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Some Affine Connexions in a Generalised Structure Manifold-II

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Abstract. In this paper we have studied some affine connexions in a generalised structure manifold. Certain theorems are also have been proved which are of great geometrical importance.

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1. Introduction

We consider a differentiable manifold V_n of differentiability class C^∞ and of dimension n. Let there exist in V_n a tensor field F of the type $(1,1)$, s linearly independent vector fields U_i, $i = 1, 2, \ldots, s$ and s linearly independent 1-forms u^i such that for any arbitrary vector field X, we have

$$\overline{\overline{X}} = b^2 X + c\, u^i(X) U_i \tag{1}$$
$$\overline{U_i} = p_i^j U_j \tag{2}$$

where $F(X) \overset{def}{=} \overline{X}$ and b^2, c are constants.
Then the structure $\{F, u^i, U_i, p_i^j; i, j = 1, 2, \ldots, s\}$ will be known as generalised structure and V_n will be known as generalised structure manifold of order s where $s < n$.

Lemma 1. *All the equations which follow hold for arbitrary vector fields $X, Y, Z, \ldots$ etc.*

Now, replacing X by $\overline{X}$ in (1),we get

$$\overline{\overline{\overline{X}}} = b^2 \overline{X} + c\, u^i(\overline{X}) U_i \tag{3}$$

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Operating F in (1), we get

$$\overline{\overline{\overline{X}}} = b^2\overline{X} + c\ u^i(X)\overline{U_i}$$

Using (2) in above, we get

$$\overline{\overline{\overline{X}}} = b^2\overline{X} + c\ u^i(X)p_i^j U_j \tag{4}$$

From (3) and (4), we have

$$u^i(\overline{X}) = p_j^i u^j(X) \tag{5}$$

Further, operating F in (2) and using (1), (2) we get

$${}^{(2)}p_i^j = b^2\delta_i^j + c\ u^j(U_i) \tag{6}$$

where

$${}^{(r)}p_j^i = {}^{(r-1)}p_k^i\ p_j^k$$

On generalised structure manifold V_n, let us introduce a metric tensor g such that 2-form $'F$ defined by $'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ is skew-symmetric,then V_n is called generalised metric structure manifold [8,12].

We have on a generalised metric structure manifold,

$$g(\overline{X},Y) + g(X,\overline{Y}) = 0$$

Replacing Y by $\overline{Y}$ in above equation and using (1), we obtain

$$g(\overline{X},\overline{Y}) + b^2 g(X,Y) + c\ u^i(X)u^i(Y) = 0 \tag{7}$$

where

$$u^i(X) = g(U_i,X) \tag{8}$$

Lemma 2. *The generalised metric structure manifold always be denoted by V_n.*

1.1. Definitions

This section consists of well known definitions required to go through the insuring sections [1,6].

1. A differentiable manifold M_n on which there a vector valued linear function F, a 2-form $'F$ defined by $'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ such that

    - $F^2 = 0$ and $'F(X,Y)$ is skew-symmetric, then M_n is called an *almost tangent metric manifold*.
    - $F^2 = -I_n$ and $'F(X,Y)$ is skew-symmetric, then M_n is called an *almost Hermite manifold*.
    - $F^2 = \lambda^2 I_n$, where λ is a non-zero complex constant and $'F(X,Y)$ is skew-symmetric, then M_n is called an *metric π-structure manifold* [13].

- $F^2 = \lambda^r I_n$ and $'F(X,Y)$ is skew-symmetric, then M_n is called an *Hsu-structure metric manifold* [4,5].
- $F^2 = I_n$ and $'F(X,Y)$ is symmetric, then M_n is said to be an *almost product Riemannian manifold* [7].

2. Let us consider a C^∞-manifold M_n $(n = 2m+1)$. Let there exist in M_n a tensor field F of the type $(1,1)$, a 1-form u, a vector field U and a Riemannian metric g satisfying

$$\overline{\overline{X}} = -X + u(X)U \tag{9a}$$

$$\overline{U} = 0 \tag{9b}$$

$$g(\overline{X},\overline{Y}) = g(X,Y) - u(X)u(X) \tag{9c}$$

where $g(X,U) = u(X)$ and $F(X) \overset{def}{=} \overline{X}$
Then M_n is called an *almost contact metric manifold* or an *almost Grayan manifold* [16,17].

3. We consider a manifold M_n of differentiability class C^∞. Let there exist in M_n, a tensor field F of the type $(1,1)$ and rank r $(1 \le r \le n)$ satisfying

$$F^3 + F = 0 \tag{10}$$

then $\{F\}$ is called F-structure and M_n satisfying (10) is called F-structure manifold [2].

If we consider $'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ where g is a Riemannian metric and $'F$ is skew-symmetric then, F-structure manifold M_n is called a metric F-structure manifold.

4. The tensor K of the type $(1,3)$ defined by [14]

$$K(X,Y,Z) \overset{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{11}$$

is called the curvature tensor of the connexion D.

5. The vector field U_i in generalised structure metric manifold V_n is called a Killing vector if it satisfies [11]

$$D_X u^i)(Y) + (D_Y u^i)(X) = 0$$

6. A connection D which satisfies

$$(D_X F)(Y) = 0,\ (D_X u^i)(Y) = 0,\ D_X U_i = 0 \tag{12}$$

is called an (F, U_i, u^i)-connexion.

7. A connection D is called an F-connexion if it satisfies

$$(D_X F)(Y) = 0 \quad \text{i.e., } D_X \overline{Y} = \overline{D_X Y} \tag{13}$$

8. Nijenhuis tensor is a vector valued bilinear function N, given by

$$N(X,Y) \stackrel{def}{=} [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[X,\overline{Y}]} - \overline{[\overline{X},Y]} \tag{14}$$

9. A vector valued, skew-symmetric, bilinear function S defined by

$$S(X,Y) \stackrel{def}{=} D_X Y - D_Y X - [X,Y] \tag{15}$$

is called torsion tensor of a connexion D.

For symmetric or torsion free connexion D, the torsion tensor vanishes, i.e.

$$D_X Y - D_Y X = [X,Y] \tag{16}$$

10. Lie derivative along any vector V in a C^∞-manifold M_n is a type preserving mapping such that [3]

$$L_V f = Vf \ ; f \text{ is a } C^\infty\text{–function} \tag{17a}$$
$$L_V X = [V,X] \tag{17b}$$
$$L_V B(X) = V(B(X)) - B([V,X]) \tag{17c}$$

where B is an arbitrary 1-form.

Remark 1. *It may be noted that V_n gives an almost tangent metric manifold, an almost Hermite manifold, metric π-structure manifold, Hsu-structure manifold, F-structure manifold, an almost product Riemannian manifold, an almost Grayan manifold and $\{F,g,u^1,u^2,U_1,U_2\}$ structure manifold according as $(b^2=0,c=0)$; $(b^2=-1,c=0)$; $(c=0)$; $(b^2=\lambda^r,c=0)$; $(b^2=-1,p_i^j=0)$; $(b^2=1,c=0)$; $(b^2=-1,c=1,p_1^1=0:i,j=1)$; and $(b^2=-1,c=1,p_i^j+p_j^i=0:i,j=1,2)$ respectively.*

1.2. Some Basic Results

1. If we put, $\rho F' = F\rho$,$U_i' =^{-1} \rho U_i$ and $u'^i = u^i \circ \rho$, where ρ is a non-singular tensor of the type $(1,1)$, then it can be easily seen that $\{F',u'^i,U_i',p_i^j;i,j=1,2,\ldots,s\}$ is also a generalised structure.
2. The eigen values of F are given by $b,-b,\sqrt{A_i},-\sqrt{A_i}$ where A_i are the roots of the equation $|\lambda^2\delta_i^j -^{(2)} p_i^j| = 0$. The multiplicity of the eigen values depends on $rank((F))$, on p_i^j and the nature of b^2,c.

2. An Affine Connexion I

In this section an affine connexion in a generalised structure manifold V_n is defined and its properties have been studied [9,10,15].

Let us define an affine connexion D such that

$$u^i(Y)(D_X U_i) + (D_X u^i)(Y) U_i = 0 \tag{18}$$

where D is an F-connexion given by (13). It can be easily seen that,

$$u^i(D_X U_i) = -(D_X u^i)(Y) \tag{19a}$$

$$({}^{(2)}p^i_j - b^2\delta^i_j)(D_X U_i) = u^i(D_X U_i)U_j \tag{19b}$$

$$(D_X u^i)(\overline{Y})(U_i) = -p^i_j u^j(Y)(D_X U_i) \tag{19c}$$

$$({}^{(2)}p^i_j - b^2\delta^i_j)\, div\ U_j = c\ u^j(D_{U_j} U_i) \tag{19d}$$

where $div(X) \overset{def}{=} (C^1_1 \nabla X)$ and $(\nabla X)(Y) = (D_Y X)$.

Theorem 1. *In V_n, let us put*

$$M(X,Y) \overset{def}{=} D_{\overline{X}}\overline{Y} + \overline{\overline{D_X Y}} - \overline{D_X \overline{Y}} - \overline{D_{\overline{X}} Y} \tag{20}$$

then,

$$M(X,Y) = 0 \tag{21}$$

Proof. Using (13) in (20), we get (21).

Theorem 2. *If connexion D is torsion free in V_n, then we have*

$$N(X,Y) = 0 \tag{22}$$

where $N(X,Y)$ is Nijenhuis tensor.

Proof. Using (16) and (13) in (14), we get (22).

Now, corresponding to the Nijenhuis tensor of an almost complex manifold, we have three tensors μ, ν and σ given by

$$\mu(X,Y) \overset{def}{=} (D_Y u^i)(\overline{X}) - (D_X u^i)(\overline{Y}) + (D_{\overline{Y}} u^i)(X) - (D_{\overline{X}} u^i)(Y) \tag{23}$$

$$\nu(X) \overset{def}{=} (D_{U_i} F)(X) - (D_X F)(U_i) - D_{\overline{X}} U_i \tag{24}$$

$$\sigma(X) \overset{def}{=} (D_X u^j)(U_i) - (D_{U_i} u^j)(X) \tag{25}$$

respectively.

Theorem 3. *If connexion D is torsion free in V_n, then we have*

$$\mu(X,Y)U_i = (D_X U_i)\,[u^i(Y)+u^i(\overline{Y})] - (D_Y U_i)\,[u^i(X)+u^i(\overline{X})] \tag{26}$$

$$\nu(X) = -(D_{\overline{X}} U_i) \tag{27}$$

$$\sigma(X) = -u^j(D_X U_i) - (D_{U_i} u^j)(X) \tag{28}$$

Proof. Using (5), (18) and (19c) in (23), we get (26). The equation (24) yields (27) on use of (13). (28) is obtained on the use of (19a) in (25).

Theorem 4. *In V_n, let us put*

$$C(X,Y) = (D_X u^i)(Y) - (D_Y u^i)(X) \tag{29}$$

Then, we have

$$C(X,\overline{Y}) + C(\overline{X},Y) = -\mu(X,Y) \tag{30}$$

$$C(\overline{X},\overline{Y})U_i = u^i(\overline{Y})\nu(X) - u^i(\overline{X})\nu(Y) \tag{31}$$

Proof. Replacing Y by $\overline{Y}$ and X by $\overline{X}$ in (29) separately and adding resulting these two equations, we get (30). Further, replacing X by $\overline{X}$, Y by $\overline{Y}$ and multiplying with U_i in (29), we get (31) on use of (5), (19c) and (27).

Corollary 1. *In V_n, we have*

$$C(X,U_i) = \sigma(X) \tag{32}$$

$$C(\overline{X},U_i) = (D_X u^j)(\overline{X}) - u^j(\nu(X)) \tag{33}$$

Proof. By replacing i by j and Y by U_i in (29), we get (32). Further, by replacing X by $\overline{X}$ in (32) and using (19c) & (27), we obtain (33).

Theorem 5. *In V_n, with U_i as a killing vector, we have*

$$C(X,U_i) = -2u^i(D_X U_i) \tag{34}$$

Proof. Considering U_i as a killing vector with respect to connexion D and using (19a) in (29) after putting U_i for Y, we get (34).

Theorem 6. *In V_n, we have*

$$(L_X u^i)(Y) - (L_Y u^i)(X) = C(X,Y) - u^i(L_X Y) \tag{35}$$

Proof. Lie derivative of u^i is given by

$$(L_X u^i)(Y) = (D_X u^i)(Y) + u^i(D_Y X) \tag{36}$$

Interchanging X and Y in the above equation and subtracting the resulting equation from above equation, we get (35) on the use of (29) and (17b).

Corollary 2. *In V_n, we have*

$$(L_X u^i)(U_i) - (L_{U_i} u^i)(X) = \sigma(X) - u^i(L_X U_i) \tag{37}$$

$$(L_{\overline{X}} u^i)(U_i) - (L_{U_i} u^i)(\overline{X}) = (D_{U_i} u^j)(\overline{X}) - u^j(\nu(X)) - u^i(L_{\overline{X}} U_i) \tag{38}$$

Theorem 7. *In V_n, we have*

$$(L_{\overline{X}} u^i)(Y) - (L_{\overline{Y}} u^i)(X) + \mu(X,Y) = (D_Y u^i)(\overline{X}) - (D_X u^i)(\overline{Y}) + p^i_j u^j([X,Y]) \tag{39}$$

Proof. Replacing X by $\overline{X}$ in (36) and using (13) & (5), we get

$$(L_{\overline{X}} u^i)(Y) = (D_{\overline{X}} u^i)(Y) + p^i_j u^j(D_Y X) \tag{40a}$$

Similarly, we can get

$$(L_{\overline{Y}} u^i)(X) = (D_{\overline{Y}} u^i)(X) + p^i_j u^j(D_X Y) \tag{40b}$$

Subtracting (40b) from (40a) and using (16) and (23), we get the required result.

3. An Affine Connexion II

In this section an affine connexion E has been defined in terms of another affine connexion D such that their torsions are equal but opposite in sign. The properties of this affine connexion E have been studied in a generalised structure manifold V_n [18].

Let us define an affine connexion E in V_n by the relation

$$E_X Y \overset{def}{=} -D_X Y + [X,Y] \tag{41}$$

where D is an (F, U_i, u^i)-connexion given by (12) and the torsions of E and D are equal but opposite in sign.

We shall study $\overset{\circ}{N}, \overset{\circ}{M}$ and curvature tensors of connexion E.

Remark 2. *Since the torsions of the connexions D and E are equal and opposite to each other, therefore, if D is half symmetric, semi-symmetric and almost symmetric, E is also half symmetric, semi-symmetric and almost symmetric respectively.*

Theorem 8. *In V_n, we have*

$$E_X Y - E_Y X = D_Y X - D_X Y + 2[X,Y] \tag{42a}$$

$$E_X U_i - E_{U_i} X = D_{U_i} X + 2[X, U_i] \tag{42b}$$

$$E_{\overline{X}} \overline{Y} - E_{\overline{Y}} \overline{X} = \overline{D_{\overline{Y}} X} - \overline{D_{\overline{X}} Y} + 2[\overline{X}, \overline{Y}] \tag{42c}$$

$$\overline{E_X \overline{Y}} - \overline{E_Y \overline{X}} = b^2(D_Y X - D_X Y) + c u^i(D_Y X - D_X Y) U_i + \overline{[X, \overline{Y}]} - \overline{[Y, \overline{X}]} \tag{42d}$$

$$\overline{E_{\overline{X}} Y} - \overline{E_{\overline{Y}} X} = \overline{D_{\overline{Y}} X} - \overline{D_{\overline{X}} Y} + \overline{[\overline{X}, Y]} - \overline{[\overline{Y}, X]} \tag{42e}$$

Proof. The equation (41) yields (42a). Putting U_i for Y in (42a) and using (12), we get (42b). Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in (42a), we get (42c) on the use of (12). The value of $(E_X\overline{Y} - E_Y\overline{X})$ is obtained by making use of (41) and operating F on the resulting equation and using (1) & (12), we get (42d). Similarly, we can get (42e).

Theorem 9. *In V_n, we have*

$$(E_X F)(Y) = [X, \overline{Y}] - \overline{[X, Y]} \tag{43a}$$

$$(E_X F)(U_i) = p^i_j [X, U_j] - \overline{[X, U_i]} \tag{43b}$$

Proof. Replacing Y by $\overline{Y}$ in (41), we have

$$E_X\overline{Y} = -D_X\overline{Y} + [X, \overline{Y}] \tag{44}$$

which on use of (41) yields (43a) and by replacing Y by U_i and using (5), we get (43b).

Theorem 10. *In V_n, we have*

$$c\{(E_X u^i)(Y) + u^i([X, Y])\} = 0 \tag{45}$$

Proof. Replacing Y by $\overline{\overline{Y}}$ in (41) and using (1), we get

$$E_X(b^2Y + cu^i(Y)U_i) = -D_X(b^2Y + cu^i(Y)U_i) + b^2[X, Y] + cu^i(Y)[X, U_i] \tag{46a}$$

Using (1.12) and (3.1) in above, we get

$$c\ U_i(E_X u^i)(Y) = -c\ U_i u^i([X, Y]) \tag{46b}$$

which implies(45).

Now, let us consider Nijenhuis tensor $N(X, Y)$ in V_n, which is given by (14). For the symmetric connexion D, it takes the following form :

$$N(X, Y) = D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X} + \overline{\overline{D_X Y}} - \overline{\overline{D_Y X}} - \overline{D_{\overline{X}} Y} + \overline{D_Y \overline{X}} - \overline{D_X \overline{Y}} + \overline{D_{\overline{Y}} X} \tag{47}$$

Using (1) in above, we get

$$\begin{aligned} N(X, Y) = D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X} + b^2(D_X Y - D_Y X) + c\ u^i(D_X Y - D_Y X)U_i \\ -\overline{D_{\overline{X}} Y} + \overline{D_Y \overline{X}} - \overline{D_X \overline{Y}} + \overline{D_{\overline{Y}} X} \end{aligned} \tag{48}$$

Similar to Nijenhuis tensor for connexion D, let us introduce a tensor $\overset{\circ}{N}(X, Y)$ for the connexion E, given by

$$\begin{aligned} \overset{\circ}{N}(X, Y) \overset{def}{=} E_{\overline{X}}\overline{Y} - E_{\overline{Y}}\overline{X} + b^2(E_X Y - E_Y X) + c\ u^i(E_X Y - E_Y X)U_i \\ -\overline{E_{\overline{X}} Y} + \overline{E_Y \overline{X}} - \overline{E_X \overline{Y}} + \overline{E_{\overline{Y}} X} \end{aligned} \tag{49}$$

Theorem 11. *In V_n, we have*

$$\overset{\circ}{N}(X,Y) = 2([\,\overline{X},\overline{Y}\,] + \overline{\overline{[X,Y]}} - \overline{[X,\overline{Y}]} - \overline{[\overline{X},Y]}) = 2N(X,Y) \tag{50}$$

Proof. Using (42a), (42c), (42d) and (42e) in (49), we obtain

$$\begin{aligned}
\overset{\circ}{N}(X,Y) = {} & (\overline{D_{\overline{Y}}X} - \overline{D_{\overline{X}}Y} + 2[\,\overline{X},\overline{Y}\,]) + b^2(D_YX - D_XY + 2[X,Y]) \\
& + c\, u^i(D_YX - D_XY + 2[X,Y])U_i - b^2(D_YX - D_XY) \\
& - c\, u^i(D_YX - D_XY)U_i - \overline{[X,\overline{Y}]} + \overline{[Y,\overline{X}]} \\
& - \overline{D_{\overline{Y}}X} + \overline{D_{\overline{X}}Y} - \overline{[\overline{X},Y]} + \overline{[\overline{Y},X]} \\
& = 2([\,\overline{X},\overline{Y}\,] + b^2[X,Y] + c\, u^i([X,Y])U_i - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]})
\end{aligned}$$

which on use of (1) and (14) yields (50).

Corollary 3. *In V_n, we have*

$$\overset{\circ}{N}(X,U_i) = 2\,p^i_j([\,\overline{X},U_j\,] - \overline{[X,U_j]}) + 2(\overline{\overline{[X,U_i]}} - \overline{[\overline{X},U_i]}) \tag{51}$$

Let us define $\overset{\circ}{\mu}$, $\overset{\circ}{\nu}$, $\overset{\circ}{\sigma}$ analogues to μ, ν, σ for connexion E.

$$\overset{\circ}{\mu}(X,Y) \overset{def}{=} (E_Yu^i)(\overline{X}) - (E_Xu^i)(\overline{Y}) + (E_{\overline{Y}}u^i)(X) - (E_{\overline{X}}u^i)(Y) \tag{52}$$

$$\overset{\circ}{\nu}(X) \overset{def}{=} (E_{U_i}F(X) - (E_XF)(U_i) - E_{\overline{X}}U_i \tag{53}$$

$$\overset{\circ}{\sigma}(X) \overset{def}{=} (E_Xu^j)(U_i) - (E_{U_i}u^j)(X) \tag{54}$$

Theorem 12. *In V_n, we have*

$$c\,\overset{\circ}{\mu}(X,Y)U_i = 2c\{u^i([\overline{X},Y]) + u^i([X,\overline{Y}])\} \tag{55a}$$

$$\overset{\circ}{\nu}(X) = \{[X,\overline{U_i}] + 2[\overline{X},U_i] - \overline{[X,U_i]} + \overline{E_{U_i}X} + D_{U_i}\overline{X}\} \tag{55b}$$

$$c\,\overset{\circ}{\sigma}(X) = 2c\{u^j([U_i,X])\} \tag{55c}$$

Proof. On account of (45) and (52), we get(55a). Due to (42b) and (43b), we obtain (55b). Finally, (55c) is obtained by using (45) in (54).

Corollary 4. $\overset{\circ}{\mu}(X,Y)$ *is skew-symmetric in both the slots X and Y, i.e.*

$$\overset{\circ}{\mu}(X,Y) + \overset{\circ}{\mu}(Y,X) = 0 \tag{56}$$

Let us define a vector valued, bilinear function $\overset{\circ}{M}$ by

$$\overset{\circ}{M}(X,Y) \overset{def}{=} E_{\overline{X}}\overline{Y} + \overline{\overline{E_XY}} - \overline{E_{\overline{X}}Y} - \overline{E_X\overline{Y}} \tag{57}$$

Theorem 13. *In V_n, we have*

$$\overset{\circ}{M}(X,Y) - [\overline{X},\overline{Y}] - \overline{\overline{[X,Y]}} + \overline{[X,\overline{Y}]} + \overline{[\overline{X},Y]} = 0 \tag{58a}$$

$$\overset{\circ}{M}(X,Y) - N(X,Y) = 0 \tag{58b}$$

Proof. Using (12), (14) and (41) in (57), we get (58a) and (58b).

Corollary 5. $\overset{\circ}{M}(X,Y)$ *is skew-symmetric in both the slots X and Y, i.e.*

$$\overset{\circ}{M}(X,Y) + \overset{\circ}{M}(Y,X) = 0 \tag{59}$$

Corollary 6. *In, V_n, we have*

$$\overset{\circ}{M}(X,U_i) = \overline{\overline{[X,U_i]}} - \overline{[\overline{X},U_i]} + p^i_j\{\overline{[\overline{X},U_j]} - \overline{[X,U_j]}\} \tag{60}$$

It can be obtained that,

$$K(X,Y,U_i) = 0 \tag{61a}$$

$$K(X,Y,\overline{Z}) = \overline{K(X,Y,Z)} \tag{61b}$$

where K is the curvature tensor of (F,U_i,u^i)-connexion.

Let us define a curvature tensor $\overset{\circ}{K}$ with respect to connexion E, by

$$\overset{\circ}{K}(X,Y,Z) \overset{def}{=} E_X E_Y Z - E_Y E_X Z - E_{[X,Y]} Z \tag{62}$$

Theorem 14. *In, V_n, we have*

$$\overset{\circ}{K}(X,Y,Z) = K(X,Y,Z) + 2D_{[X,Y]}Z - [X,D_Y Z] + [Y,D_X Z] - D_X([Y,Z]) + D_Y([X,Z]) \tag{63}$$

Proof. From (41), we have

$$E_X E_Y Z = D_X D_Y Z - [X,D_Y Z] - D_X([Y,Z]) + [X,[Y,Z]] \tag{64}$$

$$-E_Y E_X Z = -D_Y D_X Z + [Y,D_X Z] + D_Y([X,Z]) - [Y,[X,Z]] \tag{65}$$

$$-E_{[X,Y]} Z = -D_{[X,Y]} Z - [[X,Y],Z] \tag{66}$$

Adding (64), (65) and (66) and using (11), *Jacobi Identity* $([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0)$ and (62), we get (63).

Corollary 7. *In, V_n, we have*

$$\overset{\circ}{K}(X,Y,U_i) = D_Y([X,U_i]) - D_X([Y,U_i]) \tag{67a}$$

$$\overset{\circ}{K}(X,U_i,U_i) = D_{U_i}([X,U_i]) \tag{67b}$$

Proof. Replacing U_i for Z in (63) and using (12) and (61a), we get (67a). (67b) is obtained by putting U_i for Y in (67a).

4. An Affine Connexion III

In this section an affine connexion is considered which is different one from section 2 and section 3. Its properties are also have been studied.

Let us consider an affine connexion D in a generalised structure manifold V_n with torsion tensor S such that

$$(D_X u^i)(Y) + (D_Y u^i)(X) = 0 \tag{68a}$$

$$(D_X F)(Y) + (D_Y F)(X) = 0 \tag{68b}$$

$$(D_X U_i) = 0 \tag{68c}$$

Theorem 15. *In, V_n, we have*

$$D_{U_i}\overline{Y} = \overline{D_{U_i}Y} \tag{69a}$$

$$D_{\overline{Y}}\overline{X} + b^2 D_X Y - (\overline{D_X\overline{\overline{Y}}} + \overline{D_{\overline{Y}}X}) = -c\, X(u^i(Y))U_i \tag{69b}$$

$$\overline{D_X\overline{\overline{Y}}} + \overline{D_{\overline{Y}}X} - b^2(D_X Y + D_Y X) = c\, u^i(D_X Y + D_Y X)U_i \tag{69c}$$

$$b^2(D_{\overline{X}}Y + D_{\overline{Y}}X) - (\overline{D_{\overline{X}}\overline{\overline{Y}}} + \overline{D_{\overline{Y}}\overline{X}}) = -c\, \{\overline{X}(u^i(Y)) + \overline{Y}(u^i(X))\} \tag{69d}$$

$$\overline{D_{\overline{X}}Y} + \overline{D_{\overline{Y}}X} - D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X} = 0 \tag{69e}$$

Proof. Taking covariant derivative of $F(Y) = \overline{Y}$ with respect to U_i and using (68c), we get (69a). Now, (68b) is equivalent to

$$D_X\overline{Y} + D_Y\overline{X} = \overline{D_X Y} + \overline{D_Y X} \tag{70}$$

Replacing Y by $\overline{Y}$ in (70) and using (1), we get (69b). By operating F on both sides of (70) and using (1), (69c) can be obtained. Replacing X by $\overline{X}$ in (69b), we get (69d). Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in (70) separately and adding the resulting equations, we get

$$\overline{D_{\overline{X}}Y} + \overline{D_{\overline{Y}}X} - D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X} = D_Y\overline{\overline{X}} + D_X\overline{\overline{Y}} - \overline{D_X\overline{Y}} - \overline{D_Y\overline{X}} \tag{71}$$

Using (1), (68a) and (69c) in (71), we get (69e).

Let us define a tensor H of the type $(1,2)$ by

$$H(X,Y) \overset{def}{=} D_{\overline{X}}\overline{Y} - \overline{D_{\overline{X}}Y} \tag{72}$$

Theorem 16. *In, V_n, we have the following relations :*

$$H(X,Y) + H(Y,X) = 0 \tag{73a}$$

$$H(X,\overline{Y}) + H(Y,\overline{X}) = 0 \tag{73b}$$

$$H(\overline{X},\overline{Y}) - b^2 H(X,Y) = -c[\{p_i^j\overline{Y}(u^j(X))U_i\} + \overline{\overline{Y}(u^i(X))U_i}] \tag{73c}$$

Proof. (73a) follows from (69e) and (72). Now replacing Y by $\overline{Y}$ in (72) and using (1), we get

$$H(X,\overline{Y}) = b^2 D_{\overline{X}} Y + c\ \overline{X}(u^i(Y))U_i - \overline{D_{\overline{X}}\overline{Y}} \tag{74}$$

Interchanging X and Y in (74) and adding the resulting equation with (74), we get (73b), by making use of (69d). Replacing X by $\overline{X}$ in (73b) and using (1) and (5), we obtain (73c).

Theorem 17. *In, V_n, we have*

$$M(X,Y) = 2H(X,Y) - B(X,Y)U_i \tag{75}$$

where,

$$B(X,Y) = (D_X u^i)(Y)$$

Proof. Adding (69b) and (69e), we have

$$\overline{D_{\overline{X}} Y} - D_{\overline{X}}\overline{Y} - \overline{D_X \overline{Y}} + b^2 D_X Y + c\ X(u^i(Y))U_i = 0 \tag{76}$$

Using (1) in (20), we get

$$M(X,Y) = D_{\overline{X}}\overline{Y} + b^2 D_X Y + c\ u^i(D_X Y)U_i - \overline{D_{\overline{X}} Y} - \overline{D_X \overline{Y}} \tag{77}$$

Due to (76) and (77) yields (75).

Theorem 18. *The connexion D is an F-connexion if and only if*

$$H(X,Y) = 0 \tag{78}$$

Proof. Let D be an F-connexion, then (13) & (72), gives $H(X,Y) = 0$. Conversely, if (78) is true, then $D_{\overline{X}}\overline{Y} = \overline{D_{\overline{X}} Y}$. Replacing X by $\overline{X}$ in this relation, we get

$$b^2 D_X \overline{Y} + c\ u^i(X) D_{U_i}\overline{Y} = b^2\ \overline{D_X Y} + c\ u^i(X)\overline{D_{U_i} Y} \tag{79}$$

Due to (69a), (79) yields $(D_X F)(Y) = 0$, which implies that D is an F-connexion.

Theorem 19. *When D is an F-connexion ,any one of the following holds if remaining two hold :*

(a) $H(X,Y) = 0$

(b) $B(X,Y) = 0$

(c) $M(X,Y) = 0$

Remark 3. *All the results discussed in section 2, section 3 and section 4 are true in an almost tangent metric manifold, an almost Hermite manifold, metric π-structure manifold, Hsu-structure manifold, F-structure manifold, an almost product Riemannian manifold, an almost Grayan manifold and*
$\{F, g, u^1, u^2, U_1, U_2\}$ structure manifold if $(b^2 = 0, c = 0)$; $(b^2 = -1, c = 0)$; $(c = 0)$; $(b^2 = \lambda^r, c = 0)$; $(b^2 = -1, p_i^j = 0)$; $(b^2 = 1, c = 0)$; $(b^2 = -1, c = 1, p_1^1 = 0 : i, j = 1)$; *and* $(b^2 = -1, c = 1, p_i^j + p_j^i = 0 : i, j = 1, 2)$ *respectively.*

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