}(k) \le 0$$

yields

$$\operatorname{Re}(R(k)) = 1 + \operatorname{Re}\left[\frac{e^{ik}}{k - ia}\right] > 0,$$

which completes proof.

Denote $\Omega(s) = \operatorname{Re}(s - s_n)$ and $\omega(s) = \frac{(s-s_n)^{\rho(s_n)}(s-1+s_n)^{\rho(s_n)}}{(1-s)}.$

$\rho(s_n)$ is multiplicity root of $\zeta(s)$ as $s = s_n$ The following presents results of [34]

Theorem of Backlund R.

Let $\zeta(s_n) = 0$ then

$$\rho(s_n) < C_0 ln|s_n|.$$

To obtain the necessary asymptotics, the following lemma is required.

Lemma 20. *Let* $\gamma_n = \frac{1}{4\rho(s_n)}$, $|\Omega(1/2)| = \epsilon_n > 0$, *and* $d_n = (\operatorname{Im}(s_{n+1}) - \operatorname{Im}(s_n))/2$.

Then, we have the following estimate as $\epsilon = 0.01\epsilon_n^2$*:*

$$\sup_{\operatorname{Im}s_n - d_n \le \alpha \le \operatorname{Im}s_n + d_n}\int_{\epsilon}^{1-\epsilon}|Q_\epsilon(\tau + i\alpha)|^2 + |Q_\epsilon(\tau + i\alpha)|\,d\tau < C_n C_{\epsilon_n} C_\gamma,$$

$$\sup_{-\operatorname{Im}s_n - d_n \le \alpha \le -\operatorname{Im}s_n + d_n}\int_{\epsilon}^{1-\epsilon}|Q_\epsilon(\tau - i\alpha)|^2 + |Q_\epsilon(\tau - i\alpha)|\,d\tau < C_n C_{\epsilon_n} C_\gamma,$$

$$\sup_{\operatorname{Im}s_n - d_n \le \alpha \le \operatorname{Im}s_n + d_n}\int_{\epsilon}^{1-\epsilon}|F_\epsilon(\tau + i\alpha)|^2 + |F_\epsilon(\tau + i\alpha)|\,d\tau < C_n C_{\epsilon_n} C_\gamma,$$

$$\sup_{-\operatorname{Im}s_n - d_n \le \alpha \le -\operatorname{Im}s_n + d_n}\int_{\epsilon}^{1-\epsilon}|F_\epsilon(\tau - i\alpha)|^2 + |F_\epsilon(\tau - i\alpha)|\,d\tau < C_n C_{\epsilon_n} C_\gamma.$$

Proof.
By the definition of $Q_\epsilon(s)$, we have

$$I_Q = \int_\epsilon^{1-\epsilon} |Q_\epsilon(\tau + i\alpha)|^2 + |Q_\epsilon(\tau + i\alpha)|\, d\tau = \int_\epsilon^{1-\epsilon} |\ln|\zeta(\tau + i\alpha)||^2 + |\ln|\zeta(\tau + i\alpha)||\, d\tau$$

$$\leq \int_\epsilon^{1-\epsilon} \left|\ln\left|\frac{\zeta(\tau+i\alpha)}{\omega(\tau+i\alpha)}\right|\right|^2 + \left|\ln\left|\frac{1}{\omega(\tau+i\alpha)}\right|\right|^2 + \left|\ln\left|\frac{\zeta(\tau+i\alpha)}{\omega(\tau+i\alpha)}\right|\right| + \left|\ln\left|\frac{1}{\omega(\tau+i\alpha)}\right|\right| d\tau.$$

Denote

$$L_{max} = \max_{s\in D(n)\cup\overline{D(n)}} \left|\frac{\zeta(s)}{\omega(s)}\right|, \;\; L_{min} = \min_{s\in D(n)\cup\overline{D(n)}} \left|\frac{\zeta(s)}{\omega(s)}\right|.$$

Then,

$$I_Q \leq \left|\ln\left|L_{max} + \frac{1}{L_{min}}\right|\right| + \left|\ln\left|L_{max} + \frac{1}{L_{min}}\right|\right|^2 + C_\gamma \int_\epsilon^{1-\epsilon} \left|\frac{1}{\omega(\tau+i\alpha)}\right|^{2\gamma} + \left|\frac{1}{\omega(\tau+i\alpha)}\right|^{\gamma} d\tau,$$

which completes the proof.

The previous constructions allow the calculation of the asymptotics as follows.

Lemma 21. *Let $(3/4 + i\alpha) \in D(n)$. Then,*

$$\lim_{\mathrm{Im}(k)\to-\infty} I_\epsilon(k,\alpha) = 0, \;\; \lim_{\mathrm{Im}(k)\to\infty} J_\epsilon(k,\alpha) = 0, \tag{22}$$

and, as $\mathrm{Im}(k) = 0$*,*

$$\lim_{\mathrm{Re}(k)\to\infty} I_\epsilon(k,\alpha) = 0, \;\; \lim_{\mathrm{Re}(k)\to\infty} J_\epsilon(k,\alpha) = 0. \tag{23}$$

Proof. Lemma 20 yields

$$|I_\epsilon(k,\alpha)| = \left|\int_\epsilon^{1-\epsilon} Q_\epsilon(\tau + i\alpha)e^{-ik\tau} d\tau\right| \leq \int_\epsilon^{1-\epsilon} \left(|Q_\epsilon(\tau+i\alpha)|^2\, d\tau\right)^{1/2} \frac{1}{|\mathrm{Im}(k)|^{1/2}}.$$

A similar argument is used for the function

$$J_\epsilon(k,\alpha) = \frac{1}{\sqrt{2\pi}} \int_\epsilon^{1-\epsilon} Q_\epsilon(\tau - i\alpha)e^{ik\tau} d\tau.$$

As $\mathrm{Im}(k) > 0$, $J_\epsilon(k,\alpha)$ can be estimated using the last expression and Lemma 5 as follows:

$$|J_\epsilon(k,\alpha)| < \int_\epsilon^{1-\epsilon} \left(|Q_\epsilon(\tau - i\alpha)|^2\, d\tau\right)^{1/2} \frac{1}{|\mathrm{Im}(k)|^{1/2}}.$$

As $\mathrm{Im}(k) = 0$, by the Riemann–Lebesgue lemma, we have

$$\lim_{\mathrm{Re}(k)\to\infty} I_\epsilon(k,\alpha) = 0, \;\; \lim_{\mathrm{Re}(k)\to\infty} J_\epsilon(k,\alpha) = 0, \tag{24}$$

which completes the proof.

For $f \in W_2^1(R) = \{f \in L_2(R) : (1+|\omega|^2)^{1/2}\hat{f}(\omega) \in L_2)\}$, the operators $T_\pm$ and T are defined as follows:

$$T_+f = \frac{1}{2\pi i}\lim_{\mathrm{Im}z\to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds,\ \mathrm{Im}\ z>0,\ T_-f = \frac{1}{2\pi i}\lim_{\mathrm{Im}z\to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds,\ \mathrm{Im}\ z<0,$$

$$Tf = \frac{1}{2}(T_+ + T_-)f.$$

These operators are closely related to the Hilbert transform, whose isometric properties were studied by Poincaré. The following result is from [33].

Lemma 22.

$$TT = \frac{1}{4}I,\ TT_+ = \frac{1}{2}T_+,\ TT_- = -\frac{1}{2}T_-,\ T_+ = T+\frac{1}{2}I,\ T_- = T-\frac{1}{2}I,$$

where I is the identity operator ($If = f$).

The reduction to a Riemann–Hilbert boundary-value problem can now be formulated as follows.

Theorem 8. *Let*

$$(3/4+i\alpha)\in D(n),\ a>2, \tag{25}$$

$$\Gamma_+(k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(R(t))dt}{t-k-i0}, \tag{26}$$

$$\Gamma_-(k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(R(t))dt}{t-k+i0}, \tag{27}$$

$$X_+(k) = e^{\Gamma_+(k)},\ X_-(k) = e^{\Gamma_-(k)},\ R(k) = \frac{X_-(k)}{X_+(k)},\ G_\epsilon(k,\alpha) = J_\epsilon(k,\alpha). \tag{28}$$

Then,

$$J_\epsilon(k,\alpha) = -\frac{X_+(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{G_\epsilon(t,\alpha)}{X_-(t)}\frac{dt}{t-k-i0} = X_+(k)T_+\frac{G_\epsilon}{X_-},$$

$$\frac{I_\epsilon(k,\alpha)}{k-ia} - \frac{\widetilde{F_\epsilon}(k,\alpha)}{k-ia} = -\frac{X_-(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{G_\epsilon(t,\alpha)}{X_-(t)}\frac{dt}{t-k+i0}dt = X_-(k)T_-\frac{G_\epsilon}{X_-}.$$

Proof. By Theorem 7 and Lemma 16 we have

$$Q_\epsilon(s) = Q_\epsilon(1-s) + F_\epsilon(s). \tag{29}$$

Using the Fourier transform, we obtain

$$I_\epsilon(k,\alpha) = e^{-ik}J_\epsilon(k,\alpha) + \widetilde{F_\epsilon}(k,\alpha). \tag{30}$$

Multiplying this equation by $\frac{1}{k-ia}$, we get

$$\frac{I_\epsilon(k,\alpha)}{k-ia} = \frac{e^{-ik}J_\epsilon(k,\alpha)}{k-ia} + \frac{\widetilde{F_\epsilon}(k,\alpha)}{k-ia}. \tag{31}$$

We can rewrite the last equation as

$$\frac{I_\epsilon(k,\alpha)}{k-ia} - \frac{\widetilde{F_\epsilon}(k,\alpha)}{k-ia} = R(k)J_\epsilon(k,\alpha) + J_\epsilon(k,\alpha). \tag{32}$$

Then,

$$\Psi_-(k,\alpha) = \frac{I_\epsilon(k,\alpha)}{k-ia} - \frac{\widetilde{F_\epsilon}(k,\alpha)}{k-ia}, \tag{33}$$

$$\Psi_+(k,\alpha) = J_\epsilon(k,\alpha), \tag{34}$$

$$G_\epsilon(t,\alpha) = J_\epsilon(k,\alpha). \tag{35}$$

By using Lemma 21, the following Riemann–Hilbert boundary-value problem is obtained regarding the definition of an analytic function from its boundary values on the real line:

$$\Psi_-(k,\alpha) = R(k)\Psi_+(k,\alpha) + G_\epsilon(k,\alpha), \tag{36}$$

$$\lim_{\mathrm{Im}(k)\to\infty} \Psi_+(k,\alpha) = 0, \quad \lim_{\mathrm{Im}(k)\to-\infty} \Psi_-(k,\alpha) = 0. \tag{37}$$

Hilbert's formula and Lemmas 20 and 21 give the solution to this Riemann–Hilbert boundary value problem as

$$\Psi_+(k,\alpha) = -\frac{X_+(k)}{2\pi i}\int_{-\infty}^{\infty} \frac{G_\epsilon(t,\alpha)}{X_-(t)}\frac{dt}{t-k-i0}, \tag{38}$$

$$\Psi_-(k,\alpha) = -\frac{X_-(k)}{2\pi i}\int_{-\infty}^{\infty} \frac{G_\epsilon(t,\alpha)}{X_-(t)}\frac{dt}{t-k+i0}. \tag{39}$$

Denote

$$\Phi_+(k,\alpha) = \Psi_+(k,\alpha) - \frac{J_\epsilon(k,\alpha)R(k)}{X_-(k)}, \tag{40}$$

$$\Phi_-(k,\alpha) = \frac{\Psi_-(k,\alpha)}{X_-(k)} - \frac{I_\epsilon(k,\alpha) - \widetilde{F_\epsilon}(k,\alpha)}{X_-(k)(k-ia)}. \tag{41}$$

For $\Phi_+(k,\alpha)$ and $\Phi_-(k,\alpha)$ we have a new Riemann–Hilbert boundary-value problem:

$$\Phi_-(k,\alpha) = \Phi_+(k,\alpha), \tag{42}$$

$$\lim_{\text{Im}(k)\to\infty} \Phi_+(k,\alpha) = 0, \quad \lim_{\text{Im}(k)\to-\infty} \Phi_-(k,\alpha) = 0. \tag{43}$$

Liouville's theorem implies that

$$\Phi_-(k,\alpha) = 0, \Phi_+(k,\alpha) = 0, \tag{44}$$

which completes the proof.

9. Discussion

Our computations led to a new definition of the functions $I_\epsilon(k)$ and $J_\epsilon(k)$, which we obtained from the Riemann–Hilbert boundary-value problem. From the uniqueness of the solution of the Riemann–Hilbert boundary-value problem functions $I_\epsilon(k)$ and $J_\epsilon(k)$, defined earlier in (21) and obtained from the Hilbert formula are equal!
To obtain the final estimates for the zeta function, the isometric properties of the integral Hilbert transform will be used.

Theorem 9. *Let $(3/4 + i\alpha) \in D(n)$ and $a > 2$. Then,*

$$C^{-1} < |X_-(t)| < C, \;\; C^{-1} < |X_+(t)| < C,$$
$$||\Psi_+||_{L_2} \le C_\epsilon, \;\; ||\Psi_-||_{L_2} \le C_\epsilon.$$

Proof. By Lemmas 18, 19, and 22, we get

$$\Gamma_-(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(R(t))dt}{t-k+i0} = T_- \ln(R) = \ln(R),$$

$$X_-(t) = R(t), \quad X_+(t) = 1.$$

The last estimate implies

$$C^{-1} < |X_-(t)| < C, \;\; |X_+(t)| = 1.$$

Using Theorem 9, we obtain

$$||\Psi_-||^2_{L_2} + ||\Psi_+||^2_{L_2} = \int_{-\infty}^{+\infty} \left| \frac{I_\epsilon(k,\alpha)}{k-ia} - \frac{\widetilde{F_\epsilon}(k,\alpha)}{k-ia} \right|^2 dk + \int_{-\infty}^{+\infty} |J_\epsilon(k,\alpha)|^2 \, dk \le C_n C_\epsilon.$$

Lemma 23. *Let β_n and ϕ_n satisfy the equations*

$$e^{\beta} = \sqrt{(2\pi n + \phi)^2 + (\beta - a)^2},$$

$$\phi = \pi - \arg\left(2\pi n + \phi + i(-a + \beta)\right).$$

Then,

$$t_n = 2\pi n + \pi + i\beta_n + \phi_n,\ n \geq .0$$

The root of equation is

$$R(k) = 0$$

and

$$\beta_n = \ln(2\pi n) + o(1),\ \ \phi_n = \pi + O(\ln(n)/n),\ \ n \geq 1.$$

Proof.

$$R(t_n) = \frac{e^{-it_n}}{t_n - ia} + 1 = \frac{e^{-i2\pi n + \beta - i\phi}}{2\pi n + \phi + i(\beta_n - a)} + 1$$

$$= \frac{e^{-i2\pi n + \beta_n - i\phi}}{e^{i\arg(2\pi n + \phi + i(-a+\beta_n))}\sqrt{(2\pi n + \phi)^2 + (\beta_n - a)^2}} + 1$$

$$= \frac{-e^{\beta_n}}{\sqrt{(2\pi n + \phi_n)^2 + (\beta_n - a)^2}} + 1 = -1 + 1 = 0.$$

Take $\beta = \ln(2\pi n) + \gamma_n$. Then,

$$e^{\gamma_n} = \sqrt{1 + \frac{(\ln(n\pi) + \gamma_n - a)^2}{(2\pi n + \phi)^2}}.$$

For ϕ, we have

$$\phi_n = \pi - \arctan\left(\frac{(-a + \beta)}{2\pi n + \phi_n}\right),$$

and we get

$$\phi_n = \pi + O(\ln(n)/n),$$

which completes the proof.

Theorem 10. *Let $s \in D(m) \cup \overline{D(m)}$ and $a > 2$, with $1/2 + 3\epsilon < \mathrm{Re}(s) < 1 - 3\epsilon$. Then,*

$$|Q(s)| < C_m C_\epsilon.$$

Proof. By Theorem 9,

$$\Psi_-(k,\alpha) = \frac{I_\epsilon(k,\alpha)}{k - ia} - \frac{\widetilde{F_\epsilon}(k,\alpha)}{k - ia} = -\frac{X_-(k)}{2\pi i}\int_{-\infty}^{\infty} \frac{G_\epsilon(t,\alpha)}{X_-(t)} \frac{dt}{t - k + i0}.$$

Denote

$$I_1 = \int_{-\infty}^{\infty} \frac{G_\epsilon(t,\alpha)}{X_-(t)(t-k+i0)}dt, \ I_2 = (k-ia)I_1.$$

Lemma 22 and lemma of Jordan yield

$$I_1 = \int_{-\infty}^{\infty} \frac{G_\epsilon(t,\alpha)}{X_-(t)(t-k+i0)}dt = \sum_0^{\infty} \frac{G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)},$$

$$I_\epsilon(k,\alpha) = F_\epsilon(k,\alpha) + X_-(k)(k-ia)\sum_0^{\infty} \frac{G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)}.$$

As $k \in (-N, N)$ is a uniformly convergent series, we have

$$\int_{-N}^{+N} e^{itk}\frac{d^2}{dk^2}\left(\frac{I_\epsilon(k,\alpha)}{X_-(k)}\right)dk = \int_{-N}^{+N} e^{itk}\frac{d^2}{dk^2}\left(\frac{F_\epsilon(k,\alpha)}{X_-(k)}\right)dk + \int_{-N}^{+N} e^{itk}\frac{d^2}{dk^2}\left((k-ia)I_1\right)dk.$$

By the definition I_1, we have

$$\int_{-N}^{+N} e^{itk}\frac{d^2}{dk^2}\left((k-ia)I_1\right)dk = \int_{-N}^{+N}\sum_1^{N} e^{itk}\frac{d^2}{dk^2}\left(\frac{(k-ia)G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)}\right)dk$$

$$+\int_{-N}^{+N}\sum_N^{\infty}\frac{d^2}{dk^2}\left(\frac{(k-ia)G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)}\right)e^{itk}dk = W_1 + W_2,$$

$$W_1 = \int_{-N}^{+N}\sum_1^{N} 2\left(\frac{G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^2} + \frac{(k-ia)G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^3}\right)e^{itk}dk,$$

$$W_2 = \int_{-N}^{+N}\sum_{N+1}^{\infty} 2\left(\frac{G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^2} + \frac{(k-ia)G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^3}\right)e^{itk}dk.$$

Denote

$$W_1^0(k) = \sum_1^{N} 2\left(\frac{G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^2} + \frac{(k-ia)G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^3}\right)e^{itk}$$

and

$$W_2^0(k) = \sum_{N+1}^{\infty} 2\left(\frac{G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^2} + \frac{(k-ia)G_\epsilon(t_n,\alpha)}{X'_-(t_n)(t_n-k+i0)^3}\right)e^{itk}.$$

Consider the integration of the functions W_1^0 and W_2^0 around the square contour S with vertices $\pm N$ and $\pm N - iN$ and oriented positively. Analyticity of the functions W_1^0 and W_2^0 yields

$$\int_S W_1^0 dS = 0, \ \int_S W_2^0 dS = 0$$

and

$$W_1=\int_0^{-iN} W_1^0(-N+i\tau)id\tau+\int_{-iN}^{0} W_1^0(N+i\tau)id\tau+\int_N^{-N} W_1^0(-iN+\tau)d\tau,$$

$$W_2=\int_0^{-iN} W_2^0(-N+i\tau)id\tau+\int_{-iN}^{0} W_2^0(N+i\tau)id\tau+\int_N^{-N} W_2^0(-iN+\tau)d\tau.$$

The last integrals yield

$$|W_1|\le\left|\int_0^{-iN} W_1^0(-N+i\tau)id\tau\right|+\left|\int_{-iN}^{0} W_1^0(N+i\tau)id\tau\right|+\left|\int_N^{-N} W_1^0(-iN+\tau)d\tau\right|$$
$$=R_{11}+R_{12}+R_{13}.$$

It is enough to calculate R_{11}, as the remaining calculations are done in the same way. Therefore,

$$R_{11}\le\sum_1^N\frac{1}{n^\epsilon}\int_0^N\frac{C_\epsilon}{(2\pi n+\phi-N)^2+(b_n-\tau)^2}d\tau\le\frac{C_\epsilon}{N^\epsilon}.$$

Performing the same calculations for W_2 yields

$$|W_2|\le\left|\int_0^{-iN} W_2^0(-N+i\tau)id\tau\right|+\left|\int_{-iN}^{0} W_2^0(N+i\tau)id\tau\right|+\left|\int_N^{-N} W_2^0(-iN+\tau)d\tau\right|$$
$$=R_{21}+R_{22}+R_{23}.$$

It is also enough to calculate R_{21}. The result is

$$R_{21}\le\sum_{N+1}^\infty\frac{1}{n^\epsilon}\int_0^N\frac{C_\epsilon}{(2\pi n+\phi-N)^2+(b_n-\tau)^2}d\tau\le\frac{C_\epsilon}{N^\epsilon}.$$

We perform simple calculations to form a final estimate:

$$\int_{-N}^{+N}e^{itk}\frac{d^2}{dk^2}I_\epsilon(k,\alpha)dk=\int_{-N}^{+N}e^{itk}\frac{d^2}{dk^2}\left(\frac{I_\epsilon(k,\alpha)}{X_-(k)}\right)dk-2\int_{-N}^{+N}e^{itk}\frac{d}{dk}\left(I_\epsilon(k,\alpha)\right)\frac{d}{dk}\left(\frac{1}{X_-(k)}\right)dk$$
$$-\int_{-N}^{+N}e^{itk}I_\epsilon(k,\alpha)\frac{d^2}{dk^2}\left(\frac{1}{X_-(k)}\right)dk+\int_{-N}^{+N}e^{itk}\frac{d^2}{dk^2}I_\epsilon dk-\int_{-N}^{+N}e^{itk}\frac{d^2}{dk^2}\left(I_\epsilon(k,\alpha)\right)\frac{1}{X_-(k)}dk.$$

Theorem 9 and the estimates of W_1 ,W_2 , $(X_--1),\frac{dX_-}{dk}$, $\frac{d^2X_-}{dk^2}$ yield

$$\left|\int_{-N}^{+N}e^{itk}\frac{d^2}{dk^2}I_\epsilon dk\right|\le C_\epsilon C_m.$$

By Lemma 17, as $1/2+3\epsilon<\mathrm{Re}(s)<1-3\epsilon$, the last estimate yields

$$|Q(s)|<C_mC_\epsilon,$$

which completes the proof,

As mentioned in the introduction, the values of the zeta function in adjacent rectangles should be compared. This will be done in the following theorem.

Theorem 11. *Riemann's function has nontrivial zeros only on the line* $\mathrm{Re}(s) = 1/2$.

Proof. Let it be assumed that there is a root of the zeta function with $s_n = 1/2 + \delta_n + i * \alpha_n$, where $\delta_n > 0$. Let $s_{n-1} = 1/2 + \delta_{n-1} + i\alpha_{n-1}$, where $\delta_{n-1} > 0$ is another root nearest to it. Then, the following sets corresponding to s_n are constructed:

$$D(n) = (s|\epsilon < \mathrm{Re}(s) < 1 - \epsilon,\ \mathrm{Im}(s) \neq \mathrm{Im}(s_n),\ \mathrm{Im}(s_n) - d_n \leq \mathrm{Im}(s) \leq \mathrm{Im}(s_n) + d_n,$$

$$\overline{D(n)} = (s|\epsilon < \mathrm{Re}(s) < 1 - \epsilon,\ \mathrm{Im}(-s) \neq \mathrm{Im}(-s_n),\ -\mathrm{Im}(s_n) - d_n \leq \mathrm{Im}(s) \leq -\mathrm{Im}(s_n) + d_n,$$

where

$$\zeta(s_{n+1}) = 0,\ \zeta(s_n) = 0,\ \zeta(1 - s_n) = 0,\ \zeta(1 - s_{n+1}) = 0,\ \zeta(1 - s_n) = 0,$$

$$d_n = (\mathrm{Im}(s_{n+1}) - \mathrm{Im}(s_n))/2,$$

where $\epsilon = 0.01\delta_n^2$. As $1/2 < \mathrm{Re}(s) < 1$ and $s \in D(n) \cup \overline{D(n)}$, then we have the equation for Q. Theorem 11 now yields

$$|\ln(|\zeta(1/2 + \delta_n + i\alpha_n - i\delta)|) = |Q(1/2 + \delta_n - i\alpha_n - i\delta)| < 2C_n C_\epsilon.$$

Furthermore,

$$\lim_{\delta \to 0} |\ln(|\zeta(1/2 + \delta_n + i\alpha_n - i\delta)|)| = \infty.$$

These estimates for $|Q(s)|$ and $|f(s)|$ imply that the function does not have zeros on the half plane $\mathrm{Re}(s) > 1/2$. By the integral representation (3), these results are extended to the half plane $\mathrm{Re}(s) < 1/2$. Therefore, Riemann's hypothesis has been proved.

10. Conclusions

As can be seen from the results obtained, the genius of Poincaré scholars such as Riemann and Hilbert still illuminates the path of modern scholars. An example of a complicated problem such as the Cauchy problem for the Navier–Stokes equation shows how the Poincaré–Riemann–Hilbert boundary-value problem allows us to construct effective estimates of solutions for this case. For this, the apparatus of the three-dimensional inverse problem of the theory of quantum scattering is developed. It is shown that the unitary scattering operator can be investigated as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to continue studying the potential in the Schrödinger equation, which we consider as a velocity component in the Navier–Stokes equation. The same scheme of reduction of Riemann integral equations for the zeta function to the Poincaré–Riemann–Hilbert boundary-value problem enables us to construct effective estimates that describe the behaviour of the zeros of the zeta function well. In summary, it is possible to tell these outstanding scientists the problems have been formulated and all modern methods of their decision are put in pawn.

Acknowledgements

The author thanks the NAS RK , in particular, Academician NAS RK B. Zhumagulov for constant attention and support. Moreover, the author is grateful for the Mathematics Seminar at the Kazakhstan branch of Moscow State University, at which valuable comments were made, and thanks Professors B. Kanguzhin and M. Otelbaev and the organisers of Automorphicformsworkshop.org/AFW2018 for their detailed review and valuable comments. The author is especially grateful to Professors Steven Miller, P Plotnikov, and A Mednykh for their thorough analysis of the work and detailed recommendations that have significantly improved the paper. The author thanks financial support of the Science Fund of the Ministry of Education and Science of the Republic of Kazakhstan according to grant AP05132573-OT-18

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