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(p, q)-Growth of Entire Harmonic Functions in $\mathbb{R}^n$ in Terms of Approximation Errors

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Abstract. The relationship between the generalized growth parameters of an entire harmonic function in space $\mathbb{R}^n, n \geq 3$, with the rate of its best harmonic polynomial approximation error and ratios of these errors of functions harmonic in the ball of radius R has been studied.

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1. Introduction

The approximation of entire functions on compact sets was studied by Srivastava and Kumar [14,15] and obtained generalized growth parameters in terms of approximation and interpolation error. Similar studies have been done for harmonic functions. The harmonic functions play an important role not only in theoretical mathematics but also in Physics and mechanics to describe different stationary processes. Therefore, it is significant to mention here that the study of generalized growth parameters of a harmonic function in an n-dimensional spaces has relevance. Harmonic functions can be expanded into series in spherical harmonics in space $\mathbb{R}^n, n \geq 3$ and in the adjoined Legendre polynomials in space $\mathbb{R}^3$. The growth characteristics of harmonic functions in terms of the coefficients of their expansion into series as well as not related the expansion coefficients, in particular, in terms of the norm of their gradient at the origin were obtained. Also, the growth of harmonic function in terms of approximation errors by harmonic polynomials in $\mathbb{R}^n, n \geq 3$ was considered by various authors (see,[3,6-13]). The aim of the present work is to investigate

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conditions under which a harmonic function in the ball of n-dimensional space continues to the entire harmonic function, and to derive formulae for the generalized growth parameters (p, q)-order, lower(p, q)-order, (p, q)-type and lower(p, q)-type of harmonic function in space in terms of harmonic polynomial approximation errors. Here p and q are integers such that $p \geq q \geq 1$.

Let u be an entire harmonic function in $\mathbb{R}^n$ and has a Fourier-Laplace series expansion [16]

$$u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x; u) r^k, \tag{1.1}$$

where $x \in S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ a unit sphere in $\mathbb{R}^n$ centered at the origin

$$\begin{aligned} Y^{(k)}(x; u) =& a_1^{(k)} Y_1^{(k)}(x) + a_2^{(k)} Y_2^{(k)}(x) + \cdots + a_{\gamma_k}^{(k)} Y_{\gamma_k}^{(k)}(x), \\ a_j^{(k)} =& (u, Y_j^{(k)}) = \frac{\Gamma(n/2)}{2(\pi)^{\frac{n}{2}}} \int_{S^n} u(x) Y_j^{(k)}(x) dS, j = \overline{1, \gamma_k}, \\ \gamma_k =& \frac{(2k + n - 2)(k + n - 3)!}{k!(n-2)!}. \end{aligned}$$

Here dS is the element of the surface area on the sphere S^n, $(u, Y_j^{(k)})$ is the scalor product in $L^2(S^n)$ and $Y^{(k)}$ is a spherical harmonic of degree $k, k \in Z_+ = \{0, 1, 2, \ldots, \}$ on the unit sphere $S^n (n \geq 2)$ [16].

Let $B_R^n = \{y \in \mathbb{R}^n : |y| \leq R\}$ be the ball of radius R in space $\mathbb{R}^n, n \geq 3$ centered at the origin, and $\overline{B_R^n}$ be the closure of B_R^n. We denote H_R, the class of harmonic functions in B_R^n and continuous on $\overline{B_R^n}, 0 < R < \infty$.

Let π_k be the set of harmonic polynomials of degree$\leq k$. The approximation error of function $u \in H_R$ by harmonic polynomials $P \in \pi_k$ be defined as

$$E_R^{(k)}(u) = \inf_{P \in \pi_k} \{ \max_{y \in B_R^n} |u(y) - P(y)| \}. \tag{1.2}$$

For $u \in H_R$ continue to the entire harmonic function of n-dimensional space $\mathbb{R}^n, n \geq 3$, it is known [17 ,p.45] that

$$\lim_{k \to \infty} (E_R^{(k)}(u))^{\frac{1}{k}} = 0. \tag{1.3}$$

The concept of order $\rho(F)$ and lower order $\lambda(F)$ of an entire function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ was introduced by R.P. Boas [1] as

$$\rho(F) = \lim_{r \to \infty} \sup \frac{\log \log M(r, F)}{\log r}, \lambda(F) = \lim_{r \to \infty} \inf \frac{\log \log M(r, F)}{\log r}.$$

The concept of type $T(F)$ and lower type $t(F)$ has been introduced when the entire functions have same nonzero finite order. An entire function of order $\rho, 0 < \rho < \infty$, is said to be of type $T(F)$ and lower type $t(F)$ if

$$T(F) = \lim_{r \to \infty} \sup \frac{\log M(r, F)}{r^{\rho(F)}}, t(F) = \lim_{r \to \infty} \inf \frac{\log M(r, F)}{r^{\rho(F)}},$$

where $0 < \rho(F) < \infty, M(r,F) = \max_{0\leq\theta\leq 2\pi}|F(r,\theta)|$.
For the class of order $\rho(F) = 0$ and $\rho(F) = \infty$, the type can not be defined. To refine the above concept of order and type, Juneja et. al., [4,5] introduced the concept of (p,q)-orders and (p,q)-types.
Therefore, we define the (p,q)-order and lower(p,q)-order as

$$\rho(p,q,F) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,F)}{\log^{[q]} r}, \lambda(p,q,F) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,F)}{\log^{[q]} r}, \tag{1.4}$$

$b \leq \rho(p,q,F) \leq \infty, b = 0$ if $p > q$ and $b = 1$ if $p = q$. The (p,q)-type and lower(p,q)-type are defined as

$$T(p,q,F) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,F)}{\log^{[q-1]} r^{\rho(p,q,F)}}, t(F) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M(r,F)}{\log^{[q-1]} r^{\rho(p,q,F)}}, \tag{1.5}$$

where $\log^{[0]}(x) = x$ and $\log^{[m]}(x) = \log^{[m-1]}\log(x)$ for $m \geq 1$.
Notations:

$$\begin{aligned} P(\alpha) = P(\alpha,p,q) = \{\alpha \quad if \quad p > q, \\ \{1+\alpha \quad if \quad p = q = 2, \\ \{\max(1,\alpha) \quad if \quad 3 \leq p = q < \infty, \\ \{\infty \quad if \quad p = q = \infty, \} \end{aligned}$$

and

$$\begin{aligned} M(\alpha) = M(\alpha,p,q) = \{\frac{1}{e\alpha} \quad if \quad (p,q) = (2,1), \\ \{\frac{(\alpha-1)^{(\alpha-1)}}{\alpha^{\alpha}} \quad if \quad (p,q) = (2,2), \\ \{1 \quad if \quad p \geq 3, \} \end{aligned}$$

From [4] we define the relations between(p,q)-order, lower(p,q)-order, the coefficients of $F(z)$ and ratios of these successive coefficients as following:
Theorem A. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,F)$, then

$$\rho(p,q,F) = P(L(p,q,F))$$

where

$$L(p,q,F) = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} |a_n|^{-\frac{1}{n}}}.$$

Theorem B. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,F)$, then

$$\rho(p,q,F) = P(L^*(p,q,F))$$

where

$$L^*(p,q,F) = \lim\sup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} |\frac{a_n}{a_{n+1}}|}.$$

Theorem C. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,F)$ and $(|\frac{a_n}{a_{n+1}}|)$ a nondecreasing function of n for $n > n_0$ then

$$\lambda(p,q,F) = P(l(p,q,F))$$

where

$$l(p,q,F) = \lim\inf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} |a_n|^{-\frac{1}{n}}}.$$

Theorem D. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,F)$ and $(|\frac{a_n}{a_{n+1}}|)$ a nondecreasing function of n for $n > n_0$ then

$$\lambda(p,q,F) = P(l^*(p,q,F))$$

where

$$l^*(p,q,F) = \lim\inf_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} |\frac{a_n}{a_{n+1}}|}.$$

From [5] we define the relation between (p,q)-type, lower(p,q)-type and the coefficients of $F(z)$ as:

Theorem E. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,F)$ and (p,q)-type $T(p,q,F)$ if and only if $T = MV$, where

$$V(p,q,F) = \lim\sup_{n\to\infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} |a_n|^{-\frac{1}{n}})^{\rho-A}}.$$

with $A = 1$ if $(p,q) = (2,2)$ and $A = 0$ if $(p,q) \neq (2,2)$.

Theorem F. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of (p,q)-order $\rho(p,q,F)$, lower(p,q)-type $t(p,q,F)$ and $(|\frac{a_n}{a_{n+1}}|)$ a nondecreasing function of n for $n > n_0$ then $t = Mv$, where

$$v(p,q,F) = \lim\inf_{n\to\infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} |a_n|^{-\frac{1}{n}})^{\rho-A}}.$$

2. Auxiliary Results

In this section we will prove some auxiliary results which will be used in the sequel. Consider the two functions f and g of complex variable z:

$$f(z) = \sum_{k=0}^{\infty} \frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}} E_R^{(k)}(u)(\frac{z}{R})^k \tag{2.1}$$

and

$$g(z) = \sum_{k=1}^{\infty} \frac{4}{(2\nu)!}(k+2\nu)^{2\nu} E_R^{(k)}(u)(\frac{z}{R})^k. \tag{2.2}$$

In view of [17, pp. 47] we see that if u is an entire function then f and g are also entire functions of the complex variable z. Using Lemma 3 with inequality (8) of [17], we get

$$m(r,f) \leq M(r,u) \leq |Y^{(0)}(\xi,u)| + M(r,g) \tag{2.3}$$

where $m(r,f)$ is the maximum term of power series of function $f(z)$ on the circle$\{z : |z| = r\}$, and $M(r,g) = \max_{|z|=r} |g(z)|$.

Lemma 2.1. Let f and g be defined by (2.1) and (2.2). Then the (p,q)-orders and (p,q)-types of f and g respectively are equal.

Proof. First we consider the case $(p,q) = (2,1)$,

$$\begin{aligned}
\frac{1}{\rho(2,1,f)} &= \liminf_{k\to\infty} \frac{-\frac{1}{k}\log(\frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}} R^{-k} E_R^{(k)}(u))}{\log k}, \\
&= \liminf_{k\to\infty} \frac{\log R^k (E_R^{(k)}(u))^{-1} + \log \sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu} - \log \sqrt{(2\nu)!}}{k \log k}, \\
&= \liminf_{k\to\infty} \frac{\log R^k (E_R^{(k)}(u))^{-1}}{k\log k} + 1,
\end{aligned}$$

then $\frac{1}{\rho(2,1,f)} \geq 1$ and $\rho(2,1,f) \leq 1$. This implies necessarily we have $\rho(2,1,f) = 0$ to define $\rho(p,q,f)$.

Now

$$\begin{aligned}
\rho(2,1,f) = 0 &\Rightarrow \liminf_{k\to\infty} \frac{1}{\rho(2,1,f)} = +\infty \\
&\Rightarrow \liminf_{k\to\infty} \frac{\log R^k (E_R^{(k)}(u))^{-1}}{k \log k} = +\infty \\
&\Rightarrow \liminf_{k\to\infty} \frac{k\log k}{\log R^k (E_R^{(k)}(u))^{-1}} = 0 \\
&\Rightarrow \liminf_{k\to\infty} \frac{\log k!}{\log R^k (E_R^{(k)}(u))^{-1}} = 0 \\
&\Rightarrow (E_R^{(k)}(u) R^{-1})^{\frac{1}{k}} \to 0.
\end{aligned}$$

For $p \geq q > 1$, we have

$$\begin{aligned}
\frac{1}{L(p,q,f)} &= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(-\frac{1}{k}\log(\frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}R^{-k}E_R^{(k)}(u)))}{\log^{[p-1]}k} \\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(\frac{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}{k} + \frac{\log\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}{k}) - \frac{\log\sqrt{(2\nu)!}}{k}}{\log^{[p-1]}k} \\
&= \lim_{k\to\infty}\inf \frac{1}{\log^{[p-1]}k}\log^{[q-1]}(\frac{\log R^{-k}(E_R^{(k)}(u))^{-1}}{k})\times \\
&\quad (1 + \frac{\log\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}{\log[R^{-k}(E_R^{(k)}(u))]^{-1}} - \frac{\log\sqrt{(2\nu)!}}{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}) \\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))}{\log^{[p-1]}k} + \frac{\log(1+o(1))}{\log^{[p-1]}k} \\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))}{\log^{[p-1]}k},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{L(p,q,g)} &= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(-\frac{1}{k}\log(\frac{4}{(2\nu)!}(k+2\nu)^{2\nu}E_R^{(k-1)}(u)R^{-k}))}{\log^{[p-1]}k} \\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(\frac{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}{k} + \frac{\log(2\nu)!}{k} - \frac{1}{k}\log 4(k+2\nu)^{2\nu})}{\log^{[p-1]}k} \\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))}{\log^{[p-1]}k} + \frac{\log(1+o(1))}{\log^{[p-1]}k} \\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-1]}(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))}{\log^{[p-1]}k},
\end{aligned}$$

Using Theorem A, we see that the function f and g have same (p,q)-order, it leads to the fact that

$$\rho(p,q,f) = \rho(p,q,g) = \rho.$$

Now we consider the (p,q)-type for $q = 2$ as

$$\begin{aligned}
\frac{1}{v(p,q,f)} &= \lim_{k\to\infty}\inf \frac{\left(-\frac{1}{k}\log\left(\frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}R^{-k}E_R^{(k)}(u)\right)\right)^{\rho-1}}{\log^{[p-2]}k}\\
&= \lim_{k\to\infty}\inf \frac{\left(\frac{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}{k} + \frac{\log\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}{k} - \frac{\log\sqrt{(2\nu)!}}{k}\right)^{\rho-1}}{\log^{[p-2]}k}\\
&= \lim_{k\to\infty}\inf \frac{1}{\log^{[p-2]}k}\left(\frac{\log R^{-k}(E_R^{(k)}(u))^{-1}}{k}\right)^{\rho-1}\times\\
&\quad \left(1+\frac{\log\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}{\log[R^{-k}(E_R^{(k)}(u))]^{-1}} - \frac{\log\sqrt{(2\nu)!}}{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}\right)^{\rho-1}\\
&= \lim_{k\to\infty}\inf \frac{(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))^{\rho-1}}{\log^{[p-2]}k} + \frac{\log(1+o(1))^{\rho-1}}{\log^{[p-2]}k}\\
&= \lim_{k\to\infty}\inf \frac{(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))^{\rho-1}}{\log^{[p-2]}k}.
\end{aligned}$$

Similarly for g we have

$$\frac{1}{v(p,2,g)} = \lim_{k\to\infty}\inf \frac{(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))^{\rho}}{\log^{[p-2]}k}.$$

Now for the case $q \geq 3$, we have

$$\begin{aligned}
\frac{1}{v(p,q,f)} &= \lim_{k\to\infty}\inf \frac{\log^{[q-2]}\left(-\frac{1}{k}\log\left(\frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}R^{-k}E_R^{(k)}(u)\right)\right)^{\rho}}{\log^{[p-2]}k}\\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-2]}\left(\frac{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}{k} + \frac{\log\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}{k} - \frac{\log\sqrt{(2\nu)!}}{k}\right)^{\rho}}{\log^{[p-2]}k}\\
&= \lim_{k\to\infty}\inf \frac{1}{\log^{[p-2]}k}\log^{[q-2]}\left(\frac{\log R^{-k}(E_R^{(k)}(u))^{-1}}{k}\right)^{\rho}\times\\
&\quad \left(1+\frac{\log\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}}{\log[R^{-k}(E_R^{(k)}(u))]^{-1}} - \frac{\log\sqrt{(2\nu)!}}{\log[R^{-k}(E_R^{(k)}(u))]^{-1}}\right)^{\rho-1}\\
&= \lim_{k\to\infty}\inf \frac{\log^{[q-2]}(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))^{\rho}}{\log^{[p-2]}k} + \frac{\log(1+o(1))^{\rho}}{\log^{[p-2]}k}\\
&= \lim_{k\to\infty}\inf \frac{(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))^{\rho}}{\log^{[p-2]}k}.
\end{aligned}$$

In the same manner for the function g, we obtain

$$\frac{1}{v(p,q,g)} = \lim_{k\to\infty}\inf \frac{(-\frac{1}{k}\log(R^{-k}E_R^{(k)}(u)))^{\rho}}{\log^{[p-2]}k}.$$

Lemma 2.2. Let u be an entire harmonic function of an n-dimensional space $n \geq 3$ with (p,q)-order $\rho(p,q,u)$, lower (p,q)-order $\lambda(p,q,u)$, (p,q)-type $T(p,q,u)$ and lower (p,q)-type $t(p,q,u)$. If f and g are entire functions defined as in (2.1) and (2.2), then

$$\rho(p,q,f) = \rho(p,q,u) = \rho(p,q,g), \tag{2.4}$$

$$\lambda(p,q,f) \leq \lambda(p,q,u) \leq \lambda(p,q,g), \tag{2.5}$$

$$T(p,q,f) = T(p,q,u) = T(p,q,g), \tag{2.6}$$

$$t(p,q,f) \leq t(p,q,u) \leq t(p,q,g). \tag{2.7}$$

Proof. From (2.3) with (1.4) and (1.5) we have

$$\rho(p,q,f) \leq \rho(p,q,u) \leq \rho(p,q,g). \tag{2.8}$$

Since $\rho(p,q,f) = \rho(p,q,g)$, now (2.4) easily obtain by using (2.8). From (2.3) we can get (2.5) immediately. We denote the common value of (p,q)-order of f, g and u and using (2.3) we get

$$\frac{\log^{[p-1]} m(r,f)}{(\log^{[q-1]} r)^{\rho}} \leq \frac{\log^{[p-1]} m(r,u)}{(\log^{[q-1]} r)^{\rho}} \leq \frac{\log^{[p-1]} m(r,g)}{(\log^{[q-1]} r)^{\rho}}$$

It proves (2.6) and (2.7).

Now let us define $\alpha_k = \max_{x\in S^n} |Y^{(k)}(x;u)|$, $\beta_k = \frac{\sqrt{(2\nu)!}}{\sqrt{2(2\nu+1)!(k+2\nu)^{2\nu}}} E_R^{(k)} R^{-k}$ and $\gamma_k = \frac{4}{(2\nu)!}(k+2\nu)^{2\nu} E_R^{(k-1)} R^{-k}$.

3. Main Results

Theorem 3.1. Let u be an entire harmonic function in $\mathbb{R}^n, n \geq 3$ with (p,q)-order $\rho(p,q,u)$ and (p,q)-type $T(p,q,u)$. If $(\frac{E_R^{(k)}(u)}{E_R^{(k-1)}(u)})$ is a nondecreasing function of k for $k > k_0$, then

$$\rho(p,q,u) = P(L(p,q,u))$$

where

$$L(p,q,u) = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q]} (E_R^{(k)}(u) R^{-k})^{-\frac{1}{k}}} \tag{3.1}$$

and

$$T(p,q,u) = M v^*(p,q,u)$$

where

$$v^*(p,q,u) = \limsup_{k\to\infty} \frac{\log^{[p-2]} k}{(\log^{[q-1]} (E_R^{(k)}(u) R^{-k})^{-\frac{1}{k}})^{\rho - A}}, \rho(p,q,u) \equiv \rho. \tag{3.2}$$

Proof. Corresponding to an entire harmonic function $u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x;u)r^k$ we define the entire function $u(\zeta x) = \sum_{k=0}^{\infty} \max|Y^{(k)}(x;u)|\zeta^k, |x| = 1$ [2], now applying Theorem A, we have

$$\rho(p,q,u) = P(L(p,q,u))$$

where

$$L(p,q,u) = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q]}(\alpha_k)^{-\frac{1}{k}}}.$$

We know that $(\frac{\alpha_k}{\alpha_{k+1}})$ is a nondecreasing function of $k, k > k_0$ if $(\frac{E_R^{(k)}(u)}{E_R^{(k-1)}(u)})$ is a nondecreasing function of k for $k > k_0$. This implies that $(\frac{\beta_k}{\beta_{k+1}})$ and $(\frac{\gamma_k}{\gamma_{k+1}})$ are also nondecreasing function of $k, k > k_0$.
Using [17, Lemma 1] we obtain

$$\frac{\alpha_k}{\alpha_{k+1}} \leq \frac{(k+2\nu)^{2\nu} E_R^{(k-1)}(u)R}{(k+1+2\nu)^{2\nu} E_R^{(k)}(u)}.$$

Let $p(x) = (\frac{x+2\nu}{x+2\nu+1})^{2\nu}$,

$$\log p(x) = 2\nu \log(x+2\nu) - 2\nu \log(x+1+2\nu),$$

$$\frac{p'(x)}{p(x)} = \frac{2\nu}{x+2\nu} - \frac{2\nu}{x+2\nu+1},$$

taking $w(x) = \frac{2\nu}{x+2\nu}, w(x) - w(x+1) > 0$ for any $x > 0$. Hence $w(x)$ is a decreasing function and subsequently $p'(x) > 0$ for $x > 0$. Hence $(\frac{\alpha_k}{\alpha_{k+1}})$ is nondecreasing if $(\frac{E_R^{(k)}(u)}{E_R^{(k-1)}(u)})$ is nondecreasing function of k for $k > k_0$. The result (3.1) is obtain by using the relation (2.4) and the result (3.2) is found from (2.6).
Theorem 3.2. Let u be an entire harmonic function in $\mathbb{R}^n, n \geq 3$ with (p,q)-order $\rho(p,q,u)$ and $(\frac{E_R^{(k)}(u)}{E_R^{(k-1)}(u)})$ is a nondecreasing function of k for $k > k_0$, then

$$\rho(p,q,u) = P(L(p,q,u)) \tag{3.3}$$

where

$$L(p,q,u) = \limsup_{k\to\infty} \frac{\log^{[p-1]} k}{\log^{[q]}(\frac{E_R^{(k-1)}(u)R}{E_R^{(k)}(u)})}$$

Proof. For an entire function $u(zx) = \sum_{k=0}^{\infty} \max|Y^{(k)}(x;u)|z^k$, using Theorem B we have

$$\rho(p,q,u) = P(L(p,q,u))$$

where

$$L(p,q,u)=\limsup_{k\to\infty}\frac{\log^{[p-1]}k}{\log^{[q]}(\frac{\alpha_k}{\alpha_{k+1}})}$$

if $(\frac{\alpha_k}{\alpha_{k+1}})$ is a nondecreasing function of k for $k>k_0$. Applying above relation for the entire function f, we obtain

$$\rho(p,q,f)=P(L(p,q,f)), \tag{3.4}$$

$$\begin{aligned}L(p,q,f)=&\limsup_{k\to\infty}\frac{\log^{[p-1]}k}{\log^{[q]}(\frac{\beta_k}{\beta_{k+1}})}\\ =&\limsup_{k\to\infty}\frac{\log^{[p-1]}k}{\log^{[q]}(\frac{E_R^{(k-1)}(u)R}{E_R^{(k)}(u)}(\frac{k+1+2\nu}{k+2\nu})^{2\nu})}\\ =&\limsup_{k\to\infty}\log^{[p-1]}k(\log^{[q-1]}(\log(\frac{E_R^{(k)}(u)R}{E_R^{(k+1)}(u)})+2\nu\log(\frac{k+1+2\nu}{k+2\nu})^{-1}))\\ =&\limsup_{k\to\infty}\frac{\log^{[p-1]}k}{\log^{[q]}(\frac{E_R^{(k)}(u)R}{E_R^{(k+1)}(u)})}.\end{aligned}$$

Similarly for the entire function g, we have

$$L(p,q,g)=\limsup_{k\to\infty}\frac{\log^{[p-1]}k}{\log^{[q]}(\frac{E_R^{(k)}(u)R}{E_R^{(k+!)}(u)})}.$$

$\rho(p,q,g)=P(L(p,q,g))$ Now (3.3)follows from (2.4).

Theorem 3.3. Let u be an entire harmonic function in $\mathbb{R}^n, n\geq 3$ with (p,q)-order $\rho(p,q,u)$, lower (p,q)-order $\lambda(p,q,u)$, lower (p,q)-type $t(p,q,u)$ and let $(\frac{E_R^{(k)}(u)}{E_R^{(k+1)}(u)})$ is a nondecreasing function of k for $k>k_0$, then

$$\lambda(p,q,u)=P(l^*(p,q,u)) \tag{3.5}$$

where

$$l^*(p,q,u)=\liminf_{k\to\infty}\frac{\log^{[p-1]}k}{\log^{[q]}(R^{-k}E_R^{(k)}(u))^{-\frac{1}{k}}}$$

and

$$t(p,q,u)=Mv(p,q,u) \tag{3.6}$$

where

$$v(p,q,u)=\liminf_{k\to\infty}\frac{\log^{[p-2]}k}{(\log^{[q-1]}(R^{-k}E_R^{(k)}(u))^{-\frac{1}{k}})^{\rho-A}}$$

Proof. Applying Theorem D to the function f and g we can easily obtain

$$\lambda(p,q,f) = P(l^*(p,q,f))$$

and

$$\lambda(p,q,g) = P(l^*(p,q,g))$$

where

$$l^*(p,q,f) = \lim_{k\to\infty}\inf \frac{\log^{[p-1]} k}{\log^{[q]}(R^{-k}E_R^{(k)}(u))^{-\frac{1}{k}}},$$

$$l^*(p,q,g) = \lim_{k\to\infty}\inf \frac{\log^{[p-1]} k}{\log^{[q]}(R^{-k}E_R^{(k)}(u))^{-\frac{1}{k}}}$$

Now the result (3.6) follows from (2.5). Similarly applying Theorem F to the entire function f and g, we get

$$t(p,q,f) = Mv(p,q,f), t(p,q,g) = Mv(p,q,g)$$

where

$$v(p,q,f) = \lim_{k\to\infty}\inf \frac{\log^{[p-2]} k}{(\log^{[q-1]}(R^{-k}E_R^{(k)}(u))^{-\frac{1}{k}})^{\rho-A}}$$

$$v(p,q,g) = \lim_{k\to\infty}\inf \frac{\log^{[p-2]} k}{(\log^{[q-1]}(R^{-k}E_R^{(k)}(u))^{-\frac{1}{k}})^{\rho-A}}$$

The result (3.7) follows from (2.7) and above relations.

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