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On a Graph Induced by a Hyper BCI-algebra

Michelle T. Panganduyon[1], Sergio R. Canoy, Jr.[1,*]

Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra, and Analysis-PRISM, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines

Abstract. This paper introduces the notion of the zero divisor graph of a hyper BCI-algebra and investigates some of its properties.

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1. Introduction

Graph Theory and Abstract Algebra have been profoundly studied by mathematicians because of the interesting topics laid upon these branches of mathematics. Indeed, some authors studied graph theory to build connections with certain algebraic structures such as commutative semigroups, commutative rings, and non-commutative rings. Beck, in his work in [1], associated to any commutative ring R its zero divisor graph $G(R)$ whose vertices are the zero divisors of R (including an element 0 of R) and where adjacency between two distinct elements of R is defined as follows: two vertices x, y are adjacent if and only if $xy = 0$. In 2002, DeMeyer et al. [2] also pioneered the notion of zero-divisor graph of commutative semigroup S with 0. They associated an undirected graph $\Gamma(S)$ to any commutative semigroup S with 0 whose vertices are the nonzero zero divisors of S, such that two vertices x, y are adjacent if and only if $xy = 0$. More recently, Y. B. Jun and K. J. Lee [5] introduced the concept of associated graph of BCI-algebra and verified some properties of the graph. Motivated by these works, in this paper, we shall introduce the notion of the zero divisor graph of a hyper BCI-algebra and investigate some of its properties.

*Corresponding author.
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Email addresses: michelle.panganduyon@g.msuiit.edu.ph (M. Panganduyon), sergio.canoy@g.msuiit.edu.ph (S. Canoy)

2. Preliminaries

The concepts on Graph Theory are taken from [4]:

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. A graph $K = (V(K), E(K))$ is a *subgraph* of a graph $G = (V(G), E(G))$ if $V(K) \subseteq V(G)$ and $E(K) \subseteq E(G)$. Two vertices u, v of a graph G are *adjacent*, or *neighbors*, if uv is an edge of G. The set of neighbors of a vertex v of G is denoted by $N_G(v)$ and the *degree* of v in G, denoted $\deg v$, is equal to $|N_G(v)|$. The degree of G, denoted by $\Delta(G)$, is equal to the largest degree of a vertex of G. A vertex w of G is called an *isolated vertex* if $\deg_G(w) = |N_G(w)| = 0$. The set of all isolated vertices of G will be denoted by $\mathcal{I}(G)$. A graph G is called an *empty graph*, denoted by $\overline{K}_{|V(G)|}$, if $E(G) = \varnothing$, that is, $\mathcal{I}(G) = V(G)$.

A *walk* of a graph G is an alternating sequence of vertices and edges, beginning and ending with vertices, $v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$, in which each edge is incident with the two vertices immediately preceding and following it. This walk joins v_0 and v_n; and is sometimes called a v_0-v_n *walk*. It is *closed* if $v_0 = v_n$ and is *open* otherwise. It is a *path* if all the vertices (and thus necessarily all the edges) are distinct. If the walk is closed, then it is a *cycle* provided its n vertices are distinct and $n \geq 3$. We denote by C_n the graph consisting of a cycle with n vertices and by P_n a path with n vertices.

A graph is *connected* if every pair of vertices are joined by a path. A maximal connected subgraph of G is called a *component* of G. The *complete graph* K_p has every pair of its p vertices adjacent. A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every edge joining V_1 and V_2, then G is a *complete bipartite*. If V_1 and V_2 have m and n vertices, respectively, then we write $G = K_{m,n}$. A *star* is a complete bipartite $K_{1,n}$.

The *Kronecker product* $G \otimes K$ of two graphs G and K is the graph with vertex set $V(G \otimes K) = V(G) \times V(K)$ and edge set $E(G \otimes K)$ satisfying the following conditions: $(x, u)(y, v) \in E(G \otimes K)$ if and only if $xy \in E(G)$ and $uv \in E(K)$.

Let G and K be graphs and let $f : V(G) \to V(K)$ be a function. Then f is a *graph homomorphism* if $f(x)f(y) \in E(K)$ whenever $xy \in E(G)$. Two graphs G and K are *isomorphic* (written as $G \cong K$) if there exists a one-to-one correspondence between the vertex sets which preserves adjacency.

A *hyperoperation* on a nonempty set H is a map from $H \times H$ into $P^*(H) = P(H) \setminus \{\varnothing\}$. Let $\circledast$ be a hyperoperation on H and $(x, y) \in H \times H$. Then its image under $\circledast$, denoted by $x \circledast y$, is called the *hyperproduct* of x and y. If A and B are nonempty subsets of H, then $A \circledast B$ is given by $A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b$. We shall use $x \circledast y$ instead of $x \circledast \{y\}, \{x\} \circledast y$, or $\{x\} \circledast \{y\}$. When $A \subseteq H$ and $x \in H$, we agree to write $A \circledast x$ instead of $A \circledast \{x\}$. Similarly, we write $x \circledast A$ for $\{x\} \circledast A$. In effect, $A \circledast x = \bigcup_{a \in A} a \circledast x$ and $x \circledast A = \bigcup_{a \in A} x \circledast a$.

A *hyper BCI-algebra* $(H, \circledast, 0)$ is a nonempty set H endowed with a hyperoperation "$\circledast$" and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(B_1) $((x \circledast z) \circledast (y \circledast z)) \ll x \circledast y$,

(B_2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$,

(B_3) $x \ll x$,

(B_4) $x \ll y$ and $y \ll x$ imply $x = y$,

(B_5) $0 \circledast (0 \circledast x) \ll x$, $x \neq 0$,

where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a \circledast b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x \circledast y$. In such case, we call "$\ll$" the *hyper order* in H (see [7]).

A hyper BCI-algebra $(H, \circledast, 0)$ is said to be *ordered* if for each x, y, $z \in H$, $x \ll y$ and $y \ll z$ imply $x \ll z$.

Example 1. [7] Let $H = \{0, 1, 2\}$. Define the hyperoperation "$\circledast$" by the Cayley table shown below.

$\circledast$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

Then by routine calculations, $(H, \circledast, 0)$ is a hyper BCI-algebra. Further, H is ordered.

Let $(H_1, \circledast_1, 0_1)$ and $(H_2, \circledast_2, 0_2)$ be two hyper BCI-algebras. Consider a mapping $f : H_1 \to H_2$. Then f is said to be a *homomorphism* if $f(x \circledast_1 y) = f(x) \circledast_2 f(y)$, for all $x, y \in H_1$. If f is a homomorphism and $f(0_1) = 0_2$, then we call f a *hyper homomorphism*. If f is a homomorphism, one-to-one, and onto, we say that f is an *isomorphism* and $(H_1, \circledast_1, 0_1)$ and $(H_2, \circledast_2, 0_2)$ are *isomorphic*, denoted by $H_1 \cong H_2$ (see [6]).

Let $f : H_1 \to H_2$ be a hyper homomorphism of hyper BCI-algebras. If f is one to one (resp. onto) we say f is a *hyper monomorphism* (resp. *hyper epimorphism*). If f is a hyper homomorphism and a bijection, f is said to be a hyper isomorphism, denoted by $H_1 \cong_{\mathcal{H}} H_2$ (see [3]).

Throughout this study, we denote a hyper BCI-algebra $(H, \circledast, 0)$ by H, unless otherwise specified.

The following results generated previously give some of the properties of a hyper BCI-algebra.

Proposition 1. [7] *In any hyper BCI-algebra H, the following hold:*

(i) $x \ll 0$ *implies x=0,*

(ii) $0 \in x \circledast (x \circledast 0)$,

(iii) $x \ll x \circledast 0$,

(iv) $0 \circledast (x \circledast y) \ll y \circledast x$,

(v) $A \ll A$,

(vi) $A \subseteq B$ *implies* $A \ll B$,

(vii) $A \ll \{0\}$ *implies* $A = \{0\}$,

(viii) $x \circledast 0 \ll \{y\}$ *implies* $x \ll y$,

(ix) $y \ll z$ *implies* $x \circledast z \ll x \circledast y$,

(x) $x \circledast y = \{0\}$ *implies* $(x \circledast z) \circledast (y \circledast z) = \{0\}$ *and* $x \circledast z \ll y \circledast z$,

(xi) $A \circledast A = \{0\}$ *implies* A *is a singleton,*

(xii) $A \circledast \{0\} = \{0\}$ *implies* $A = \{0\}$.

for all $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H.

Theorem 1. [3] *Let* $f : H_1 \to H_2$ *be a hyper homomorphism. Then the following hold:*

(i) *If* $x \ll y$, *where* $x, y \in H_1$, *then* $f(x) \ll f(y)$.

(ii) *If* $A, B \subseteq H_1$ *such that* $A \ll B$, *then* $f(A) \ll f(B)$.

3. Zero Divisor Graph of a Hyper BCI-algebra

Let H be a hyper BCI-algebra and $A \subseteq H$. We will use the notation $L_H(A)$ to denote the set

$$L_H(A) := \{x \in H | x \ll a, \forall\, a \in A\} = \{x \in H | 0 \in x \circledast a, \forall\, a \in A\}.$$

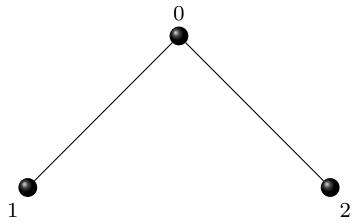
If $A = \{a\}$, we write $L_H(\{a\}) = L_H(a)$. For any $x \in H$, the set of *zero divisors of* x is $Z_x = \{y \in H | L_H(\{x, y\}) = \{0\}\}$.

Let H be a finite hyper BCI-algebra. The *zero divisor graph* $\Gamma(H)$ of H is the graph whose vertex set $V(\Gamma(H)) = H$ and edge set $E(\Gamma(H))$ satisfying the following condition: for every distinct $x, y \in H$, $xy \in E(\Gamma(H))$ if and only if $L_H(\{x, y\}) = \{0\}$ (equivalently, $x \in Z_y$ or $y \in Z_x$).

Although there are infinite hyper BCI-algebras, this paper only considers zero divisor graphs of finite hyper BCI-algebras.

Example 2. Consider the hyper BCI-algebra H defined in Example 1.

Then $L_H(\{0, 1\}) = L_H(\{0, 2\}) = \{0\}$ and $L_H(\{1, 2\}) = \{0, 1\}$. The zero divisors of $x \in H$ are $Z_0 = \{y \in H | L_H(\{0, y\}) = \{0\}\} = \{1, 2\}$ and $Z_1 = \{0\} = Z_2$. Thus, the zero divisor graph $\Gamma(H)$ of H is given by the following figure:



The next result gives some properties of the operator L_H.

Proposition 2. *Let A and B be subsets of H. Then the following hold:*

(i) $L_H(\varnothing) = H$

(ii) $L_H(\{0\}) = \{0\}$

(iii) *If $A \subseteq B$, then $L_H(B) \subseteq L_H(A)$.*

(iv) $L_H(A) = \bigcap_{a \in A} L_H(\{a\})$

(v) *If $x \in H$, then $x \in L_H(\{x\})$. Furthermore, $L_H(\{x\}) = \{0\}$ if and only if $x = 0$.*

Proof.

(i) Suppose $L_H(\varnothing) \neq H$. Then $\exists\, h \in H$ such that $h \notin L_H(\varnothing)$; i.e., $\exists\, a \in \varnothing$ such that $a \not\ll h$, a contradiction. Therefore, $L_H(\varnothing) = H$.

(ii) By definition, $L_H(\{0\}) = \{x \in H | x \ll 0\} = \{0\}$, by Proposition 1.

(iii) Let $x \in L_H(B)$. Then $x \ll b$, $\forall\, b \in B$. Since $A \subseteq B$, $x \ll a$, $\forall\, a \in A$. Thus, $x \in L_H(A)$. Hence, $L_H(B) \subseteq L_H(A)$.

(iv) Follows from the definition of $L_H(A)$:

$$\begin{aligned} L_H(A) &= \{x \in H | x \ll a, \forall\, a \in A\} \\ &= \{x \in H | x \in L_H(\{a\}), \forall\, a \in A\} \\ &= \bigcap_{a \in A} L_H(\{a\}). \end{aligned}$$

(v) Let $x \in H$. By (B_3), $x \ll x$. Hence, $x \in L_H(\{x\})$. Furthermore, $x \in L_H(\{x\}) = \{0\}$ implies $x = 0$ and if $x = 0$, then $L_H(\{x\}) = L_H(\{0\}) = \{0\}$ by (ii).

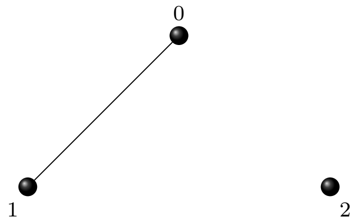
□

The zero divisor graph of a hyper BCI-algebra is not always connected:

Example 3. Consider the hyper BCI-algebra H with '$\circledast$' defined by the following Cayley table:

$\circledast$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

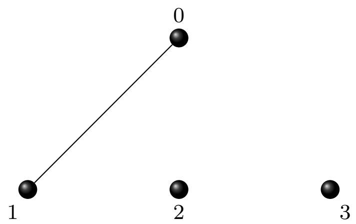
Then $L_H(\{0,1\}) = \{0\}$; $L_H(\{0,2\}) = \varnothing = L_H(\{1,2\})$. Thus, the zero divisor graph $\Gamma(H)$ of H is given below:



Example 4. Consider H defined by the following Cayley table:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{2\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0,1\}$

Then $L_H(\{0,1\}) = \{0\}$, $L_H(\{0,2\}) = L_H(\{0,3\}) = L_H(\{1,2\}) = L_H(\{1,3\}) = \varnothing$, and $L_H(\{2,3\}) = \{2\}$. The zero divisor graph $\Gamma(H)$ of H is given below



Proposition 3. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then*

(i) $\deg_{\Gamma(H)}(0) = |\{x \in H \setminus \{0\} : 0 \in L_H(x)\}| = \Delta(\Gamma(H))$;

(ii) $\Gamma(H) = \overline{K}_{|H|}$ *if and only if* $0 \notin L_H(x)$ *for all* $x \in H \setminus \{0\}$; *and*

(iii) *if* $\Gamma(H) \neq \overline{K}_{|H|}$, *then* $\mathcal{I}(\Gamma(H)) = \{x \in H \setminus \{0\} : 0 \notin L_H(x)\}$.

Proof.

(i) Note that for any $x \in H \setminus \{0\}$, $0x \in E(\Gamma(H))$ if and only if $L_H(\{0, x\}) = \{0\}$. Hence, by Proposition 2(ii), $0x \in E(\Gamma(H))$ if and only if $0 \in L_H(x)$. Thus,

$$\begin{aligned}\deg_{\Gamma(H)} 0 &= |\{x \in H \setminus \{0\} : 0x \in E(\Gamma(H))\}| \\ &= |\{x \in H \setminus \{0\} : 0 \in L_H(x)\}|.\end{aligned}$$

Let $x \in H \setminus \{0\}$ and let $y \in N_{\Gamma(H)}(x)$. Then $L_H(\{x, y\}) = \{0\}$. By Proposition 2(ii) and 2(iv), it follows that $L_H(0, y) = \{0\}$, that is, $y \in N_{\Gamma(H)}(0)$. Thus,

$$\deg_{\Gamma(H)}(x) = |N_{\Gamma(H)}(x)| \leq |N_{\Gamma(H)}(0)| = \deg_{\Gamma(H)}(0).$$

Since x was arbitrarily chosen, it follows that $\Delta(\Gamma(H))| = \deg_{\Gamma(H)}(0)$.

(ii) Suppose that $\Gamma(H) = \overline{K}_{|H|}$. That is, $\mathcal{I}(\Gamma(H)) = V(\Gamma(H))$. This implies that $\deg_{\Gamma(H)} 0 = 0$. Hence, $0x \notin E(\Gamma(H))$ for all $x \in H$. Thus, $0 \notin L_H(x)$ for all $x \in H \setminus \{0\}$.

For the converse, suppose that $0 \notin L_H(x)$ for all $x \in H \setminus \{0\}$. By (i), it follows that $\deg_{\Gamma(H)} 0 = \Delta(\Gamma(H)) = 0$. Therefore, $\Gamma(H) = \overline{K}_{|H|}$.

(iii) Suppose that $\Gamma(H) \neq \overline{K}_{|H|}$. Then $\deg_{\Gamma(H)} 0 = \Delta(\Gamma(H)) \neq 0$, i.e., $0 \notin \mathcal{I}(\Gamma(H))$. Let $x \in H \setminus \{0\}$. If $0 \notin L_H(x)$, then $L_H(x, y) \neq \{0\}$ for all $y \in H \setminus \{x\}$. Thus, $\deg_{\Gamma(H)}(x) = 0$, i.e., $x \in \mathcal{I}(\Gamma(H))$. Conversely, if $x \in \mathcal{I}(\Gamma(H))$, then $0x \notin E(\Gamma(H))$, i.e., $L_H(0, x) \neq \{0\}$. By Proposition 2(ii) and 2(iv), $0 \notin L_H(x)$. Therefore, $\mathcal{I}(\Gamma(H)) = \{x \in H \setminus \{0\} : 0 \notin L_H(x)\}$.

□

Next, we give equivalent statements for connectedness of the zero divisor graph.

Proposition 4. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then the following are equivalent:*

(i) $\Gamma(H)$ *is connected.*

(ii) $L_H(\{x, 0\}) = \{0\}$ *for all* $x \in H \setminus \{0\}$.

(iii) $0 \in L_H(x)$ *for all* $x \in H \setminus \{0\}$.

(iv) $\Delta(\Gamma(H)) = |H| - 1$

(v) $\mathcal{I}(\Gamma(H)) = \varnothing$

Proof.

(i)⇔(ii) Suppose $L_H(\{x, 0\}) \neq \{0\}$ for some $x \in H$. Then $0 \notin L_H(x)$, by Proposition 2(ii) and 2(iv). Thus, $0 \notin L_H(\{x, y\}) = L_H(\{x\}) \cap L_H(\{y\})$ for all $y \in H$. That is, for all $y \in H$, $xy \notin E(\Gamma(H))$. This implies that $\Gamma(H)$ is disconnected. For the converse, suppose that $L_H(\{x, 0\}) = \{0\}$ for all $x \in H \setminus \{0\}$. Then $\deg_{\Gamma(H)}(0) = |H| - 1$. Therefore, $\Gamma(H)$ is connected.

(ii)⇔(iii) This follows from Proposition 2(ii) and 2(iv).

(iii)⇔(iv) This follows from Proposition 3(i).

(iv)⇔(v) By Proposition 3(i), $\deg_{\Gamma(H)}(0) = |H| - 1$. This implies that $0x \in E(\Gamma(H))$ for each $x \in H \setminus \{0\}$. Hence, $\mathcal{I}(\Gamma(H)) = \varnothing$. Conversely, suppose that $\mathcal{I}(\Gamma(H)) = \varnothing$ and let $x \in H \setminus \{0\}$. Since $x \notin \mathcal{I}(\Gamma(H))$, there exists $y \in H \setminus \{x\}$ such that $L_H(x, y) = \{0\}$. Hence, $0 \in L_H(x)$ by Proposition 2(iv). By Proposition 3(i), it follows that $\deg_{\Gamma(H)}(0) = \Delta(\Gamma(H)) = |H| - 1$.

□

Remark 1. *Let H be a hyper BCI-algebra with $|H| \geq 2$. If $\Gamma(H)$ is connected, then*

(i) $\mathrm{diam}(\Gamma(H)) = 2$;

(ii) $\deg_{\Gamma(H)} 0 = |H| - 1 = \Delta(\Gamma(H))$.

Proposition 5. *Let H be a hyper BCI-algebra such that $L_H\{x, 0\} \neq \varnothing$ for all $x \in H$. Then $L_H\{x, 0\} = \{0\}$.*

Proof. Suppose that $L_H\{x, 0\} \neq \varnothing$ for all $x \in H$. Then there exists $y \in L_H\{x, 0\}$. Note that $y \in L_H\{0\} = \{0\}$ means that $y = 0$. It follows from Proposition 2(ii) and 2(iv) that $L_H\{x, 0\} = \{0\}$. □

Remark 2. *Let H be a hyper BCI-algebra such that $L_H\{x, 0\} \neq \varnothing$ for all $x \in H$. Then $0x \in E(\Gamma(H))\ \forall\ x \in H \setminus \{0\}$.*

Proposition 6. *If $|H| > 3$, then $\Gamma(H)$ is neither a cycle nor a path.*

Proof. Case 1. $\exists\ x \in H \setminus \{0\}$ such that $0 \notin L_H(x)$. Then $\Gamma(H)$ is disconnected, and the result follows. Case 2. $0 \in L_H(x)\ \forall\ x \in H$. Then $0x \in E(\Gamma(H))\ \forall\ x \in H \setminus \{0\}$. Evidently, $\Gamma(H)$ is neither a cycle nor a path. □

Corollary 1. *If a graph G is a cycle or a path of order $n \geq 4$, then there is no hyper BCI-algebra H such that $\Gamma(H) \cong G$.*

Proof. Immediate from Proposition 6. □

Theorem 2. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then $G = \Gamma(H)$ cannot have two nontrivial components; that is, G can only have at most one non-trivial component.*

Proof. If G is connected, then we are done. Suppose that G is disconnected. Suppose further that G has two distinct non-trivial components, say G_1 and G_2. Let G_3 be a component of G with $0 \in V(G_3)$ (G_3 may be G_1 or G_2). If G_3 is different from G_1, then $0 \notin L_H(x)$ for all $x \in V(G_1)$. Similarly, if G_3 is not G_2, then $0 \notin L_H(y)$ for all $y \in V(G_2)$. Hence, by Proposition 4, G_1 or G_2 is the trivial graph, a contradiction. □

Proposition 7. *Let H be an ordered hyper BCI-algebra. Then the following hold:*

(i) *For any subset A of H, $L_H(L_H(A)) \subseteq L_H(A)$.*

(ii) *For any $a, b \in H$, if $a \ll b$, then $L_H(\{a\}) \subseteq L_H(\{b\})$ and $Z_b \subseteq Z_a$.*

Proof.

(i) Let $x \in L_H(L_H(A))$. Then $x \ll b$ for all $b \in L_H(A)$. Since $b \ll a$ for all $a \in A$ and H is ordered, it follows that $x \ll a$ for all $a \in A$. Thus, $x \in L_H(A)$ and the result follows.

(ii) Suppose $x \in L_H(\{a\})$. Then $x \ll a$. Since H is ordered and $a \ll b$, $x \ll b$. That is, $x \in L_H(\{b\})$. Hence, $L_H(\{a\}) \subseteq L_H(\{b\})$. Now, suppose $x \in Z_b$. Then $L_H(\{b, x\}) = \{0\}$. Since $L_H(\{a, x\}) \subseteq L_H(\{b, x\})$, we have $L_H(\{a, x\}) = \{0\}$. This means that $x \in Z_a$. Thus, $Z_b \subseteq Z_a$.

□

Proposition 8. *Let H be a hyper BCI-algebra. Then*

(i) $\deg_{\Gamma(H)} x = |Z_x|$ *for all nonzero $x \in H$.*

(ii) *$y \in Z_x$ if and only if $x \in Z_y$ for all $x, y \in H$.*

Proof. Let $x, y \in H$.

(i) if $x \neq 0$, then

$$\begin{aligned} |Z_x| &= |\{y \in H \setminus \{x\} : L_H(\{x, y\}) = \{0\}\}| \\ &= |\{y \in H : xy \in E(\Gamma(H))\}| \\ &= \deg_{\Gamma(H)} x. \end{aligned}$$

(ii) $y \in Z_x$ means that $L_H(\{x, y\}) = \{0\}$, which further means that $x \in Z_y$.

□

Lemma 1. *Let $f : H_1 \to H_2$ be a hyper monomorphism of hyper BCI-algebras. Then for any $x, y \in H_1$, $x \ll y$ if and only if $f(x) \ll f(y)$.*

Proof. The sufficiency part is done by Theorem 1(i). Now, suppose $f(x) \ll f(y)$. Then $0_2 \in f(x) \circledast_2 f(y) = f(x \circledast_1 y)$. Thus, $0_1 = f^{-1}(0_2) \in f^{-1}f(x \circledast_1 y) = x \circledast_1 y$. Hence, $x \ll y$. □

Proposition 9. *Let $f : H_1 \to H_2$ be a hyper monomorphism of hyper BCI-algebras. Then $L_{H_2}(f(A)) = f(L_{H_1}(A))$ where $A \subseteq H_1$.*

Proof. Let $f : H_1 \to H_2$ be a hyper monomorphism. Let $A \subseteq H_1$.

$$\begin{aligned} y \in f(L_{H_1}(A)) &\iff f^{-1}(y) \in L_{H_1}(A) \\ &\iff f^{-1}(y) \ll a \text{ for all } a \in A \\ &\iff y \ll f(a) \text{ for all } a \in A, \text{ by Lemma 1} \\ &\iff y \in L_{H_2}(f(A)) \end{aligned}$$

Therefore, $L_{H_2}(f(A)) = f(L_{H_1}(A))$. □

Theorem 3. *Let H_1 and H_2 be hyper BCI-algebras. If $H_1 \cong_{\mathcal{H}} H_2$, then $\Gamma(H_1) \cong \Gamma(H_2)$.*

Proof. Suppose $H_1 \cong_{\mathcal{H}} H_2$, say $f : H_1 \to H_2$ is a hyper isomorphism. Since $V(\Gamma(H_1)) = H_1$ and $V(\Gamma(H_2)) = H_2$, there exists a one-to-one correspondence between the vertex sets. Note that for any distinct elements $x, y \in H_1$, $xy \in E(\Gamma(H_1))$ if and only if $L_{H_1}(\{x, y\}) = \{0\}$. By Proposition 9, $xy \in E(\Gamma(H_1))$ if and only if $L_{H_2}(\{f(x), f(y)\}) = \{0\}$. Thus, $xy \in E(\Gamma(H_1))$ if and only if $f(x)f(y) \in E(\Gamma(H_2))$. Consequently, $\Gamma(H_1) \cong \Gamma(H_2)$. □

3.1. On zero divisor graphs involving hyperatoms

Definition 1. *An element a of a hyper BCI-algebra H is called a hyperatom if for each $x \in H$, $x \ll a$ implies $x = 0$ or $x = a$.*

Denote by $A(H)$ the set of all hyperatoms of H, and by $A^*(H)$ the set of all nonzero hyperatoms of H; i.e., $A^*(H) = A(H) \setminus \{0\}$. Obviously, $0 \in A(H)$.

Definition 2. *A hyper BCI-algebra H is said to be hyperatomic if each element of H is a hyperatom, that is, $A(H) = H$.*

Remark 3. *A hyper BCI-algebra H is hyperatomic if and only if $L_H(x) = \{x\}$ or $L_H(x) = \{0, x\}$ for each $x \in H$.*

Remark 4. *A hyperatomic hyper BCI-algebra is ordered.*

Proof. Suppose H is a hyperatomic hyper BCI-algebra. Let $x, y, z \in H$ such that $x \ll y$ and $y \ll z$. Then by Remark 3, $L_H(z) = \{z\}$ or $L_H(z) = \{0, z\}$. Thus, $y \ll z$ implies that $y = 0$ or $y = z$. If $y = 0$, then $x \ll 0$ since $x \ll y$. By Proposition 1, $x = 0$. Since $0 \ll z$ and $x = 0$, we have $x \ll z$. If $y = z$, then the assumption $x \ll y$ implies that $x \ll z$. Hence, H is ordered. □

Example 5. Consider the hyper BCI-algebra defined by the Cayley table:

$\circledast$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

H is hyperatomic since all its elements are hyperatoms.

Example 6. The hyper BCI-algebra H in Example 1 is not hyperatomic since 2 is not a hyperatom of H: $\exists x = 1 \in H$ with $1 \ll 2$ but $x = 1 \neq 0$ and $x = 1 \neq 2$. However, the hyper BCI-algebra H in Example 3 is hyperatomic.

Proposition 10. *Let H be a hyper BCI-algebra such that $|H| \geq 2$. If x and y are distinct nonzero hyperatoms of H, then $L_H(\{x,y\}) = \{0\}$ or $\varnothing$.*

Proof. The result depends on whether or not $0 \in L_H(x)$ for all $x \in H$. If $0 \notin L_H(x)$, then $L_H(x) = \{x\}$. Hence, $L_H(\{x,y\}) = \varnothing$. If $0 \in L_H(\{x\})$, then $L_H(\{x\}) = \{0,x\}$. Since $L_H(\{y\}) = \{0\}$ or $\{0,y\}$ by Remark 3, we have $L_H(\{x,y\}) = \{0\}$ or $\varnothing$. □

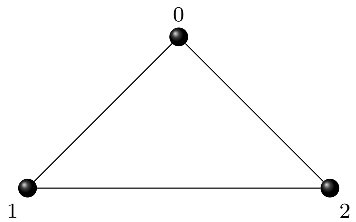
We have the following characterization for a complete graph:

Proposition 11. *Let H be a hyper BCI-algebra such that $|H| \geq 2$. Then $\Gamma(H)$ is a complete graph if and only if $\Gamma(H)$ is connected and H is hyperatomic.*

Proof. If $\Gamma(H)$ is disconnected, then $\Gamma(H)$ is not a complete graph, and we are done. Assume that $\Gamma(H)$ is connected. By Proposition 4, $0 \in L_H(x)$ for all $x \in H$. Since $x \in L_H\{x\}$, we now have $0, x \in L_H(x)$. If H is not hyperatomic, then there exists $z \in H \setminus \{0\}$ such that $y \ll z$ with $y \notin \{0,z\}$. Since $y \in L_H\{y\}$, $y \in L_H\{y,z\}$. This means that $L_H\{y,z\} \neq \{0\}$, implying that $yz \notin E(\Gamma(H))$. Therefore, $\Gamma(H)$ is not complete.

Conversely, suppose $\Gamma(H)$ is connected and H is hyperatomic. Then by Remark 3, $L_H\{x\} = \{0,x\}$ for all $x \in H \setminus \{0\}$. Thus, for any distinct nonzero elements x, y of $H = V(\Gamma(H))$, $L_H\{x\} \cap L_H\{y\} = \{0\}$, that is, $xy \in E(\Gamma(H))$. Consequently, $\Gamma(H)$ is a complete graph. □

Example 7. The hyper BCI-algebra in Example 5 has a complete zero divisor graph:



Remark 5. *Given an ordered hyper BCI-algebra H, it is not always true that there exists $a \in A^*(H) = A(H) \setminus \{0\}$ such that $a \ll x$ for all $x \in H \setminus \{0\}$.*

Example 8. Consider the hyper BCI-algebra H defined in Example 3. H is hyperatomic and hence, ordered and $A^*(H) = \{1,2\}$. Notice that neither $1 \ll x$ nor $2 \ll x$ for all $x \in H \setminus \{0\}$. But for each $x \in H \setminus \{0\}$, there exists $a \in A^*(H)$ such that $a \ll x$.

Theorem 4. *Let H be an ordered hyper BCI-algebra with $|H| \geq 2$. Then the following hold:*

(i) *For each $x \in H$, there is $a_x \in A^*(H)$ such that $a_x \ll x$. In particular, $A^*(H) \neq \varnothing$.*

(ii) *There exists $a \in H \setminus \{0\}$ such that $a \ll x$ for all $x \in H \setminus \{0\}$ if and only if $|A^*(H)| = 1$ (that is, $A^*(H) = \{a\}$).*

Proof.

(i) Let $x \in H \setminus \{0\}$. If x is a hyperatom, then take $a_x = x$. If x is not a hyperatom, there exists $x_1 \in H \setminus \{0, x\}$ such that $x_1 \ll x$. Again, if x_1 is a hyperatom, then take $a_x = x_1$. Otherwise, there exists $x_2 \in H \setminus \{0, x, x_1\}$ such that $x_2 \ll x_1 \ll x$. Since H is finite, continuing in this fashion yields a terminal point $x_n \in H \setminus \{0, x, x_1, \dots, x_{n-1}\}$ with $x_n \ll x_{n-1} \ll \cdots \ll x_2 \ll x_1 \ll x$ such that only $z = 0$ (provided $0 \in L_H\{x\}$) or $z = x_n$ satisfies $z \ll x$. This implies that $a_x = x_n \in A^*(H)$ and $a_x \ll x$.

(ii) Suppose that there exists $a \in H \setminus \{0\}$ such that $a \ll x$ for all $x \in H \setminus \{0\}$. Choose any $b \in A^*(H)$. Then $a \ll b$. Since $b \in A(H)$ and $a \neq 0$, it follows that $a = b$. Thus $a \in A^*(H)$. Since b was arbitrarily chosen, we have $A^*(H) = \{a\}$. Conversely, suppose that $|A^*(H)| = 1$, say $A^*(H) = \{a\}$. Let $x \in H \setminus \{0\}$. Then by (i), $a \ll x$.

□

Example 9. Consider the ordered hyper BCI-algebra H defined in Example 1. Note that 1 is the only nonzero hyperatom of H and the zero divisor graph $\Gamma(H)$ of H is a star.

As a generalization of Example 9, we have the following theorem.

Theorem 5. *Let H be an ordered hyper BCI-algebra with $|H| \geq 2$. Then $\Gamma(H)$ is a star if and only if $\Gamma(H)$ is connected and $|A^*(H)| = 1$.*

Proof. Suppose that $\Gamma(H)$ is a star. Then $\Gamma(H)$ is connected. If $|H| = 2$, then clearly, $|A^*(H)| = 1$. Suppose that $|H| \geq 3$. By Proposition 3(i), 0 is the central vertex of $\Gamma(H)$. Suppose further that $|A^*(H)| \geq 2$, say a, $b \in A^*(H)$ with $a \neq b$. Since $0a, 0b \in E(\Gamma(H))$, $0 \in L_H(a) \cap L_H(b)$. By Proposition 10, $ab \in E(\Gamma(H))$. This implies that $\Gamma(H)$ is not a star, a contradiction. Therefore, $|A^*(H)| = 1$.

Conversely, suppose that $\Gamma(H)$ is connected and $|A^*(H)| = 1$, say $A^*(H) = \{a\}$. If $|H| = 2$, then $\Gamma(H) = P_2$, a star. Suppose that $|H| \geq 3$ and let $y, z \in H \setminus \{0\}$. By Theorem 4(ii), $a \ll y$ and $a \ll z$. That is, $a \in L_H(\{y, z\})$. This implies that $yz \notin E(\Gamma(H))$. Since $\Gamma(H)$ is connected, by Proposition 4, $0x \in E(\Gamma(H))$ for all $x \in H \setminus \{0\}$. Consequently, $\Gamma(H)$ is a star. □

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